Radiative Heat Transfer
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Lecture - 20
Approximate Methods – I

Hello friends, In the previous lecture, we discussed the solution of radiative heat transfer in one dimensional media. In this lecture, we will study approximate methods for the solution of radiative transfer equation.

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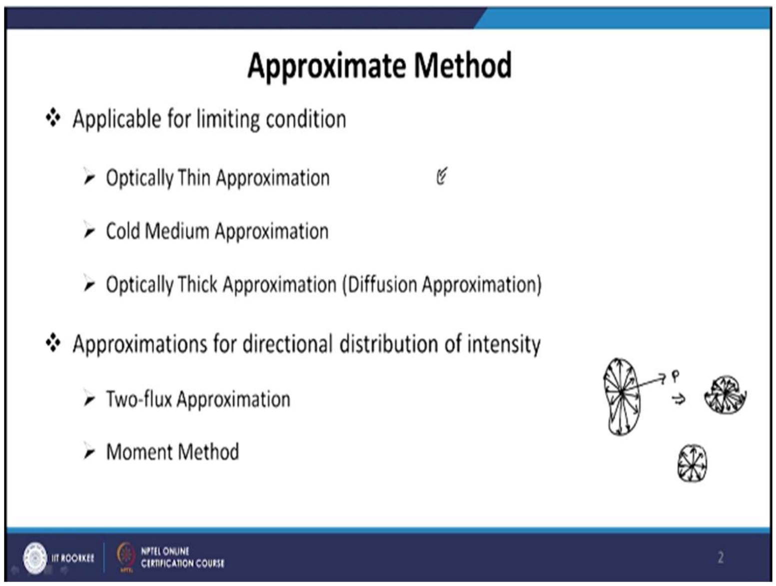

Now when we talk about Approximate Methods, there are number of ways we can approximate the radiative transfer equation. Some of these approximations are based on the properties of the media such as the absorption coefficient and scattering, we can assume non-scattering media, we can assume non-observing media, we can assume isotropic media, we can have the choice to assume that the medium is optically thin that is κ_λ is very small.

We can have the choice to assume that the medium is optically thick or cold. So these are basically the approximation based on the properties of the medium the observing catering medium. The second way to approximate the radiative transfer equation is by approximating the geometry of the problem. As we discussed in the previous lecture, we can have one dimensional problems, plain parallel slab or cylindrical media where the radiative intensity is a function of one space coordinate.

In plain parallel slab the intensity was just a function of Z and θ and θ while for the cylindrical media it was function of r, θ and ψ, the polar and azimuthal angle. The third level of approximation is based on the intensity itself, and this is the approximation which is mostly used in many commercial radiative transfer solvers. The reason is the approximation on radiative properties assuming them to be optically thin or thick is very restrictive.

We have very less practical problems where we have optically thin or thick media. So this approximation is theoretically good but has very limited practical value. Similarly, one-dimensional problem like cylindrical problems or plain parallel slab also have very limited applications. On the other hand, most of the practical problems two-dimensional or three-dimensional with continuously varying properties and therefore the approximations that I just mentioned are not valid.

So the third approximation based on the direction of the intensity is used in many, many radiative solver, so solve the radiative transfer equation. So I will just quickly explain you the fact of this intensity. So in practical problem at any point the intensity may vary in a certain manner. So this is that is a point P and the intensity may continuous vary along the solid angle; it may be very large at certain angle and then it continuously varies.

So the intensity is a function of θ and μ, okay. So we see that there is a continuous variation of intensity with respect to direction. Now many methods basically approximate this dependence. For example, the Two-flux Approximation assumes that the intensity can be; assume to have this type of variation where we have intensity, one value for the downward direction and another value in the upward direction that is for $\mu = 0$ to 1 we have one value of intensity and for $\mu = -1$ to 0 we have another value.

So this is basically two-flux approximation where we have approximated the intensity to have two values. Similarly, we can always assume the intensity to be isotropic that is say in all direction. So this is the approximation which is used in many, many methods which we will discuss. But in this lecture we will focus our attention on the approximate methods where we are approximating the properties of the medium. The absorbing is scattering medium and the first in this category is optically thin approximation.

