Radiative Heat Transfer
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Module - 5
Lecture - 21
Approximate Methods - II

Hello friends, we are discussing the solution of radiative transfer equation. In the previous lecture, we discussed approximate methods. We discussed methods where we have approximated either the dimension of the problem; that is, 1-dimensional problems, plane parallel slab or cylindrical geometry. Or we approximated the properties of the medium as optically thin or optically thick.

And we also discussed that these 2 type of approximations are little restrictive in the sense that, in practical problems the solution or the domain of the problem is not 1-dimensional nor optically thick or thin. The third approximation that we are going to discuss in this lecture is based on the dependence of intensity on direction.

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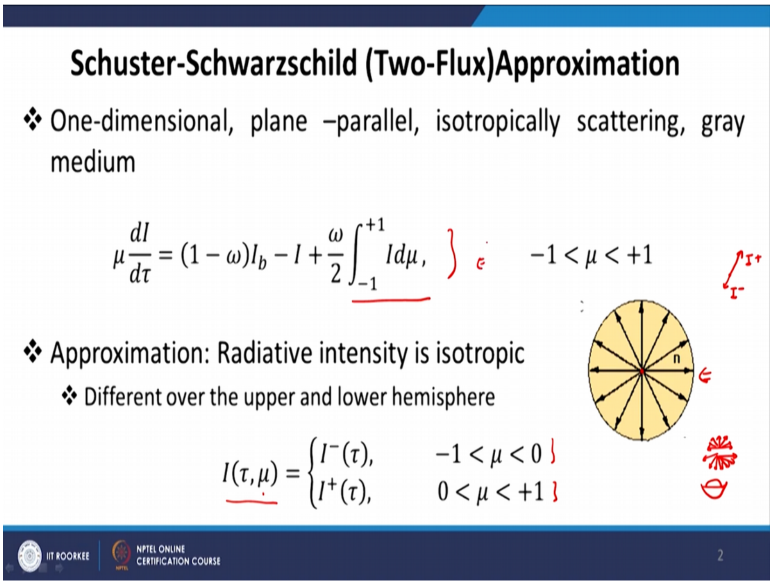

So, the radiative transfer equation in a given direction; that means, in a single direction, it given by mu times dI by d tau is = 1 – omega I b – I omega by 2 integration from – 1 to 1 I times d mu. Of course, this is an integral differential equation and the problem is little difficult to solve analytically, if we do not have some way to approximate the intensity function that appears in the integration.

So, for 1-dimensional case, it has been argue bide by Schuster and Schwarzschild after which this method is based, the method is basically called two-flux approximation or Schuster-Schwarzschild approximate methods. So, what this method basically assumes is that the intensity is isotropic. It, but the isotropic intensity may have different magnitudes in the upward direction and downward direction.

So, how does an isotropic intensity looks like? As it is represented by this image, at any point, the intensity has same magnitude in entire 4 pi solid angle. If this is the case, this is called isotropic intensity. What Schuster-Schwarzschild assumed that although the intensity is isotropic, but it has different magnitudes in the upward and bottom, downward direction. So, the intensity will follow certain distribution in the upward direction with same intensity.

And in the downward direction, the intensity will be different, but it will be isotropic in all the downward directions. Okay. So, this distribution looks like this. We have different intensity in the upward direction and different intensity in the downward direction. Mathematically we can write this as, intensity at any point in the medium bounded by 2 parallel plates and in any direction, mu is basically has 2 components I – and I +.

So, at any point in a given direction, we have 2 components, I + and I –. And I + and I – may be different. I + is the intensity going in the upward direction. I – is the intensity going in the downward direction. So, for I –, that is mu < 0 and for I +, the value of mu is > 0, between 0 and 1. The intensity is different. So, when we do that, this integration that appears in the integral differential equation is simplified. So, substituting the expression of I in this equation;

