Radiative Heat Transfer
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Lecture - 19
Radiative Heat Transfer in Cylindrical Media

Hello friends. In the previous lecture, we formally solved the radiative transfer equation for a plane parallel slab. It turned out that the temperature distribution needs to be known if we assume radiative equilibrium and if temperature distribution is known, we need to find out the radiative heat flux, which is a function of space and the divergence of heat flux, the radiative heat source term as a function of space.

In this lecture, we will take one example of the solution of radiative transfer equation in plane parallel slab, where the temperature distribution is known.

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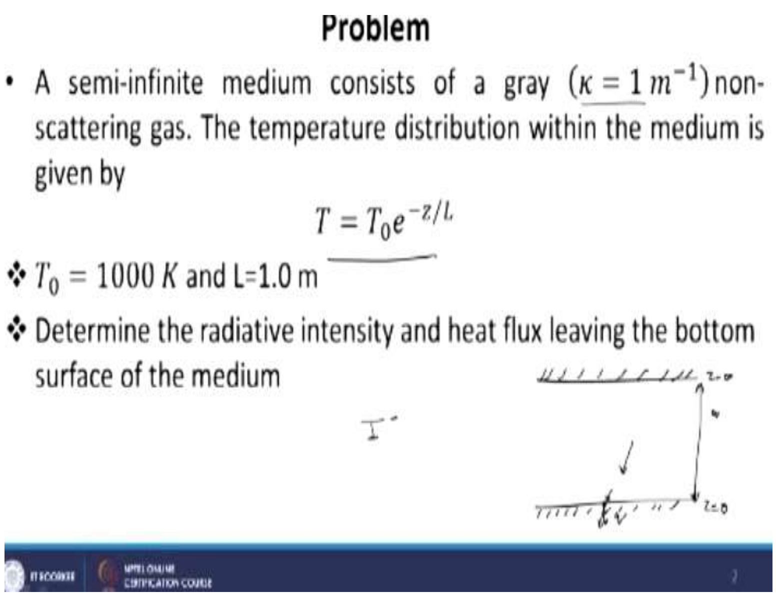

So we have basically a semi-infinite medium that consists of a grey, grey means the absorption coefficient is independent of wavelength, so we have absorption coefficient in the medium. So we have a plane parallel slab. Now it is said that it is semi-infinite, that means the separation between these plates is very large, okay, let us call this infinity, okay. So this is semi-infinite medium. So this may be located at z=0 and this plate is located at z=infinity.

This is a semi-infinite problem. So we have gone one step further simplification. In the lecture before this, we made certain simplifications that the plates are infinite in x and y direction and now have made one more simplification that it is semi-infinite that it extends to infinity. The purpose of doing this is basically any radiation from the top plate cannot reach the plate at z=0. Now there is a temperature distribution.

The temperature distribution T=T_0 where T_0 is 1000 Kelvin $e^{-(\frac{z}{L})}$ that is the temperature is decreasing as we move away from the plate. The plates themselves are cold and black, that means plates do not emit any radiation. They are cold and black. So we have to find out the intensity of radiation leaving the bottom plate, of course it is going to be function θ and we have to find out the heat flux Q at the bottom surface.

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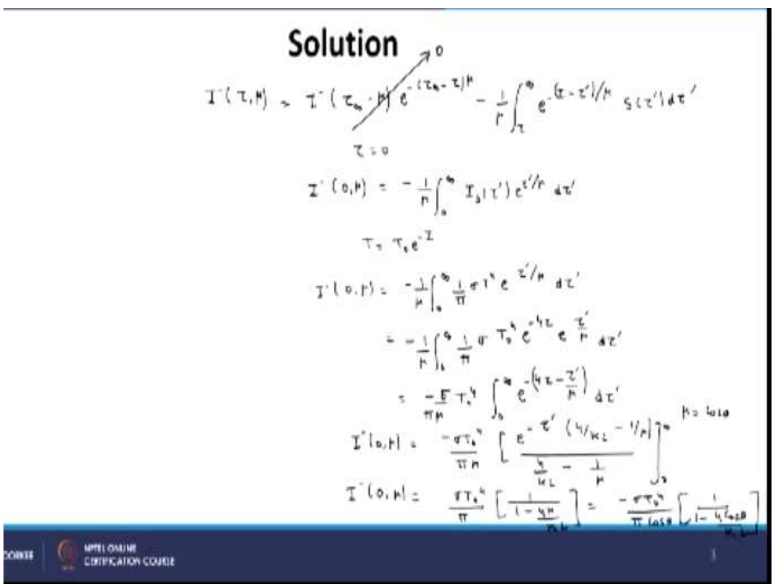