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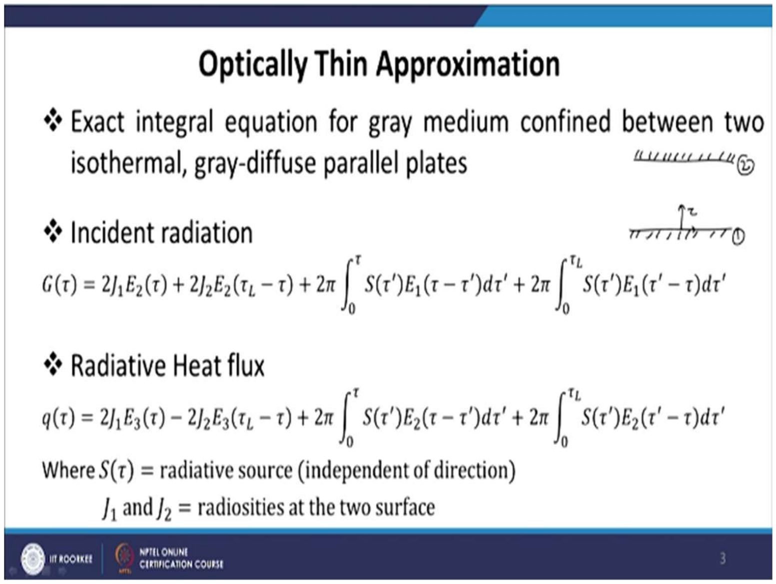

So optically thin approximation is a case where the absorption coefficient is small the optical path length is very small, so we assume; now in this case there is a medium bounded between two parallel plates; these plates are gray and they are diffusely emitting and reflecting surfaces. The temperature of the plate is uniform so they are isothermal. So we have already derived a solution for this problem in the previous lecture.

So the expression for incident radiation $G(\tau)$, so we know that the radiation is; the Incident radiation and Heat flux is going to vary with coordinate τ, so τ =0 at the bottom plate, this is plate 1, this is plate 2 so τ is 0 at the bottom plate and τ = τ_L at the top plate. so the Incident variation vary with τ and expression is $2J_1E_2(\tau)$ (exponential integral of second order) + $2J_2E_2(\tau_L - \tau)$.

And then the 2 integrations on the right hand side each having the source term as an integrant. So this was the solution we derived in the previous class and similarly for radiative heat flux we have this expression, $q(\tau) = 2J_1$; we have exponential integral of order 3 ($E_3(\tau)$) $-2J_2E_3(\tau_L - \tau)$ and then the two terms for the source term.

So this was the expression we derived in the previous class for heat flux and incident radiation as a function of τ between two parallel plates. Now let us see if we can simply this solution. So this solution we have the only problem in this solution is what we have to evaluate the integrations and exponential integrals as appearing in these two equations.

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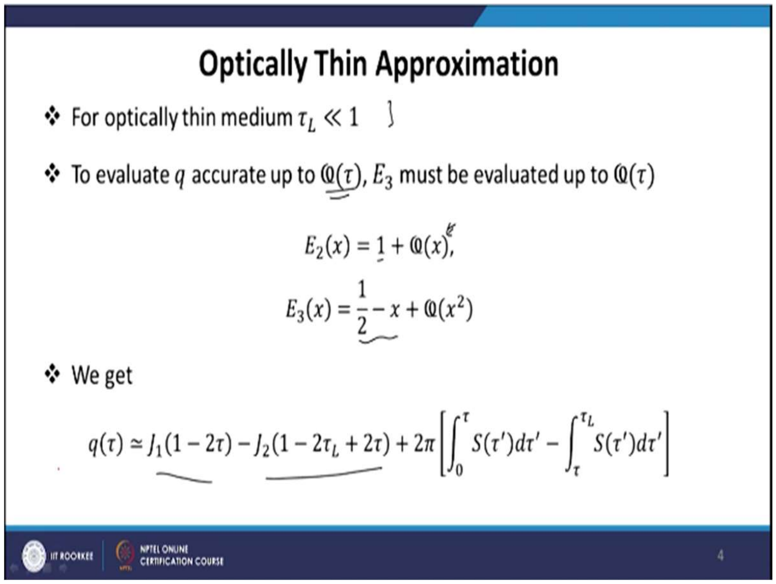

So we assume that $\tau_L \ll 1$, that is the medium is optically thin. So when the medium is optically thin all we need to do is we have to evaluate the exponential integrals for order 1, so this is the term of order x and above which are neglected. So we write exponential integral in terms of a series and the first term is only retained rest of the terms order x and higher are basically ignored.

Similarly, in exponential three integrals the term $\frac{1}{2}$ and $-x$ is retained that is all the terms of order x^2 and higher are basically ignored. When we do such an approximation the heat flux evaluated will be accurate up to order τ. So we want that heat flux should be accurate up to order τ and that is why we have approximated the exponential integrals and when we do that we substitute the value of $E_2(\tau)$ and $E_2(\tau_L) - \tau$ in this expression for $G(\tau)$, and exponential three integrals in $q(\tau)$.

So our heat flux is basically simplified to $J_1(1-2\tau) - J_2(1-2\tau_L+2\tau)$. Okay. So the first two terms in the expression of heat flux have been simplified. We do not need to evaluate the exponential integrals and the solution is relatively simple. Now let us see what we can do for the next two terms that is integration over the source term for an optical thickness 0 to τ and τ to τ_L. So let us see.