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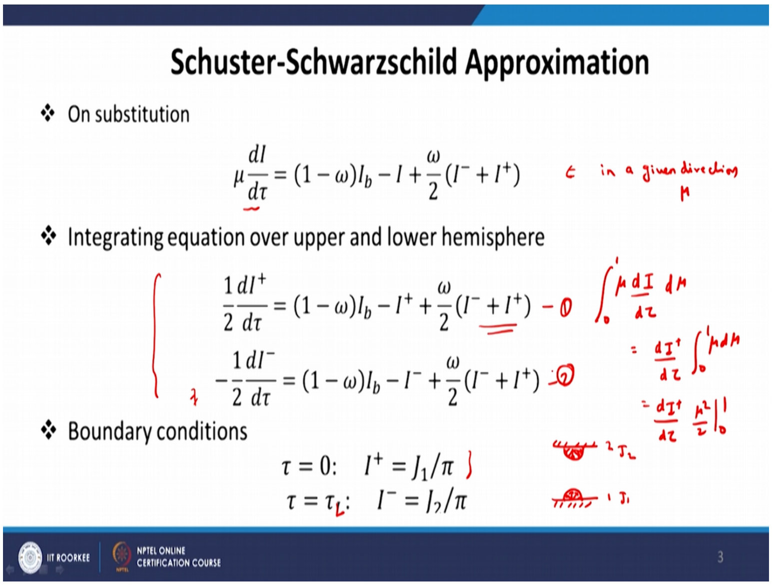

We get mu times dI by d tau is = 1 – omega I b – I + omega by 2 I – + I +. So, we see that, with this assumption or approximation on intensity, the integral differential equation has reduced to just differential equation. But this differential equation has to be solved on all the solid angles. This is in a given direction. This is in a given direction mu. So, we still have to integrate this equation over all the solid angles.

So, we integrate over upper and lower hemisphere. When we do that, we basically get; for this one, for I + half d I + upon d tau is = 1 – omega I b – I + + omega by 2 I – + I +. This is constant; so, integration on this quantity does not affect. Similarly, on lower hemisphere, we get – 1 by 2 d I – upon d tau is = 1 – omega I b – I – + omega by 2 I – + I +. Now, why this factor, half is coming? I will just solve it for you. So, mu dI by d tau.

We have to integrate over the upper hemisphere. That is, 0 to 1 d mu. Now, on the upper hemisphere, the intensity is constant as I +. So, d I +, we can take out. And in the integration, we are just left with mu d mu. And 0 to 1. So, this will be mu square by 2. So, d I + upon d tau mu square by 2 0 to 1. And this becomes half. So, this is why this half is coming. And in the negative direction, – is also coming, because mu is < 0.

The boundary conditions again will be isotropic. On the upper plate we have diffuse and isotropic intensity, given by radiosity J 1. Sorry, J 2 by pi. And similarly, on the bottom plate, the intensity is isotropic, given by radiosity J 1 and J 2. So, I + is J 1 upon pi at tau is = 0. And I – is J 2 by pi at tau is = tau L. So, these are the boundary conditions.

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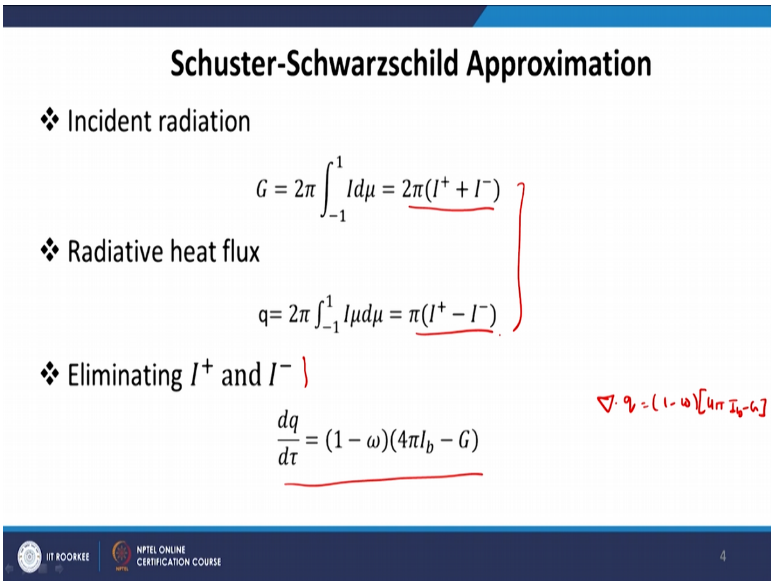

Now, we define incident radiation. So, incident radiation is integration of I over all the solid angles. 2 pi – 1 to 1 pi times d mu. And this can be integrated by substituting the value of I from 0 to 1 and – 1 to 0. So, this will be simply = 2 pi I + + I –. Radiative heat flux similarly is defined as to pi integration over 1 – 1 to 1 mu times I d mu. And this simplifies to pi times I + – I –. Okay. So, this is the expression for radiative heat flux and radiative incident radiation.

