Radiative Heat Transfer

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Lecture - 12
Solution Methods for Governing Integral Equations

Welcome friends, in the previous lectures we have discussed radiative heat transfer between flat surfaces, we discussed black surfaces, we discussed gray surfaces, we also solved a number of numerical problems for which analytical solution was easy to calculate. We also introduced the method of electric networks to find a solution for the gray surfaces, but these methods, the electric network method and the analytical method may not be applied when we are dealing with a large problem.

A large problem with large number of enclosure surfaces makes it difficult to solve these problems by hand. So we have to either employ a computer, we have to go for some kind of numerical technique to solve these problems. So in this lecture we will study a few of these techniques to solve problems where we have the size of the problem is large or where we cannot approximate the enclosure with flat surfaces.

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Solution Methods for Governing Integral Equations

- General equation for radiative exchange in Gray enclosure

$$q(\mathbf{r}) = \epsilon(\mathbf{r})E_b(\mathbf{r}) - \alpha(\mathbf{r})\left[\int_A J(\mathbf{r}')dF_{dA-dA'} + H_0(\mathbf{r})\right]$$

- Black surface

$$q(\mathbf{r}) = E_b(\mathbf{r}) - \left[\int_A E_b(\mathbf{r}')dF_{dA-dA'} + H_0(\mathbf{r})\right]$$

So just a quick recap to what we have done, in enclosure the radiative heat flux q at any location r. So this is the vector of the location r, if we take it with respect to some kind of origin, the heat flux at this location is made up of 2 components, one is the emitted energy, and one is the absorbed energy. So this is the energy balance for small element or at any location given by the vector r.

In the second term we have what we call irradiation, so this is irradiation okay, so for a gray surface we define radiosity (J) so irradiation can be written in terms of radiocity that is the total energy leaving at any location is called radiosity. It has 2 components. The emission from the surface and reflection of any radiation coming on to that element. So radiosity has 2 components emitted component and reflective component and we have to integrate over the entire enclosure with proper view factor.

So with proper view factor we have to integrate over the enclosure and this will give you the total irradiation, total incoming radiation at surface dA from all the enclosure and we may have also an external component of irradiation coming from outside the enclosure. So these relation we have already discussed and this relation reduces to $E_{br} - \int_A E_b(\mathbf{r}')dF_{dA-dA'}\, H_o(r)$ (the second term) which is very similar to the previous term.

The only thing is the epsilon and alpha for the black surface have value = 1 and the radiosity (J) will simply reduces to emissive power okay. So these relations we have already developed. Now we will see how we can basically use this method to solve for an large enclosure okay.

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Matrix Method (Black Enclosure)

- Let some surfaces have known Temperature and some have known flux

$$\sum_{j=1}^{n}(\delta_{ij} - F_{ij})E_{bj} = q_i + H_{0i} + \sum_{j=n+1}^{N} F_{ij}E_{bj}$$

$$A \,.\, e_b = b$$

$$A = \begin{bmatrix} 1-F_{11} & -F_{12} & \cdots & -F_{1n} \\ -F_{21} & 1-F_{22} & \cdots & -F_{2n} \\ \cdots & \cdots & \ddots & \cdots \\ -F_{n1} & -F_{n2} & \cdots & 1-F_{nn} \end{bmatrix} \quad e_b = \begin{bmatrix} E_{b1} \\ E_{b2} \\ \cdots \\ E_{bn} \end{bmatrix} \quad b = \begin{bmatrix} q_1 + H_{01} + \sum_{j=n+1}^{N} F_{ij}E_{bj} \\ q_2 + H_{02} + \sum_{j=n+1}^{N} F_{ij}E_{bj} \\ \cdots \\ q_n + H_{0n} + \sum_{j=n+1}^{N} F_{nj}E_{bj} \end{bmatrix}$$

- Easy to solve using a computer (Gauss-Siedel method)

So in the previous lectures on gray and black surfaces we have derived algebraic equations, we assume that the enclosure, any enclosure can be divided into finite number of surfaces, okay, so this is approximation which sometimes is good, but sometimes it may give you wrong results. We will see we cannot and we should not approximate the enclosure with flat surfaces, but for the time being we assume that the enclosure is approximated with flat surfaces.