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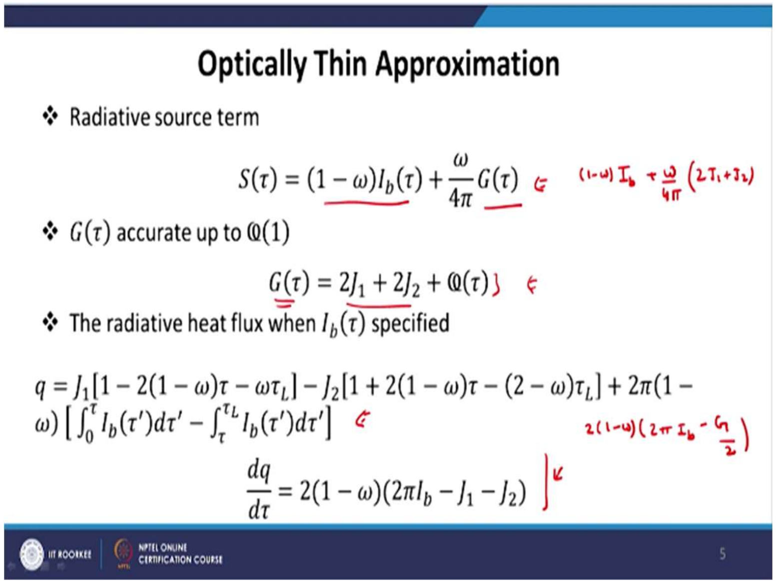

Now for this we write the radiative source term first. So radiative source term basically has contribution from emission and contribution from scattering, so these terms we have to approximate, if you want to approximate $G(\tau)$ or order 1 and higher so one thing you should note that if these two terms are order τ, the order of accuracy of first two term is order τ then the integration need to be evaluated for order 1 only okay because when we integrate an order of accuracy is increased.

So we need to evaluate the incident radiation $G(\tau)$ for order 1, okay. And we do that from this expression for $G(\tau)$. So in this expression we approximate this expression for order 1, so the integrations this two integrations are ignored because their accuracy is less so we ignore the last two terms in the expression of $G(\tau)$. And our $G(\tau)$ is simply written as $2J_1 + 2J_2$ and rest of the terms are ignored, okay.

So this $G(\tau)$ is accurate to order 1, okay. So this expression of $G(\tau)$ is substituted in this equation. And when we do that we receive the value of the source term as $(1-\omega)I_b(\tau) + \frac{\omega}{4\pi}$ and then this will be $2J_1 + J_2$. So now this expression we have simplified significantly. Now this value of source term as $S(\tau)$ we will put in the expression of heat flux here. So the expression for heat flux contains two integrations and the integrant is source term and this source term we have simplified as $(1-\omega)I_b(\tau)+\frac{\omega}{4\pi}(2J_1+J_2)$.

So when we substitute this, the expression basically is simplified significantly and this is the expression for the radiative heat flux. And the only quantity now appearing in the integration is just blackbody emissive power, emissive intensity that is I_b. Okay. So this is the approximate result for the heat flux in under optically thin approximation. Now the expression for divergence of heat flux $\nabla.q$ is equal to as we already know this is given by 2 (1-ω) $2\pi I_b$ and then we have to put the value of G here, so this will be G/2.

So substituting for G from this equation this expression for $\nabla.q$ is basically found. Okay. This is the expression for heat flux and $\nabla.q$ under the assumption that we have optical thickness very, very less than 1 that is the optically thin approximation.

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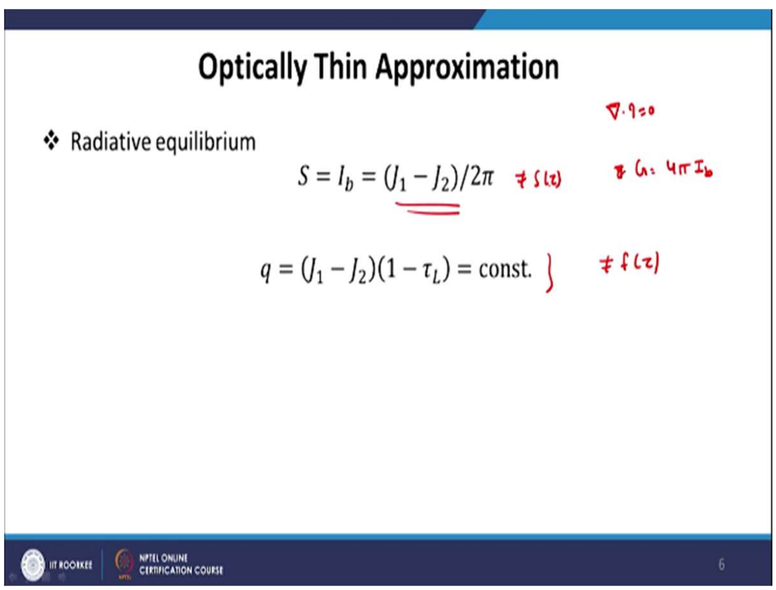