Now we want to eliminate I + and I –, because we do not know the, these quantities. So, what we do is, we add these 2 equations and subtract. So, let us call this equation 1 and let us call this equation 2. So, adding equation 1 and 2 will give us the value of radiative heat flux. And subtracting will give us the value of G. So, we get; eliminate I + and I –. So, we get this expression for del dot q.

dq by d tau is = 1 – omega 4 pi I b – G, which is same expression which we have already discussed earlier; the relation between del dot q is related to incident radiation. We have already derived this result before. Okay. So, we get the same equation by adding, by eliminating I + and I –.

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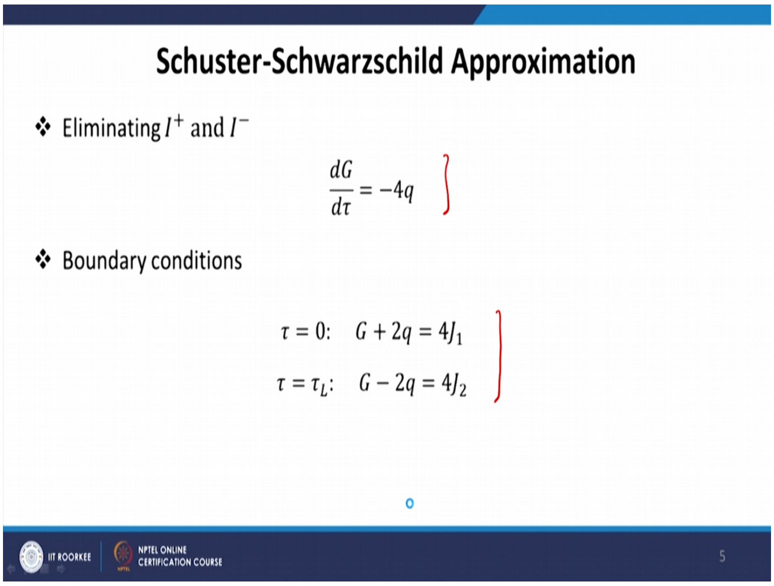

In, together we get 1 more equation. That is dG by d tau is = – 4 q. So, by adding the 2 equations and subtracting the 2 equations, we get 2 equations. 1 is for del dot q and 1 is 4 dG by d tau. These 2 equations, we have to solve. These are now ordinary differential equations which are very easy to solve. So, just by assuming the intensity to be isotropic, we have eliminated the integration in the differential equation and our equation has simplified to simple ordinary differential equations to coupled ordinary differential equations.

The boundary condition can similarly be found. So, our boundary conditions was: I + is = J 1 upon pi and I – is = J 2 upon pi. So, eliminating I + and I – in terms of G and q, we get G + 2q is = 4 times J 1. And G – 2q is = 4 times J 2. So, where we have added these 2 equations. So, G + 2q will give us I +, which is basically the boundary condition at the bottom surface. And subtracting this giving us I –, which is the boundary condition at the top surface. So, in terms of G and q, we have represented our equations and boundary conditions. And now, these can be easily solved.

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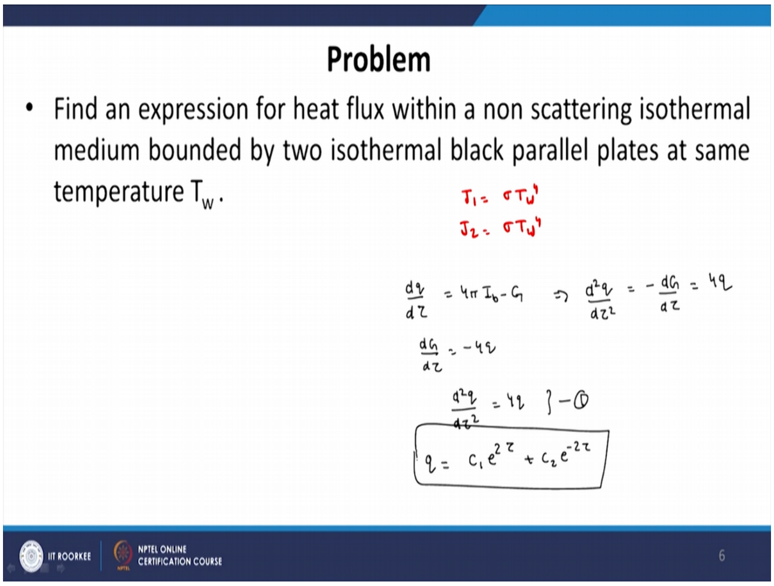

Now, let us see how this method can be applied to simple problem where we have to find out heat flux within a non-scattering isothermal medium bounded by 2 isothermal black parallel plates at the same temperature T w. So, plates are parallel and they are black. So, J 1 is simply = sigma T w power 4. And J 2 is simply = sigma T w power 4. They are black; so, the radiosity is simply = the emissive power of the black body. Okay.