So if there are n flat surfaces which we have divided the enclosure into. Some of the surfaces have known heat flux value. So some of the surfaces, the q is known, while on some other surfaces the temperature maybe known okay and the heat flux maybe unknown. So let us say from i = 1 to n, we know q values and for i = n+1 to N the temperature is known that is emissive power is known but that flux is not known.

So total number of unknowns will be N and some of these unknowns maybe flux and some of these unknowns maybe temperature. So we write our energy balance equation that we have derived earlier. We introduce variable called Kronecker delta. So δ_{ij} is simply = 1, for i = j and = 0 for i≠j. So this is just for computer notation, we have introduced this parameter. So on the left hand side we have unknown fluxes for the unknown emissive power or the unknown temperature for the first n surfaces.

On the right hand side, we have known fluxes qi external irradiation and irradiation from the surfaces n + 1 to N for which the temperature is known okay. So we can write down the energy balance equation in this form. Now one thing to note here is there are total n variables here. So first we are trying to solve for the temperature here and once we know the temperature or the emissive power we can calculate the heat flux for surfaces n + 1 to N okay.

Now this equation can be written in matrix form A times e_b (Ae_b)= b where the matrix A contains the coefficients on the left hand side. So the diagonal coefficients we have i = j. So we have 1-Fij from the first term, off diagonal elements will have i≠j, so delta ij will be = 0. So we have – Fij terms in the off diagonal and the e_b vector contains the emissive power, the unknown emissive power for the first n surfaces.

So this will have dimension of n x 1, this will have dimension n x n and the right hand side the vector b contains the non-fluxes for the first n surfaces, external irradiation on the surfaces and irradiation from all other surfaces for which the temperature is known okay. This also has dimension (n x 1). Now this is the system of linear equation which is very easy to solve.

We can program this equation and we can apply any linear solver like Gauss–Seidel to solve for the unknown temperature or emissive power e . Once the emissive power is known we can find out the unknown heat fluxes by just substituting this values in the original equation okay.

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Matrix Method (Gray Surface)

$$\sum_{j=n+1}^{N}\left(\frac{\delta_{ij}}{\epsilon_j}-\frac{\rho_j}{\epsilon_j}F_{ij}\right)q_j-\sum_{j=1}^{n}(\delta_{ij}-F_{ij})E_{bj}$$
$$=-\sum_{j=1}^{n}\left(\frac{\delta_{ij}}{\epsilon_j}-\frac{\rho_j}{\epsilon_j}F_{ij}\right)q_j-H_{0i}+\sum_{j=n+1}^{N}(\delta_{ij}-F_{ij})E_{bj}$$

$$A\,.X=B$$

So this was for the black surfaces okay, for the gray surfaces similarly we can write down the energy balance equation, the only difference between black surfaces and gray surfaces is that the unknowns contain the heat fluxes. For some surfaces unknown may be heat fluxes. For other surfaces the radiosity may be unknown okay. So the radiocity contains both reflection and the emission okay.

So we have to solve for the entire N surfaces simultaneously. We have to solve for this simultaneously for the entire N surfaces. So as before we write the equation for the unknowns, the first N unknowns contain the unknown emissive power. The second term on the left hand side contains N unknowns. So we have this N unknowns here in terms of the emissive power okay.

And the rest n + 1 to N unknowns contain the unknown heat flux as the variable. So the total size of the matrix that we will form in this case for the gray surfaces will be N okay. So we have to solve simultaneously for the unknown fluxes and the emissive power okay. So right hand side contains the known terms, so we know the heat flux for the first N surfaces.