So one thing is clear that we are only interested in I^-. There is no I^+here. All the radiation is moving as far as any point in the bottom surface is concerned, all the radiation is moving down. So there is no forward or upward movement of radiation. The plate itself is cold, so it does not emit any radiation. So there is no I^+ here. So we have only I^-, which is function of τ and μ, but we are interested in finding I^- at 0, that is bottom plate τ=0 and μ, which is dependent on θ.

So we start instead of the solution of heat flux, we start with the solution of intensity $I^-(\tau, \mu)$ = I^- (τ_∞). I have written it infinity because the top surface is at infinity μ, $e^{-(\tau_\infty - \tau)\mu}$ –

$\frac{1}{\mu}\int_{\tau}^{\infty} e^{-\frac{\tau-\tau'}{\mu}} S(\tau')d\tau'$. So this is the equation or the solution of the radiative transfer equation for intensity in direction mu at any location τ for the downward movement of intensity, downward moving wave.

Now this term is 0. There is no radiation that is going to reach from the top surface to the bottom surface and we are going to put τ =0. So we are interested in the solution $I^{-}(0,\mu)$ and this will be equal to $-\frac{1}{\mu}\int_{0}^{\infty}$ Now there is no scattering, so we will just write $I_b(\tau')$. So this is going to be e power, tau is 0, so $\frac{\tau'}{\mu}$ and $d\tau'$, okay. This is what is basically we have got. Now the temperature distribution is known T=$T_0 e^{-z}$.

So L is basically equal to 1, so I will just leave the L part here. So $T_0 e^{-z}$ okay, so we get $I^{-}(0,\mu) = -\frac{1}{\mu}\int_{0}^{\infty}$ and then we have to find out the temperature dependence of this Ib, so we are dealing with a grey problem. So we write $\frac{1}{\pi}\sigma T_0^4 e^{\frac{\tau'}{\mu}} d\tau'$ and this will be equal to $-\frac{1}{\mu}\int_{0}^{\infty}\frac{1}{\pi}\sigma T^4 e^{-4z} e^{\frac{\tau'}{\mu}} d\tau'$, okay.

So now looking at this, you may think okay now this looks easy to integrate, right. So we can basically write like this -1/μ and we can just put π here and σ here. So T_0^4, so $-\sigma T_0^4/\mu\pi$, then 0 to infinity $e^{-(4z-\frac{\tau'}{\mu})}$ and $d\tau'$, okay. Now we have to solve it, then what we will do is we just write down this expression like this $I^{-}(0,\mu)$ =$-\sigma T_0^4/\mu\pi$, by the way μ is basically nothing but cosθ.

So we get this as $e^{-\tau'(4\kappa L-\frac{1}{\mu})}$ and then in the denominator we get $4\kappa L-\frac{1}{\mu}$, so this will be just simply μ and then we have to put the limits 0 to infinity, okay. So we get basically $I^{-}(0,\mu)$ as – =$-\sigma T_0^4/\mu\pi e^{-\tau'}$ where z value is from 0 to infinity. In multiplication, we have $4/\kappa L-\frac{1}{\mu}$ in the denominator we $4/\kappa L-\frac{1}{\mu}$. Now we see that the numerator will be 0 for z value infinity.