Now if we have Radiative equilibrium then the analysis is pretty simple for optically for radiative equilibrium we have $\nabla.q$ = 0 so we put the expression of $\frac{dq}{d\tau}$ as 0 and we get J_1 from this expression basically we get G=2π so from this we get G=4πI_b okay, so that means the source term is equal to simply the blackbody intensity and this can be written in terms of wall radiocities J_1 and J_2. Okay.

So this is very much simple were simplified for optically thin approximation. Similarly, the radiative heat flux under radiative equilibrium simply a function of J_1 and J_2 and the expression $q=J_1-J_2$ multiplied by $1-\tau_L$. And this is basically constant. So that means radiative heat flux is not a function of; it is not a function of τ anymore, okay. So this is constant and here also radiative source is not a function of τ that is it is constant, okay.

So what we have done here is we have found the expressions for radiative heat flux and radiative heat source term for optically thin cases where the absorption coefficient is very small and we have derived simplified expression.

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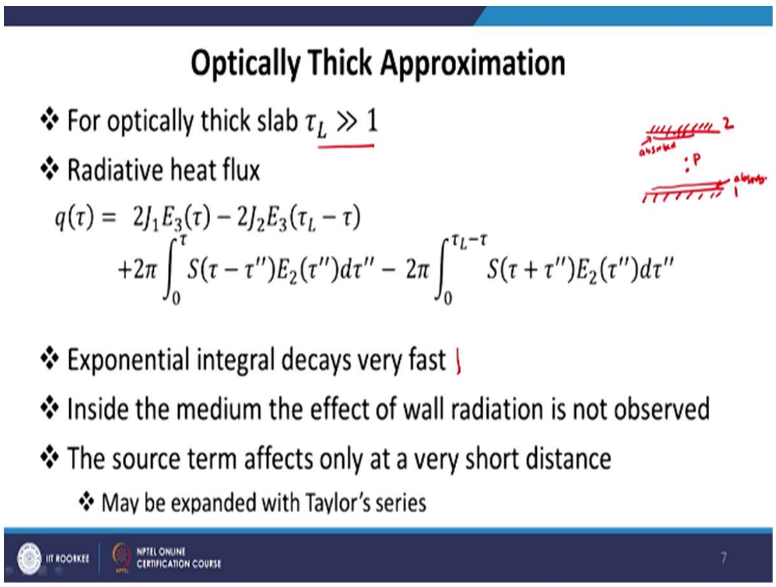

The other end of this analysis is optically thick approximation. In optically thick approximation, the absorption coefficient is going to be very large. So for optically thick cases the τ_L is going to be large and the value of absorption coefficient is large so τ_L is going to be very greater than 1. Now what we observe here that for optically thick cases the exponential integral will decay very fast, that means for the point under consideration if you move little bit further from the point of interest then the exponential integral will decay very fast.

So let us say we are interested in finding the expression for heat flux or intensity at this point T then only certain small distance away from P the source term will become negligible that mean the effect of intensity coming from this point will be negligible and based on the same argument the radiative intensity coming from plate 2 and plate 1 will not reach P. It will be absorbed in the small distance, okay. So it will be absorbed here and it will be absorbed here.

So radiation starting from plates 1 and 2 will not reach point P it will be absorbed within small distance away from the plates and the reason is the absorption coefficient is large, the attenuation is very large and the radiation is basically not reaching point P, the radiation from plates is not reaching point P. So what we have is basically only the local effect. So point P finds itself in a position where the intensity at point P is affected by only local phenomenon, and long distance effects are basically negligible because of the significant amount of attenuation taking place.

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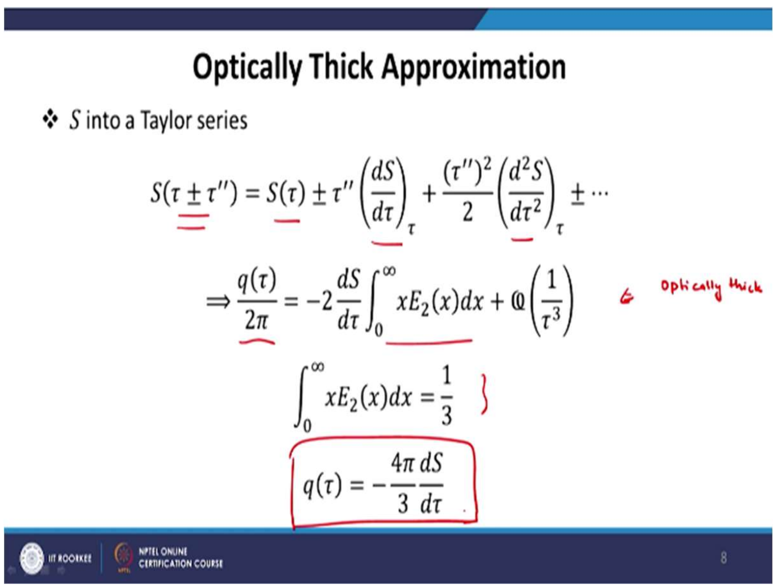