So the first term on the right hand side is known, external irradiation is known and the last term on the right hand side is the emissive power because the temperature for surface is n + 1 to N is known, so this term is also known. So again we write this as a system of linear equation A X = B.

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Matrix Method (Gray Surface)

$$
A = \begin{bmatrix}
-(1-F_{11}) & \cdots & F_{1n} & -\frac{\rho_{n+1}}{\epsilon_{n+1}}F_{1-n+1} & \cdots & -\frac{\rho_N}{\epsilon_N}F_{1-N} \\
\vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\
F_{n1} & \cdots & -(1-F_{nn}) & -\frac{\rho_{n+1}}{\epsilon_{n+1}}F_{n-n+1} & \cdots & -\frac{\rho_N}{\epsilon_N}F_{n-N} \\
F_{n+1,1} & \cdots & F_{n+1,n} & \left(\frac{1}{\epsilon_{n+1}}-\frac{\rho_{n+1}}{\epsilon_{n+1}}F_{n+1,n+1}\right) & \cdots & -\frac{\rho_N}{\epsilon_N}F_{n-N} \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
F_{N1} & \cdots & F_{N,n} & -\frac{\rho_{n+1}}{\epsilon_{n+1}}F_{N-n+1} & \cdots & \left(\frac{1}{\epsilon_N}-\frac{\rho_N}{\epsilon_N}F_{NN}\right)
\end{bmatrix}
\quad
X = \begin{bmatrix} E_{b1} \\ \vdots \\ E_{bn} \\ q_{n+1} \\ \vdots \\ q_N \end{bmatrix}
$$

But this has little expanded matrix, the matrix looks like this. So this part of the matrix contains the view factors that relates to the emissive power. The first n terms contains emissive power as the unknowns and the rest n + 1 to N contains heat flux as unknown okay. So similarly the matrix structure is like this. The left top corner contains elements which directly relates to the emissive power as in the previous example of black surfaces we have diagonal terms 1-Fii.

And off diagonal terms as Fij while this part of the matrix in the right hand side of the matrix contains terms which relates to the heat flux okay. So it has the emittance and reflectance appearing in the terms because it relates to the heat flux. Just looking at the equation it will be clear that reflectance appears in the heat flux term and it does not appear in the emissive power term. So this is the structure of the matrix.

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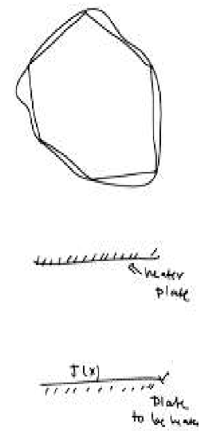
Matrix Method (Gray Surface)

$$
B = \begin{bmatrix}
\sum_{j=1}^{n}\left(\frac{\delta_{ij}}{\epsilon_j}-\frac{\rho_j}{\epsilon_j}F_{1j}\right)q_j - H_{01} - \sum_{j=n+1}^{N} F_{1j}E_{bj} \\
\vdots \\
\sum_{j=1}^{n}\left(\frac{\delta_{ij}}{\epsilon_j}-\frac{\rho_j}{\epsilon_j}F_{nj}\right)q_j - H_{0n} - \sum_{j=n+1}^{N} F_{nj}E_{bj} \\
\sum_{j=1}^{n}\left(-\frac{\rho_j}{\epsilon_j}F_{n+1,j}\right)q_j - H_{0,n+1} + \sum_{j=n+1}^{N}\left(\delta_{n+1,j}-F_{n+1j}\right)E_{bj} \\
\vdots \\
\sum_{j=1}^{n}\left(-\frac{\rho_j}{\epsilon_j}F_{Nj}\right)q_j - H_{0N} + \sum_{j=n+1}^{N}\left(\delta_{Nj}-F_{Nj}\right)E_{bj}
\end{bmatrix}
$$



The right hand side again contains the non-fluxes for the first hand surfaces, external irradiation and known emissive power from surfaces n + 1 to N and we can apply the same or similar method for the system of linear equation to solve this type of problems. So this is how we can basically write the energy balance equation for surfaces in an enclosure which is an idealized case and solve it on a computer okay.