So numerator will basically vanish for the value of z=infinity and it will be simply equal to 1 for the value of z=0. So what we get is $I^{-}(0,\mu)$ =$-\sigma T_0^4/\mu\pi$ and then we will just have 1/1-4μ/ κL, okay. So this is intensity for the radiation leaving the bottom surface. So this intensity if you want to write it in terms of theta will be simply =$-\sigma T_0^4/\mu\pi$ and this will be equal to cosθ 1/[(1-4 cosθ/ κL)].

So this is the expression for intensity leaving the bottom surface from the semi-infinite medium. Now we have to find out the radiative heat flux.

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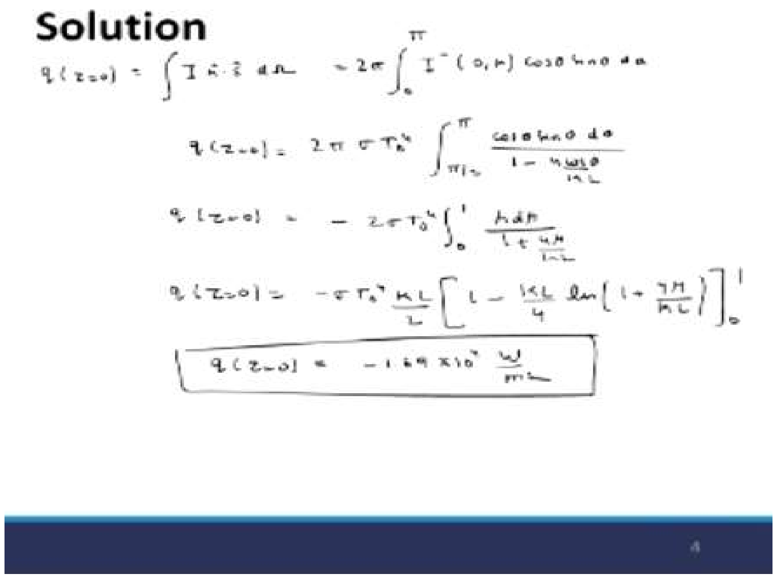

So we write Q at z=0, so we have to basically multiply by cosθ and integrate overall the solid angles, so we write this as $I\,\hat{n}.\hat{s}$dΩ, now this will be simply equal to $I^{-}(0,\mu)$ and then will be multiplied by $\cos\theta$ or $\sin\theta\, d\theta$ and then we have just taken the integration over the azimuthal angle outside and this will vary from 0 to π, okay. So we will get q at z=0 as equal to $2\pi\sigma T_0^4$.

We just substitute the expression for intensity from this relation. So we get $2\sigma\pi T_0^4$ and then integration from π/2 to π because we are just taking the downward direction, so we have taken from π/2 to $\pi\cos\theta\sin\theta\, d\theta$/[1-(4cos$\theta$/ κL)], okay. So we have written the heat flux as an integration over polar angle multiplied by cosθ and multiplied by intensity.

So after we do this integration, which is relatively easy to do, we get q at z=0 as equal to -$2\sigma\pi T_0^4 \int_0^1 \frac{\mu d\mu}{1+\frac{4\mu}{\kappa L}}$, okay. So this is relatively easy to solve. I will just give you the solution here q(τ=0)= -$\frac{\sigma\pi T_0^4}{2}[1-\frac{\kappa L}{4}\ln(1+\frac{4\mu}{\kappa L})$ and this will be 0 to 1. So this is all we can calculate the radiative heat flux by integrating over the limits and putting the limits we get q at τ=0 or z=0 as simply equal to -1.69*10^4 W/m^2.

So this is the expression, this is the value of radiative heat flux leaving the bottom surface and for this semi-infinite medium, we could solve it with some little bit of maths, still the math was manageable and we could integrate it by hand. For other problems, we will have to apply medical procedures to solve the problem. The integration in this problem was relatively easy to handle and do it by hand and that is why I could solve for the radiative heat flux leaving the bottom surface for this semi-infinite medium.