So, let us solve this equation. We write this equation first. So, we have, so, first equation is dq by d tau. So, let me just show you the equation. So, dq by d tau is = 1 – omega 4 pi I b – G. And omega is 0. So, this equation is basically simply = 4 pi I b – G. 4 pi I b – G. And the second equation is, dG by d tau is = – 4q. Okay. To solve this system of ordinary differential equation, we differentiate this equation with respect to tau.

So, we get d square q by d tau square is =; now, the I b is constant, this is isothermal medium; so, the first term derivative, first term will be 0 and this will be = – dG by d tau. And this will be = simply 4 times q. Okay. So, our equation becomes d square q by d tau square is = 4q. Let us call this equation 1. Now, this equation can be solved by complementary function method. So, we have this value of q using standard approach of solving ordinary differential equation which we are familiar.

So, q is = C 1 e power 2 tau + C 2 e power – 2 tau. This is the solution of radiative heat flux at any point in the medium. So, although the medium temperature is uniform, the heat flux is not. Okay. It is varying at different locations. Now, let us simplify this expression further.

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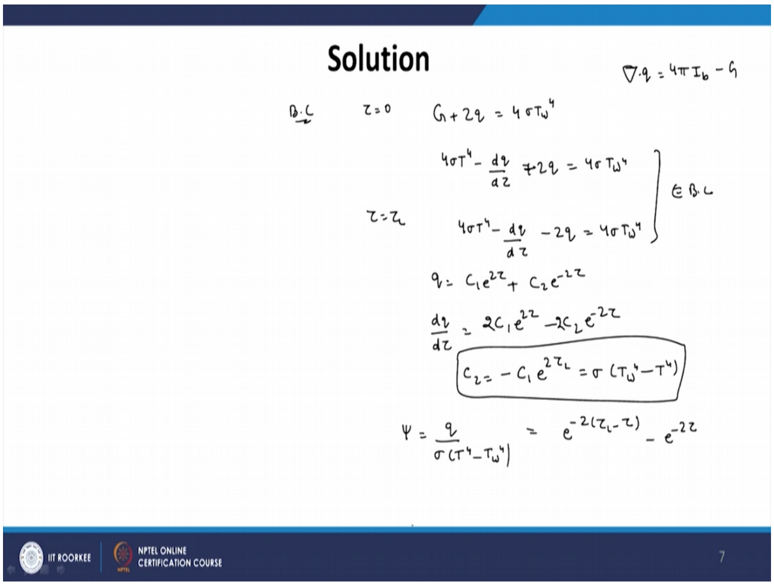

So, boundary conditions we apply. So, boundary condition will be at tau is = 0. So, the first boundary condition was G + 2q is = 4 J 1. G + 2q is = 4 times J 1. J 1 is simply sigma T power 4. Okay. Now, we substitute for the expression for G. Okay. So, we substitute for the expression for G. And we get 4 sigma T 4 – dq by d tau is = + 2q is = 4 sigma T w 4; where we have used the expression G is = 4 pi I b or 4 sigma T power 4 –; sorry; this is del dot q.

This is del dot q or dq by d tau. So, del dot q is 4 pi I b – G. So, G is = dq by d tau 4 pi I b – dq by d tau. That is what we have done. We have eliminated G in the expression by substituting for dq by d tau. Similarly, tau is = tau L. The condition will be 4 sigma T power 4 – dq by d tau – 2q is = 4 sigma T w power 4. So now, this is the 2 boundary conditions we have. Okay. So, applying the boundary conditions, okay. So, this was the solution.

So, we apply the boundary condition here. Now, q is = C 1 e power 2 tau + C 2 e power – 2 tau. And dq by d tau is simply = C 1 2 C 1 e power 2 tau – 2 C 2 e power – 2 tau. And we substitute the value of q and dq by d tau in these 2 boundary conditions. So, we get the expression for C 1 and C 2. So, this we get C 2 is = – C 1 e power 2 tau L. Okay. And this will be = sigma T w power 4 – T power 4.

So, this is the expression for the coefficient, unknown coefficient C 1 and C 2. So, C 1 and C 2 are related. And the value of C 2 is sigma T w 4 – T power 4. Now, the non-dimensional radiative flux psi is = q by sigma T power 4 – T w power 4. Okay. So, this value will be simply = e power – 2 tau L – tau. And – e power – 2 tau. Okay. So, this is the expression for the radiative heat flux. And we can show the result.