So under this assumption we can express our source term with the Taylor series where we are saying that at point P the source term is affected by intensity within small optical distant + $(-\tau'')$, that is point P is affected by radiation within a small circle of optical thickness represented by τ''. So which we expression this with the help of Taylor series, so we write S(τ) + τ''as S(τ) and then the first derivative and the second derivative terms.

So this is the Taylor approximation of the radiative source term. When we substitute this in the expression of heat flux so we have the expression of heat flux, the first two quantities are automatically 0 because the effect of wall radiation is not absorbed as I already explained. J_1 and

J_2 will not appear in the expression of heat flux because the effect is all local. The long distance effect is not felt in the heat flux.

So for any point within the medium the wall radiation will not take place and the second two terms the last two terms will basically appear and these two terms also we have approximated using the Taylor series. When we do that the expression of radiative heat flux q tau contains only this integration and derivative of the source terms $\frac{dS}{d\tau}$. So we have neglected the second order higher attempts and our expression for heat flux becomes simply $-2\frac{dS}{d\tau}$ and then $\int_0^{\infty} xE_2(x)dx$ okay.

So with this, this expression is basically for optically thick case. Okay. Now evaluation of; the integration in this equation is relatively straightforward. So this integration basically value to 1/3 and with this we get the radiative heat flux as equal to -4; $-\frac{4\pi}{3}\frac{dS}{d\tau}$. So for optically thick cases the expression of heat flux is very simple and it is given by $q(\tau) = -\frac{4\pi}{3}\frac{dS}{d\tau}$ where S is not the source term at any location of τ.

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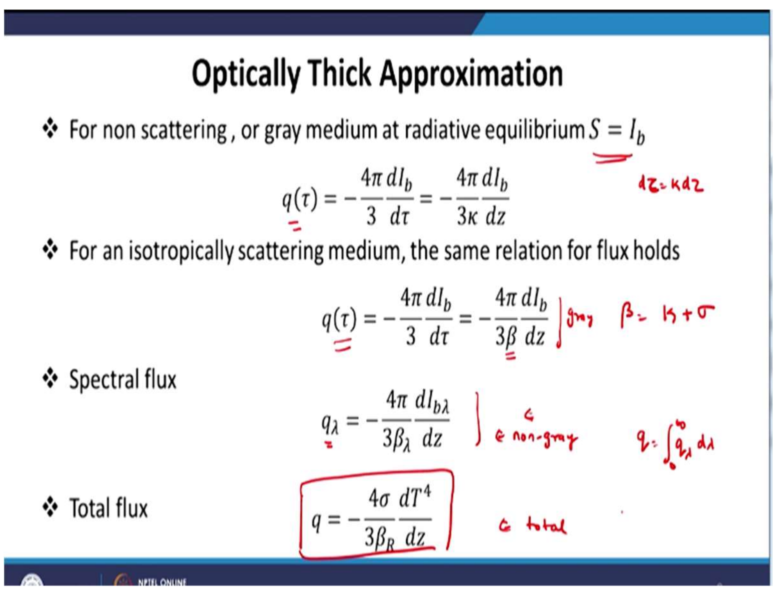

Now if the medium is not scattering then S is simply equal to I_b and we can just write down our heat flux equation substituting $S = I_b$, so $q(\tau) = -\frac{4\pi}{3}\frac{dI_b}{d\tau}$ and if you write in terms of absorption coefficient then we have dτ = κ times dZ so this is τ. So dτ= κ times dZ that expression for radiative heat flux at any location within the medium is $= -\frac{4\pi}{3\kappa}\frac{dI_b}{dZ}$.

For an isotropically scattering medium, same thing, if you do not have purely absorbing media, we have scattering also, the expression remains the same only thing is κ is replaced by β where beta is now the absorption coefficient (κ) + scattering coefficient (σ). So β is the extension coefficient rest of the expression remains the same. The same thing is valid for Spectral basis also, this we are where we have not use the wavelength but we can write the same expression for wavelength also, so $q_\lambda = -\frac{4\pi}{3\beta_\lambda}\frac{dI_{b\lambda}}{dZ}$.