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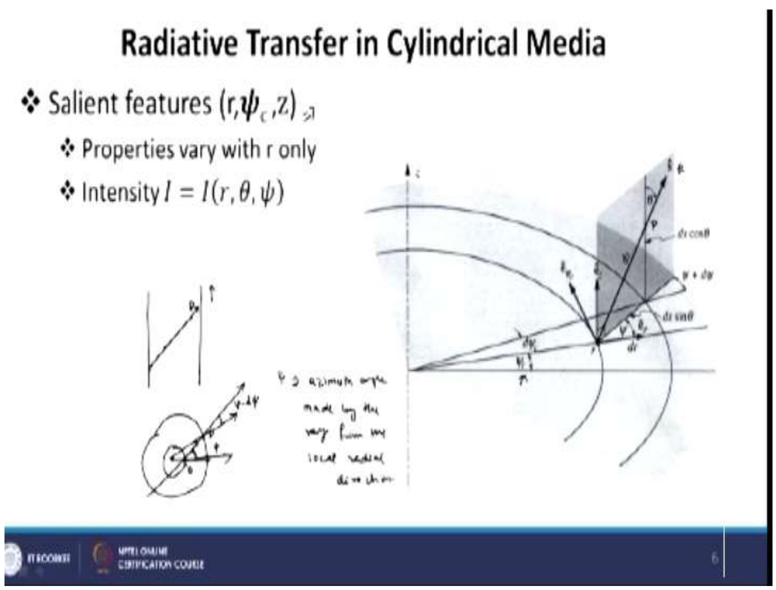

The next thing that we will develop is the radiative heat transfer in cylindrical media. So radiative heat transfer in cylindrical media appears in many, many examples, because many times, the furnaces are also cylindrical in shape, then amount of energy lost from pipes, so radiative transfer in cylindrical geometry is of great importance. Here we will take the coordinate system as r, ψ_c. Now ψ_c is different from ψ where ψ was the azimuthal angle.

Here ψ_c is basically the cylindrical coordinate system. The azimuthal angle in the cylindrical coordinate system measured from some radial direction, so this is ψ_c. This is a one dimensional problems of properties are allowed to vary only in the radial direction. The cylinder is infinite in length in z direction, so the properties do not vary in the z direction. Now in this case, even though this problem is one dimensional, the intensity is going to be function of radius, theta as well as the azimuthal angle.

So this problem is relatively more involved than the problem we discussed previously that is plane parallel slab. Now looking at the notation that we are using, so let us say we have a cylinder extending infinite in both the direction and we are interested in radiation intensity at certain point P. Let us say this is point P. This intensity in this direction it is going to be function of θ, ψ and r comes from certain direction like this.

So this is the intensity direction that is it is basically it is coming. This direction we can look at this direction in the projection. So we have this is the direction of movement of radiation ray and we are basically interested in some point P here where we want to find out intensity. Now at certain point the point was somewhere here, okay and this is basically another cylinder I have drawn here.

So the point P where we are interested in finding the intensity as the function of r, θ and ψ the ray is traveling in a certain path as is also shown here. The ray is traveling at a certain path. Sometime before it reaches P, the wave intersects the cylinder at this circle. So we define the direction of movement of this radiation within azimuthal angle ψ. So this angle basically represents the direction psi. So I will just note down psi. Let us call this point Q.

So ψ is at azimuthal angle made by the ray from the local radial direction, okay. That means at point Q, this was the radial direction as is shown here also. This is the local point here and from this direction, the ray makes an angle ψ. After certain time, the wave reaches the point P. At P, this is the radial direction, this is the new radial direction and this angle is going to be now $\psi - d\psi$. So the point that I am on to say is when the ray is traveling in space as is shown in this image, its azimuthal angle is going to decrease.