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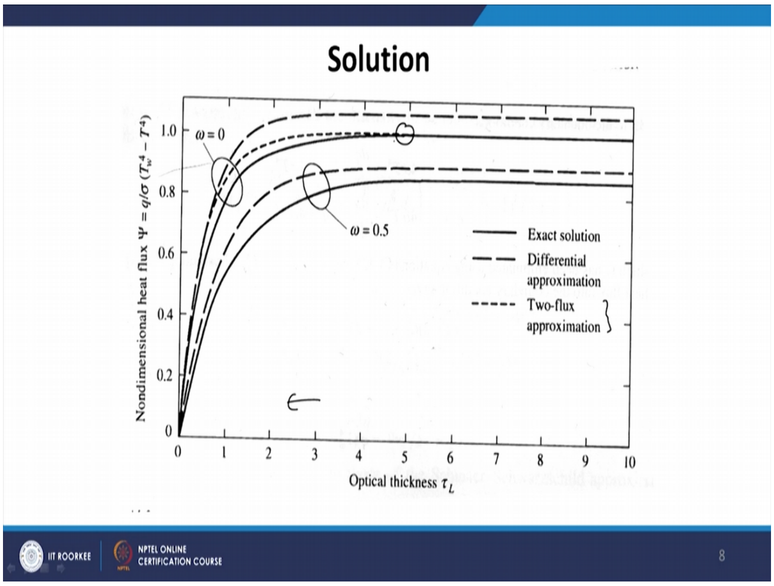

In this chat, the result for the two-flus approximation method is shown. And you can observe here. So, two-flux approximation is basically the Schuster-Schwarzschild method. And we see that, for optically thin case, that is tau L value is very small, the method is exact. That means, it always goes to correct optically thin limit. The values are correct for optically thin limit. And for optically thick cases, we see that the method is relatively more accurate than the differential approximation method, which is our, which is the method we are going to discuss next.

But still it gives good agreement for optically thick cases also. So, for this 1-dimensional problem between parallel plates, the two-flux approximation method gives very good results for optically thin as well as for optically thick cases. The next method that we will discuss is;

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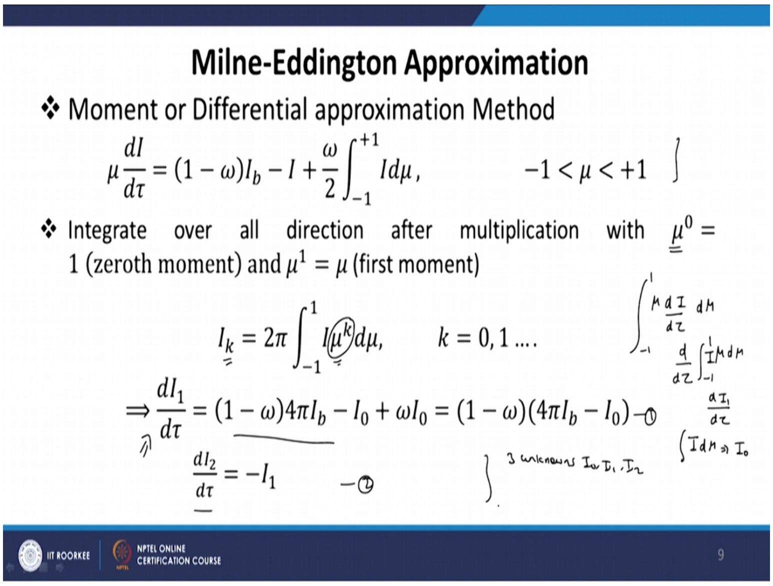

The differential approximation or Milne-Eddington method. So, this method again is based on the approximation of intensity. As we discussed in the previous method, we approximated the intensity using isotropic with different upper and lower hemispherical intensity. This method is also based on the same concept where we are approximating the intensity. The mathematical procedure is little different than the previous method.

So, what we do in this method is, we take, the starting point is the same integro differential equation for radiative transfer in a given direction mu, where mu varies from – 1 to 1 or theta varies from – pi to pi. We integrate over all directions, but before integration we multiply by mu power k. So, we are multiplying by mu power k. Okay. If k is 0, it is called zeroth moment and if k is = 1, we call it the first moment.