So the expression is valid for spectral basis also and we have replaced the κ or β with spectral absorption or extension coefficient β_λ and the blackbody intensity is replaced by spectral black-body intensity $I_{b\lambda}$. The expression remains the same on spectral basis also. Now we can integrate this to find out the total value of heat flux. So this expression was gray; this is for non-gray that is on an spectral basis and this is for total.

So total means we have to integrate over all the wavelength so q will be simply equal to q_λdλ, 0 to infinity. So this will be the total heat flux. So when we integrate this expression the expression for heat flux q on total basis is simply equal to 4σ; now π $I_{b\lambda}$ we write as equal to $E_{b\lambda}$ that is the blackbody emissive power ; and when we integrate over all the wavelength we get σT^4.

So the expression becomes $q = -\frac{4\sigma}{3\beta_R}\frac{dT^4}{dZ}$. Now β_R is basically Rosseland-mean extinction absorption coefficient.

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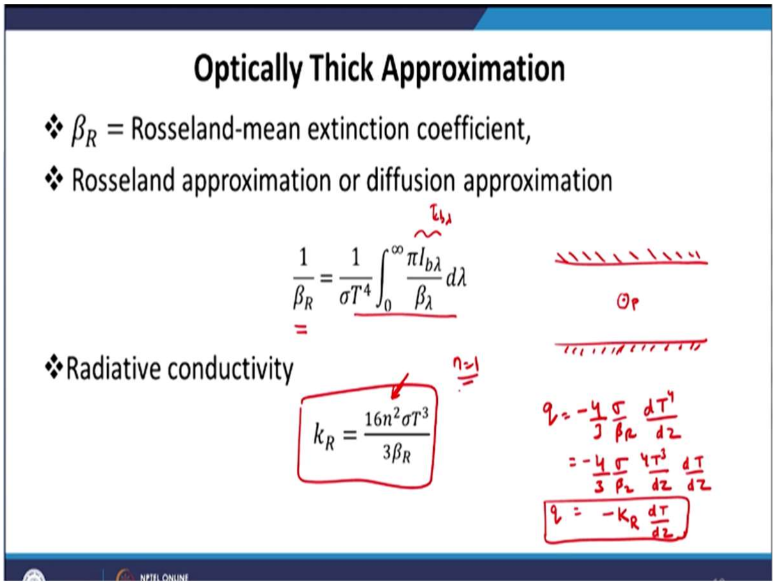

So β_R is defined as Rosseland-mean extinction coefficient. We define this extinction coefficient as basically the mean value integrated our all the wavelengths. So $1/\beta_R$ where beta is Rosseland-mean extinction coefficient is equal to $1/\sigma T^4$ then integrating from our 0 to infinity on all wavelengths π times $I_{b\lambda}$ over β_λ, that is π times $I_{b\lambda}$ is nothing but $E_{b\lambda}$. Okay.

So this is how we calculate this mean extinction coefficient. So $1/\beta_R$ is basically given by this relation and this relation is basically called Rosseland-mean extinction coefficient. And this method for optically thick case is called Rosseland Approximation or Diffusion Approximation. Now why it is called Rosseland diffusion approximation is because is because the radiation effect is local that is the intensity at a given point intensity at a given P is effective by only small region around P. What is happening at the wall is not affecting P.

So much energy is coming from the wall is not basically affecting P. So radiation at P is affected by locally and this is very similar to what we have in conduction where heat transfer basically takes place due to phonon-phonon interaction at a local level, so that's why it is the method is also basically called diffusion approximation. And we can also define because the method under optically thick cases is very similar to conduction cases.

We also define a quantity called radiative conductivity. And this can be calculated based on the expression of radiative heat flux so $q = -\frac{4\sigma}{3\beta_R}\frac{dT^4}{dZ}$. Now this can be written as$= -\frac{4\sigma}{3\beta_R}$ and this will be equal to $\frac{4T^3}{dZ}\frac{dT}{dZ}$. So we can write this as K, now this is not κ it is absorption coefficient or rather called radiative conductivity, radiative conductive K_R, K subscript R dT/dZ.

So this is very similar to the expression of conduction thermal conduction. We can define a radiative conductivity for optically thick cases because the radiative heat transfer in optically thick cases is very similar to the conduction, and this K_R is nothing but $\frac{16\sigma T^3}{3\beta_R}$ and just represents the refractive index which is normally equally to 1. So the expression is simply $\frac{16\sigma T^3}{3\beta_R}$. So this is the radiative conductivity.