So when the ray hits Q, the azimuthal angle was let us say ψ, and then the ray moves to P at a new location, the angle becomes $\psi - d\psi$, so the angle basically changes. Now we will use the trigonometry here. We have this distance, let us call this distance, this is P, so RP is ds. The movement of the ray is ds, so distance RP is ds and we have this distance, this distance as ds cosθ. So let us go into the trigonometry here.

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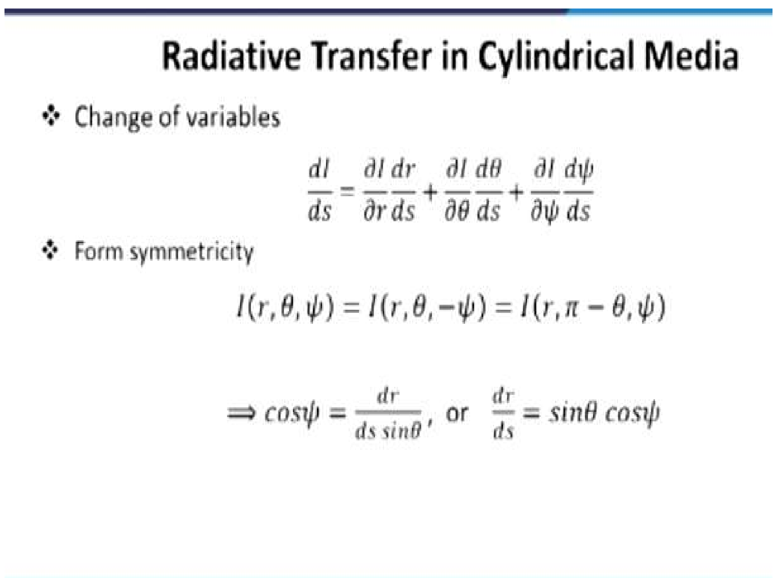

So from the triangle, we have dr/ds. So this is ds sin theta, this distance is ds sinθ , where this angle is theta, the polar angle, which is constant. So polar angle is not changing. In the previous example also the parallel media, we have seen the polar angle does not change when the ray moves. In this example also, the polar angle is not going to change. So this distance is ds sin theta. So from here we can write down ds sinθ /drs cosψ.

So this is the angle ψ. So looking at this triangle cosψ is dr/ds . So $cos\psi$ is $\frac{dr}{ds\ sin\theta}$ or dr/ds is simply equal to $sin\theta\ cos\psi$. So why we are doing this exercise, because we want to change the coordinates as we have changed the coordinates in plane parallel media from ds to z. Here we will change the coordinate system from s the distance along the ray to radial coordinate system, cylindrical coordinate system r, θ and ψ.

So we have found a relation between r and s as dr/ds= $sin\theta\ cos\psi$. Now along a beam, as I said that is ψ is going to decrease. This ψ is going to decrease, so earlier the angle was ψ and after movement the distance S, the angle decreases from ψ to $\psi - d\psi$ and we can write down $d\psi$ /dr as -$-\frac{tan\psi}{r}$. So look at this triangle. This distance is $rd\psi$, okay. So $rd\psi$ is nothing but dr $tan\psi$ and minus sign denotes that the angle ψ is going to decrease.

So $\frac{d\psi}{dr} = -\frac{tan\psi}{r}$, this is from trigonometry. So we can write down a relationship between S and ψ as $\frac{d\psi}{ds} = \frac{d\psi}{dr}\frac{dr}{ds}$ and dr/ds we have already calculated and $\frac{d\psi}{dr}$ also we have calculated, so our relation becomes $-\frac{sin\theta\ sin\psi}{r}$.