So I again recap why we call it idealized enclosure because we have divided the enclosure into number of finite surfaces okay, which may not always be good in all type of problems. For example, if we have surface okay and let us say we want to maintain or heat the surface uniformly over the entire area, we want let us say the temperature of the surface to be maintained at 1500 kelvin and we do not want the temperature variation of the surface should not be greater than let us say 1 degree centigrade.

So that entire surface should be maintained at uniform temperature and the surface is heated using some heater, let us call this is heater plate okay and we want to heat this plate. So this is plate to be heated and maintain at certain temperature uniformly. So one thing that we should do is we cannot approximate the surface as single surface because if we do it then this will just give us an average heat flux or average temperature.

But if we want to know the temperature variation along the plate we have to take into account how the radiocity J varies along the surface okay. So this J has to be calculated at each and every point and we should not approximate this surface with finite width or finite dimension plates okay. So we start with the same equation that we develop for the heat flux for a gray surface. The emitted part, the absorbed part okay.

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Methods For Variable Radiosity

$$q(\mathbf{r}) = \epsilon(\mathbf{r})E_b(\mathbf{r}) - \alpha(\mathbf{r})\left[\int_A J(\mathbf{r}')dF_{dA-dA'} + H_0(\mathbf{r})\right]$$

- Black surface (Integral evaluated before heat flux)

$$q(\mathbf{r}) = E_b(\mathbf{r}) - \alpha(\mathbf{r})\left[\int_A E_b(\mathbf{r}')dF_{dA-dA'} + H_0(\mathbf{r})\right]$$

- Gray Surface (Heat Flux appears in the Integral)

$$\frac{q(\mathbf{r})}{\epsilon(\mathbf{r})} - \int_A \left(\frac{1}{\epsilon(\mathbf{r}')} - 1\right) q(\mathbf{r}')dF_{dA-dA'} + H_0(\mathbf{r}) = E_b(\mathbf{r}) - \int_A E_b(\mathbf{r}')dF_{dA-dA'}$$

And for black surface this will be just equal to 1 and radiocity will be simply equal to emissive power. Now if the emissive power is constant, if temperature is known, if temperature over all the surface is known then the integral in this equation can be evaluated first and then subsequently we can solve for the flux. So please note if the temperature at all the surfaces is known we can solve for this integral before we solve for the flux okay.

So this integral part of the equation can be separated from the algebraic part okay and the equation becomes pretty simple okay. On the other hand, if we have heat flux as unknown then definitely the problem is going to be complicated. So if temperature is unknown then we have basically a variable appearing in integral. So we have a variable appearing in integral and then it becomes an integral equation.

It is not a algebraic equation anymore, it becomes an integral equation and this fact is much more manifested in gray surfaces. So for gray surfaces we write the same energy balance equation as we have done before and we see that the unknown heat flux is appearing in the integral and known emissive power may also appear in the integral although if temperature for some surfaces is known then this integral can be evaluated very easily.

So the only unknown appearing in the integral will be then the heat flux okay. So we have what we call integral equation in this cases when we are finding the heat flux by not dividing the enclosure into finite number of surfaces. The good advantage of dividing the enclosure into finite number of surfaces was to convert the integral equation into algebraic equation, but if we do not do that we have to solve for the integral equation and there are number of methods to solve this equation.