In mathematics, this procedure of multiplying a quantity by certain quantity raised to power k, and then integrating over all the values of that quantity is called moment. Okay. So, for example, I subscript k is basically known as kth moment of intensity, where the moment is defined as integration over all possible values of mu and multiplied by mu power k with intensity. Okay. So, this is the kth moment of the intensity.

k is = 0 means zeroth moment and k is = 1 is known as the first moment. Okay. You must have heard this moment in moment of inertia, the first moment of area, second moment of area and so on. Okay, so this, in mathematics, this is called moment method. Okay. So, when we multiply by mu 0, that is 1. It is called zeroth moment. So, we multiply first by 1, this equation, the radiative transfer equation and integrate over all the solid angles.

When we integrate over all the solid angles, the equation is basically transformed. d I 1 by d tau where I 1 is the first moment. And then, the second, right-hand side simply becomes 1 – omega 4 pi I b – I 0 + omega I nought. So, again we see that, we have got rid of the integro differential equation and we have simplified this equation in a simple ordinary differential equation.

So, 1 – omega 4 pi I b – I nought is the first equation that we have obtained by taking zeroth moment of this radiative transfer equation. Now, just to demonstrate you, let us take this quantity mu dI by d tau. So, we have to integrate this quantity by multiplying by 1. So, we multiplied by 1. And then, we have to integrate it with respect to mu. And the mu value varies from – 1 to 1. Okay. Now, this quantity d by d tau, we can take out.

So, it becomes – 1 to 1 mu I times mu d mu. Okay. So, this becomes the first moment. So, we just call it d I 1 d tau. Okay. And that is what basically we have got in this equation. Similarly, if we have just I and we want to integrate it with respect to mu, this becomes the zeroth moment and we can write it simply I nought. Okay. So, we got this equation; let us call this equation 1. Now, what we do is, we multiply by mu.

That is, we take the first moment and then integrate overall the solid angles. So, the first quantity becomes d I 2 by d tau. And the right-hand side becomes simply = – I 1. Okay. So, this is the quantity that we have obtained. So, we have, let us call this equation as 2. Okay. So, we have obtained 2 equations. Okay.

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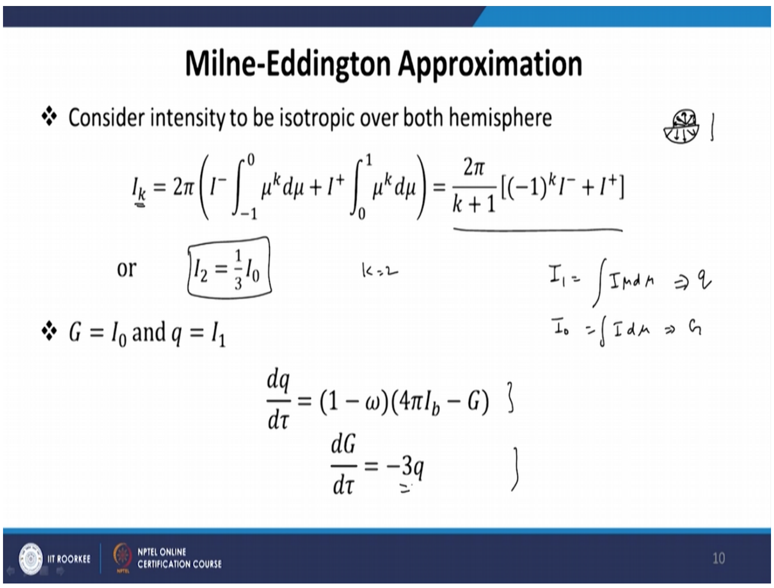

Now, we have to define the boundary conditions and we have to do simplification. Because in these 2 equations, we have 3 unknowns: I nought, I 1 and I 2. And we have 2 equations. So, we have to remove a 1 variable here. So, what basically Milne-Eddington did, they did the same thing what basically Schuster-Schwarzschild did; assuming the intensity to be isotropic over both the hemisphere.

That is, the upper intensity is isotropic and the bottom intensity is also isotropic, although they may have same, similar or different magnitude. So, this approximation is basically the same. And we will see that it leads to similar equation that we developed for two-flux method. So, we write the kth moment as, integration over – 1 to 0, where intensity is going to be – I –. So, 2 pi I – – 1 to 0 mu k d mu + integration over 0 to 1 mu k d mu. And this quantity becomes 2 pi k + 1 – 1 power k I – + I +.

So, with value of k is = 2, this is simply leads to I 2 is = 1 by 3 I nought. That is, second moment of intensity is one by third of first moment of intensity. If we assume that the method the intensity is isotropic in such a way that upward intensity is I + and the downward intensity is I –. We should also observe that the first moment I 1 is basically I times mu d mu; is basically nothing but heat flux.