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Radiative Transfer in Cylindrical Media

- Change of variables

$$\frac{dI}{ds} = \frac{\partial I}{\partial r}\frac{dr}{ds} + \frac{\partial I}{\partial \theta}\frac{d\theta}{ds} + \frac{\partial I}{\partial \psi}\frac{d\psi}{ds}$$

- From symmetry

$$I(r,\theta,\psi) = I(r,\theta,-\psi) = I(r,\pi-\theta,\psi)$$

$$\Rightarrow cos\psi = \frac{dr}{ds\, sin\theta}, \text{ or } \frac{dr}{ds} = sin\theta\ cos\psi$$

So now we applied the change of coordinates from $\frac{dI}{ds}$ chain rule, $\frac{dI}{ds}$ is partial with respect to r $\frac{\partial I}{\partial r}$ $\frac{dr}{ds}$ and similarly partial with respect to θ and partial with respect to ψ. So we substitute for $\frac{dr}{ds}$ dθ/ds, this is going to be 0 because the polar angle is not changing. So we have to substitute for $\frac{dr}{ds}$. So dr/ds is $sin\theta cos\psi$ and dψ/ds is already calculated as $-\frac{sin\theta\ sin\psi}{r}$.

So with this change in coordinate system, we get $\frac{dI}{ds} = sin\theta\left[cos\psi\frac{\partial I}{\partial r}(r,\theta,\psi) - \frac{sin\psi}{r}\frac{\partial I}{\partial \psi}(r,\theta,\psi)\right]$, intensity is going to be function of r, θ, ψ. Now the radiative transfer equation in cylindrical media, so we substitute for $\frac{dI}{ds}$ in the radiative transfer equation.

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Radiative Transfer in Cylindrical Media

- Radiative transfer equation

$$\frac{dI}{ds} = \kappa I_b - \beta I + \frac{\sigma_s}{4\pi}\int_{4\pi} I(\hat{s}_i)\phi(\hat{s}_i,\hat{s})d\Omega_i$$

$$sin\theta\left[cos\psi\frac{\partial I}{\partial \tau}(\tau,\theta,\psi) - \frac{sin\psi}{r}\frac{\partial I}{\partial \psi}(\tau,\theta,\psi)\right] = (1-\omega)I_b - I(\tau,\theta,\psi) + \frac{\omega}{4\pi}\int_{4\pi} I(\hat{s}_i)\phi(\hat{s}_i,\hat{s})d\Omega_i$$

Where $\tau = \int_0^r \beta dr$

- Have to be solved numerically

So left hand side becomes sin theta $sin\theta\left[cos\psi\frac{\partial I}{\partial \tau}(\tau,\theta,\psi) - \frac{sin\psi}{r}\frac{\partial I}{\partial \psi}(\tau,\theta,\psi)\right]$and in the non-dimensional optical depth coordinates, the right hand side is simply $(1-\omega)I_b - I$ and then the In-scattering source term. Now this can be solved numerically. We tried to solve the problem under certain simplification for the plane parallel slab, but this problem is going to be little more difficult to solve analytically even after simplifications.

So we will just leave it here. this problem can be solved numerically. What we have done is derived the equation in cylindrical coordinates and we have simplified the equations in terms of partial derivatives with respect to the optical coordinates τ and ψ. This is a partial differential equation and needs to be solved. This is partial differential equation along with this integral term. So this is again much more difficult, but it can be programmed and it can be solved using the numerical methods.

So we will just leave it here. In the next lecture, we will discuss approximate methods for the solution of RTE. This lecture and the previous lecture focused on exact method for simplified geometries. The next lecture will focus on approximate methods where we will simplify the dependence of intensity. Here we have not made any simplification or approximation. The intensity variation in three dimensional space was taken as it was.

The geometry itself simplified the case. For example, in plane parallel media, the geometry itself simplified the intensity variation with respect to the azimuthal angle. So there was no dependence on azimuthal angle in this cylindrical media also, we have simplified geometry. So intensity is not a function of z. All it is function of basically are radial coordinate and of course θ and ψ. So we have more complications and we got a partial differential equation.

But still we could obtain an analytical solution for plane parallel slab and this problem we have to solve numerically, but the solution is exact, although the method is not analytical, the solution is exact. The next topic we will discuss is approximate methods and we will see how to simplify the expression for intensity or dependence of intensity on θ and ψ. So thank you for your attention.