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Methods For Variable Radiosity

- ***Fredholm integral equation of the second kind***

$$\emptyset(\mathbf{r}) = f(\mathbf{r}) + \int_A K(\mathbf{r}, \mathbf{r}')\emptyset(\mathbf{r})dA'$$

Where $K(\mathbf{r}, \mathbf{r}')$ is *kernel* of the integral equation

$f(\mathbf{r})$ is a known function

$\emptyset(\mathbf{r})$ is the function of radiosity or heat flux

This type of equation where we have a variable, let us say the unknown function phi ($\emptyset$), it could be radiocity, it can be heat flux, appearing in the integral on the right hand side also. So this type of equation is called integral equation and this particular equation is called Fredholm integral equation of second kind. The K appearing in this equation is called the kernel function.

The F is the known function and $\emptyset(\mathbf{r})$ is the function that we want to evaluate, it could be radiocity, it could be heat flux. So this Fredholm equation we need to solve if we want to find out the exact distribution of heat flux over the enclosure. So there are number of methods. Analytical methods are also possible and numerical methods are also possible.

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Solution of Fredholm Equation

- Analytical method
  - Kernel Approximation Method
- Numerical methods
  - Successive Approximation Method
  - Quadrature Method
  - Variational Calculus Method (Galerikin Method)

Although the analytical methods are mathematically quite involved and time consuming. So most of the time we prefer to solve these type of problems using numerical methods and the numerical methods provide good accuracy also. So in analytical methods we have what we call Kernel approximation method and in numerical methods there are many methods but 3 most important methods used is successive approximation method, quadrature method and variational methods or Galerikin method okay.

These methods are basically used extensively to solve problems in gray enclosures. So I will discuss the successive approximation method and quadrature method in this lecture.

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Kernel Approximation Method (Analytical Solution)

$$\emptyset(r) = f(r) + \int_A K(r, r')\emptyset(r)dA'$$

- Determines approximate analytical solution
- The kernel $K(r, r')$ approximated with liner series of functions
  - e^{-ax}, $cosax$, $cosh\ ax$ etc.
- Integral equation is converted to differential equation that can be solved explicitly

So I just give you an outline of the kernel approximation method, the method when applied to real problems becomes the mathematics become quite involved, so I will not take an example of this method. The outline of this method is basically we approximate the Kernel function, the K using some kind of simple functions like exponential function or trigonometric function or hyperbolic functions.

So these type of functions are used, the series of these function is used to approximate the kernel function okay and these functions are then substituted in the integral and the integrals are then evaluated very easily by calculating the integral of these trigonometric or hyperbolic functions, the integral equation is converted into a differential equation which can be solved easily as compared to the integral equation.

So the method in itself is straightforward, but the solution of these integrals and the subsequent differential equation becomes quite involved okay. So I will not take any example on this method. The next method is successive approximation method. In successive approximation method, this is the numerical method, what we do is we basically make a guess of this function and substitute this into the integral okay.

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Successive Approximation Method (Numerical Solution)

- First guess $\emptyset(\mathbf{r})^{o} = f(\mathbf{r})$
  - Integral evaluated analytically or through numerical quadrature
- The improved value of $\emptyset(\mathbf{r})$ substituted back into the integral
- Converge for all surface radiation problem

So let us say we first guess that the distribution function for the heat flux or radiosity the unknown function is given by this function f okay and we substitute this function in to the integral and solve the integral either analytically or through numerical quadrature. So we solve this, in analytical or using numerical quadrature. Once the integral is solved we get a new value of the function $\emptyset(r)$ okay.

And then this value of $\emptyset(r)$ is again substituted into the integral and a new solution is found. So this is an iterative method okay. So let me just again highlight. So we have unknown function phi which is basically given by k $\emptyset(dr)$as is given here. We can just write it as A okay. So we first assume some value of let us call $\emptyset(i)$and substitute it here and solve the right hand side.

So we get $\emptyset_2$ as f + k $\emptyset_1$ dA, this integral we can solve analytically or through numerical quadrature. Once this solved we get a new value of the function $\emptyset_2$ and then we substitute it back. So we get $\emptyset_3$ as f + k $\emptyset_2$ dA and this keeps on, we keeps on doing this until we get a desired result okay. So this method also is quite involved if we try to solve this integral using analytical methods.