And we also observe that I nought which is zeroth moment, is nothing but the radiative intensity radiation G. So, we replace I nought with G and q with I 1. And what we get is, this is the first equation that we had. dq by d tau is = 1 – omega 4 pi I b – G. So, this is same equation we had in the two-flux method. And the second equation, d I 2 by d tau is = – I 1 is reduced to dG by d tau is = – 3q.

Now, this is slightly different from the previous method of two-flux, where we had – 4q. So, the two-flux method had – 4q and this differential approximation method has – 3q. Rest of the things are same.

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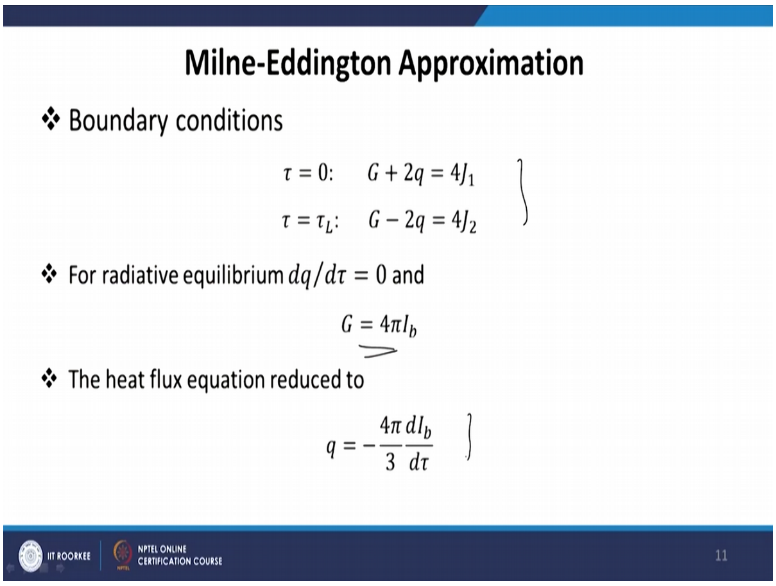

The boundary conditions are also same. Okay. So, for radiative equilibrium, dq by d tau is = 0. And we get G = 4 pi I b. And the heat flux equation reduces to – 4 pi by 3 d I b by d tau. Okay. So, this thing, we have already developed. So, the method; it turns out to be, the mathematical procedure is entirely different. Here we have used the approach of moments. But because the intensity was approximated as isotropic, the governing equations are very similar. In fact, 1 equation is exactly the same as the two-flux method.

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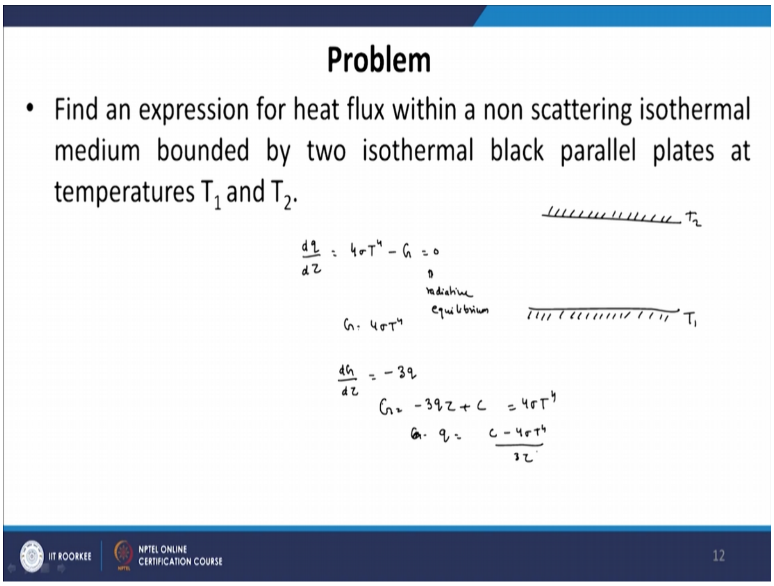

Okay now, let us solve the problem and demonstrate how this method can be used to solve a practical problem. So, in this case, we have again a non-scattering medium bounded by 2 plates. And the 2 plates are maintained at different temperature. Okay. So, let us say we have 2 plates. The bottom plate is at temperature T 1, it is black. And the above plate is at temperature T 2. Okay. It is again black.