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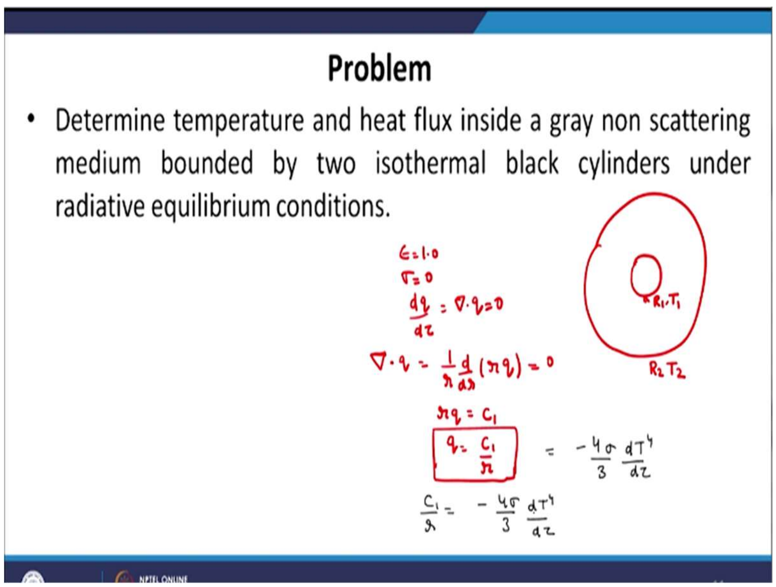

So we learned in this chapter two methods the optically thin approximation and optically thick approximation. We will solve one problem. And in this problem we will see how to find out the temperature and heat flux inside non-scattering medium bounded by two isothermal black cylinders. So let us say we have two cylinders.

This is the outer cylinder and we have an inner cylinder, okay. So we have; both the cylinders are black that means the emittance is equal to 1 okay. The inner cylinder has radius R_1 and temperature is T_1. Outer cylinder has radius R_2 and temperature is T_2, okay. So we have to find out temperature and heat flux inside a gray non-scattering medium. So this is non-scattering medium σ=0.

The surfaces are black and we assume radiative equilibrium that means dq/dτ or $\nabla . q$ = 0. Okay. So we will use cylindrical coordinate system. In cylindrical coordinate system we have $\nabla .$ as written by $\frac{1}{r}\frac{d}{dr}(rq)$. So this is the expression for $\nabla . q$ in cylindrical coordinates and this will be equal to 0 because of radiative equilibrium this will be equal to 0. So this means r times q is constant, let us call this constant C_1.

So rq is constant or radiative heat flux varies inversely with respect to r. So q = C_1/r, radiative heat flux varies inversely with respect to r. Okay. Now from the previous result for optically thick cases we have the expression for radiative heat flux was $q = -\frac{4\sigma}{3}$ this was a expression $-\frac{4\pi}{3}\frac{dI_b}{d\tau}$, so we can write this as; this will be equal to -4; if we are talking about total basis so $-\frac{4\sigma}{3}\frac{dT^4}{d\tau}$, okay. So this was the expression for the heat flux under optically thick cases that is the diffusion approximation. Now we will write this as C_1/r as equal to $-\frac{4\sigma}{3}\frac{dT^4}{d\tau}$. So now let us solve it this.

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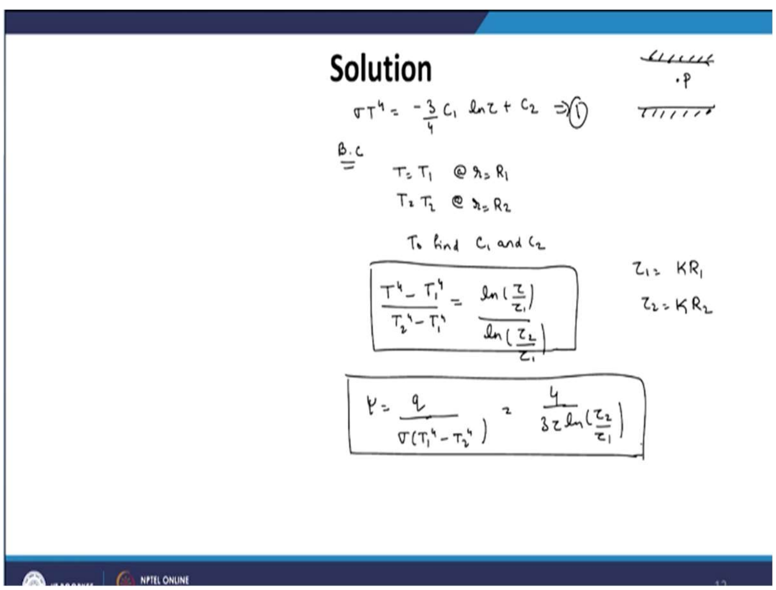

So we write this as σT^4 as equal to $-\frac{3}{4}C_1$, this was expression for C_1 so C_1 lnτ + C_2, okay. So what we have done is we have integrated with respect to tau and we got this result $\sigma T^4 - \frac{3}{4}C_1\, ln\tau + C_2$. So this gives an idea of how the emissive power within the medium is changing with respect to, the coordinate τ.