But it can be an easy method if we use numerical quadrature to solve the integrals. We will do one example to demonstrate this method. In this case we have 2 infinite parallel plates separated by a distance h.

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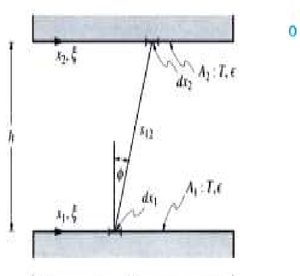
Problem

Problem-1: Consider two large parallel plates of width ω as shown. Both plates are isothermal at the same temperature T, and both have a gray, diffuse emittance of ϵ. The plates are separated by a distance h and are placed in a large, cold environment. Determine the local radiative heat fluxes along the plate using the method of successive approximation.



Both the plates are maintained at temperature T and they are exposed to cold environment that means we are not including the effect of external irradiation, both the plates are diffuse with emittance epsilon. So we have to find out local radiative efflux. So we have to find out q as a function of r on both the plates okay. So this is the desired quantity that we are interested in. As is clear, we are not interested in average heat flux.

Average heat flux we could have found by the methods we have already discussed okay, but we are interested in the actual value of flux as it varies over radius r okay, although this problem we have taken is quite trivial due to the one dimensionality of the problem, it is an infinite plate. This dimension in third direction is infinite. So the result would be quite similar to the average values but still as a purpose of demonstrating the method this problem serves a good example. So let us solve this problem.

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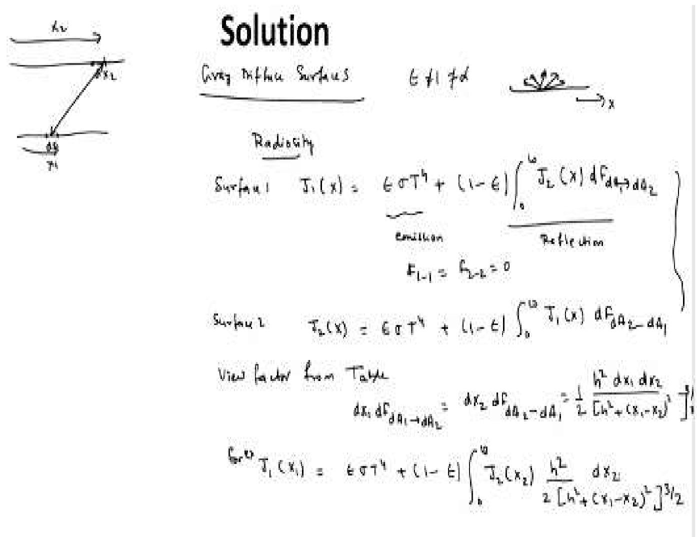

So first of all we write down the expression for radiosity, this is going to be gray, diffuse surfaces. So the problem involves gray diffuse surfaces that means the value of emittance is not equal to 1 and we have diffusely emitting and reflecting surfaces okay. So we write down the radiocity surface 1, $J_1(x)$ which is going to be function of x okay, x is the coordinate along the direction of the plate , $J_1(x) = \epsilon\sigma T^4 + (1-\epsilon)\int_0^{\omega} J_2(x)\mathrm{d}F_{dA1-d}$ and we can just emission and absorption.

So this is our radiosity, so radiocity contains the emission term and reflection term. So this is emission from the plate and this is the reflection from the plate okay. In this expression you might have noticed that we have only J_2 appearing in this because there is not going to be irradiation from the same surface, so we have F11 = F22 = 0. So there is no self-irradiation, so the only unknown appearing in the integral is for the J_2 okay.