So, we write down the equation. dq by d tau is =; now, omega is 0, it is not, non-catering medium. So, we get 4 sigma T power 4 – G. And this is = 0. Why 0? Because, radiative equilibrium. So, G is simply = 4 sigma T power 4. Okay. Now, the second equation. dG by d tau is = – 3 times q. Now, substituting for; from this, we get G is = – 3q tau + some constant C. And this value is simply 4 sigma T power 4. Okay. So, we get G is, q is =; basically, we get C – 4 sigma T power 4 upon tau. So, this is the expression for the radiative heat flux. Now, we apply the boundary condition.

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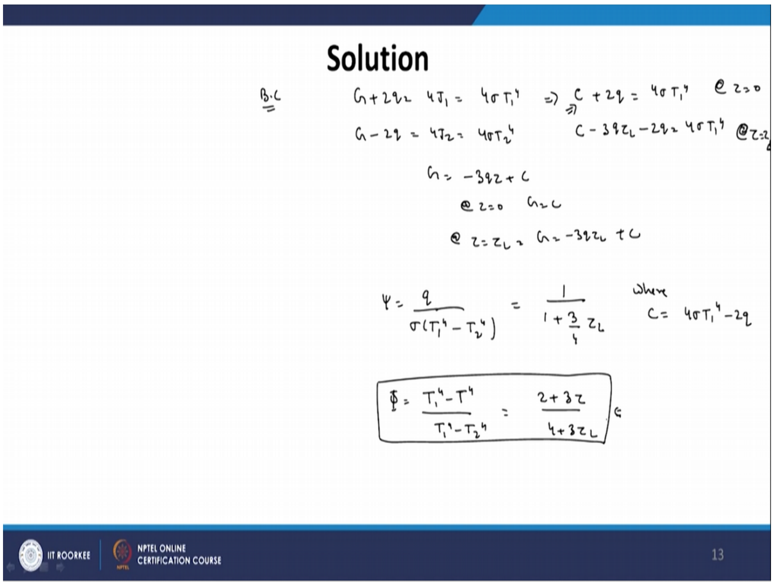

So, boundary condition are basically the same. G + 2 q is = 4 times J 1. That is, 4 sigma T 1 power 4. And G – 2q is = 4 times J 2 is = 4 sigma T 2 power 4. Now, the 2 plates are at different temperature. So, that is why J 1 and J 2 are not the same. But they are black. So, J 1 is simply = sigma T 1 power 4. And J 2 is simply = 4 sigma T 2 power 4. Okay. Now, we substitute the value of G in this case.

So, we get; this equation basically reduced to C + 2q is = 4 sigma T 1 power 4. Because G is constant, so C + 2q is = 4 sigma T 1 power 4. And this equation reduces C – 3q tau L – 2q is = 4 sigma T 1 power 4. So, please note, G is =, the expression for G is – 3q tau + C. So, at tau is = 0, G is simply = C. So, this is at tau is = 0. So, we have substituted G is = C. And at tau is = tau L, G is = – 3q tau L + C.

And that is why we have this at tau is = tau L. Okay. So, we have written our boundary conditions in terms of q. Okay. Now, we will; so, we have the solution for q and we have to solve for this constant C. So, let us solve this. So, we get, non-dimensional heat flux psi is = q by sigma T 1 power 4 – T 2 power 4. Okay. So, from this equation, we have to find out the expression for C using the boundary condition.

We get this value as = 1 by 1 + 3 by 4 tau L. Okay. Where, now C has been found as = 4 sigma T 1 4 – 2q. The constant C from here, 4 sigma T 1 power 4 – 2q is with the, basically the value of the constant C. Okay. And the non-dimensional emissive power phi is = T 1 power 4 – T power 4 upon T 1 power 4 – T 2 power 4. That is non-dimensional emissive power, temperature power 4 is = 2 + 3 tau 4 + 3 tau L. Okay.

So, this is the expression for non-dimensional emissive power or basically a measure of temperature distribution between the 2 parallel plates. So, temperature or temperature power 4 or the emissive power varies with this expression like, as in this expression. So, we have basically calculated in this lecture the radiative heat flux and non-dimensional temperature distribution between 2 parallel plates using 2 different methods.

The method of Schuster-Schwarzschild, two-flux approximation and the method of Milne-Eddington, the differential approximation and the results that we have already shown before in an image show that the method of two-flux is a relatively more accurate for optically thick cases than this method on differential approximation. So, Schuster-Schwarzschild method gives better results than the Milne-Eddington method as far as the optically thick heat flux is concerned.

So, thank you very much. In the next lecture, we will discuss approximate methods spherical harmonics and discrete ordinate method. These 2 methods that we are going to discuss in the next lecture are most widely used method for radiative heat transfer equation and are available in many commercial packages like Ansys and Star CCM. So, thank you for now.