Now boundary condition, although the expression for optically thick cases that we derived was valid for any point away from the surface. So remember we derived the optically thick case for a point P away from the plates, okay. If it is at the plate, then expression is not valid. But we are trying to apply this to cylindrical configuration assuming that the cylinder can be opened up in the form of parallel plates. So we have approximated the cylinder in the form of parallel plates. And we are applying the boundary conditions at the wall.

So with boundary condition we say T = T_1 at r = R_1 and T = T_2 at r = R_2. And with this substitution in the equation, let us call this equation 1, put this boundary condition to define to find unknown coefficient C_1 and C_2, okay. And when we find C_1 and C_2 we get this expression for the temperature $T^4 - T_1^4; T_2^4 - T_1^4$.

This is basically the non-dimensional emissive power, so $\frac{\ln(\frac{\tau}{\tau_1})}{\ln(\frac{\tau_2}{\tau_1})}$, where τ_1 is nothing but the gray absorption coefficient κ times R_1 and τ_2 is gray absorption coefficient κ times R_2, so we get this distribution for the emissive power non-dimensional emissive power or temperature power 4. And similarly, the radiative heat flux $\frac{q}{\sigma(T_1^4 - T_2^4)} = \frac{4}{3\tau\ln(\tau_2/\tau_1)}$ okay.

So this is the expression for the non-dimensional radiative heat flux. We have found this expression using C_1/r so we have calculated the unknown coefficient C_1 already, so the expression for the non-dimensional radiative heat flux $\frac{q}{\sigma(T_1^4 - T_2^4)}$ is simply equal to $\frac{4}{3\tau\ln(\tau_2/\tau_1)}$. So these two solutions we have found for the case of cylindrical media approximated using T diffusion approximation that is optically thick approximation.

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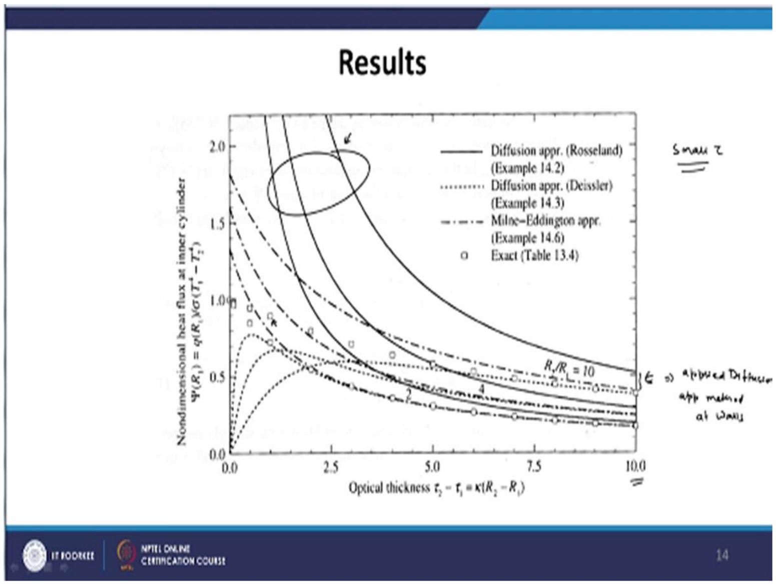

Now these results are plotted in a chart and as you see that this is the results for the diffusion approximation, Rosseland approximation. For small values of τ that is small τ optically thick case is not valid and definitely the results obtained from this method are not accurate. Why? Because this method is derived for optically thick cases that is τ values large.

For τ values small, the optically thick approximation is less accurate and as it seen the square represents the exact solution and this represents the Rosseland or diffusion approximation and the results are very much inaccurate. On the other hand, we see that the results are relatively good for optically thick cases that is the value of optical thickness is large and the results are relatively more accurate and they match relatively better with the exact results.

Although, there is still some problem at the wall; there is a jump at the wall. And the reason for this jump is that we have applied this method Diffusion Approximation to at wall. So we have applied this method at wall while the method itself was not derived for walls. The method was derived for finding radiative heat flux away from the walls where the effects of walls was negligible. Why?

In this problem, we have applied this method to find out radiative heat flux very much near the wall and that is why there is a jump here; and this jump needs to be taken care otherwise this method is good only for points away from the walls for optically thick case. So thank you. In the next lecture, we will introduce other methods other approximate methods and these methods will be based on approximating the intensity variation as oppose to the approximation of properties optically thin or optically thick assumed in this lecture. Thank you.