Similarly, we will write this for surface 2 also. So surface 2 the J_2 value can be written as , $J_2(x) = \epsilon\sigma T^4 + (1-\epsilon)\int_0^{\omega} J_i(x)\mathrm{d}F_{dA2-dA1}$, both the plates are of same temperature with same emittance, okay. So this is basically the relation. Now we have to find out the view factors first. So let us, so I will not do the view factor calculation, we have already done that and this is a simple geometry.

So view factor from look up tables can be calculated, so this value will be = dx_1, we are trying to find out the small element view factor $dx_1.dF_{dA1\text{-}dA2} = dX_2.\ dF_{dA2\text{-}dA1}$, this is readily available from the look up table, from the tables for the view factor and this value ($dx_1.dF_{dA1\text{-}dA2} = dX_2.\ dF_{dA2\text{-}dA1}$) $= \frac{1}{2}\frac{h^2 dX_1 dX_2}{(h^2+(X_1-X_2)^2)}$okay. So let me tell you what basically this represent.

So we have these 2 tables, we have a small element area at a distance of x_1, the thickness of this is dx1 another area we have the distance from the left side is x_2 and the area thickness is dx_2. So the view factor between these 2 surfaces is what we are basically talking in this case okay. So now our radiocity relation are both very similar, the expression for J_1 and expression for J_2 are very similar.

So we are going to substitute for the view factors in this relation. So for example J_1 (x_1), now we have introduced specific x coordinate x_1 for the plate 1 $J_1(x) = \epsilon\sigma T^4 + (1-\epsilon)\int_0^{\omega} J_2(x)\frac{h^2 dX_2}{(h^2+(X_1-X_2)^2)^{3/2}}$ okay. So as you see the integral is quite involved okay. So we have to introduce.

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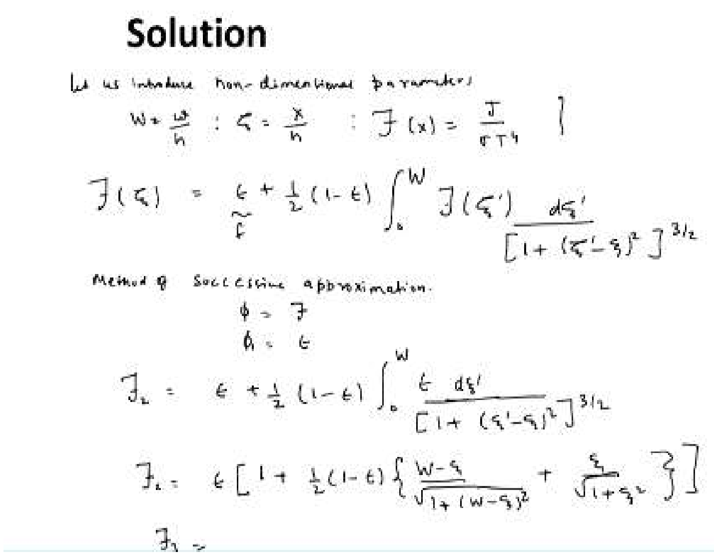

So let us introduce non-dimensional parameters okay, so we introduce non-dimensional parameters as W as w/h and zeta $\xi(x)$as = x/h okay, so these non-dimensional parameters we have introduced and we have also introduced function, this is a non-dimensional function which is $J(x) = \sigma T^4$ okay. So the governing equation for the radiocity then becomes, will be a function of zeta and this will $J(\xi) = \epsilon + \frac{1}{2}(1-\epsilon)\int_0^W J(\xi')\frac{d\xi'}{[1+(\xi'-\xi)^2]^{\frac{3}{2}}}$. okay

So we have just non-dimensionalized the radiocity using this non-dimensional parameters okay. Now how do we solve it? So we will use method of successive approximation where we have to make a guess. So our in this case phi is basically this function (φ= ξ)and the first guess for ϕ_1 will be = epsilon (ϵ). The function f okay. So the first guess we make is the $\phi_1 = \epsilon$.

So the second guess for this function will be $[J2 = \epsilon + \frac{1}{2}(1-\epsilon)\int_0^W J(\xi')\frac{d\xi'}{[1+(\xi'-\xi)^2]^{\frac{3}{2}}}]$= epsilon + 1/2 1- epsilon okay and then integrate it over 0 to W, so we substitute it here, it is constant epsilon d zeta prime and then in the denominator 1 + zeta prime – zeta square power 3/2 okay. So this is a simple integral, we can solve it and this gives you the value as $J2 = \epsilon$ $[1 + \frac{1}{2}(1-\epsilon)\left\{\frac{W-\xi}{\sqrt{1+(1+(W-\xi^2))}} + \frac{\xi}{\xi^2}\right\}]$

Now because of symmetry, the radiocity of the 2 plates will be similar okay. Now we can integrate this for the third time it is possible by putting the second iteration into the integral, but the integration becomes quite complicated. So we will not evaluate psi 3 okay, it will be very complicated to evaluate the integral, we will just approximate the solution by this relation.

So the value of J = J1 = J2 both are going to be similar because of the symmetry is simply =σT^4, the non-dimensional radiocity. Once the non-dimensional radiocity is known we can easily solve for the heat flux Q as a function of r okay. So this was the method of successive approximation. The next method is the method of numerical quadrature.

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Numerical Quadrature Method

$$\int_a^b f(x)dx = \frac{b-a}{2}\sum_{j=1}^{N} w_j f(x_j)$$

- Method to evaluate complex integrals using numerical method
  - Weighted sum of function values at selected nodal points
  - The nodal point may be equally spaced (Newton-Cotes quadrature) or optimized for accuracy (Gaussian quadrature)
- This method is easily applied to any arbitrary 3-D geometry.

$$\emptyset(\mathbf{r}) = f(\mathbf{r}) + \int_A K(\mathbf{r},\mathbf{r}')\emptyset(\mathbf{r})dA' = f(\mathbf{r}) + \frac{b-a}{2}\sum_{j=1}^{N} w_j K(\mathbf{r},r_j)\emptyset(\mathbf{r}_j)$$

This is the very powerful method which can easily be implemented in any computer. Now the method of numerical quadrature is very popular in radiation. So what this basically includes suppose we want to solve for an integration numerically. So we have a function $f(x)$ that we want to integrate from limits or boundaries A to B, so what we do is we write this integral as algebraic sum or weighted sum of the function value f at selected number of node points.

So we evaluate the function at selected number of node points and we multiply by width and then sum over those points and that is what basically this integral. For example, let us say we want to find out the area under this curve okay. This area will be (A) = $\int f(x)\, dx$ from whatever limits A to B let us say. Now what we are saying is we have selected few number of points on this.

We know the value of the function at these points and the area can be calculated based on the value at these points multiplied by some weights assigned to each point okay. Now these points may be equally spaced that means these points may be equally spaced or they may not be equally spaced okay. If they are equally spaced the method of numerical quadrature are based on this approach is called Newton-Cotes quadrature.

And if the location of these point is optimized based on either Legendre polynomial or Chebyshev polynomial, this method is called accordingly Gauss-Chebyshev quadrature or Gauss Legendre quadrature. In short it is called basically the Gaussian quadrature. So Gaussian quadrature is different from Newton-Cotes quadrature in the sense that the points used to evaluate this integral are optimized according to the problem okay.

So we can apply this method easily to evaluate the integral in the Fredholm's equation. So what we do is let us say we have a one dimensional program where the limits are a to b, so the kernel function has to be integrated from a to b. So we approximate this $\varphi(x)$= $f(r)$, the term in the integral is approximated as per the formula given above, b –a/2 weights w_j then summed over 1 to N.

Now N is the order of quadrature, we may have 4-point quadrature, we may have 8-point quadrature. So mostly 8-point quadrature for Gaussian works fine. So the value of n can be then 8, then the kernel function valuated at the nodes r_j and the function phi also evaluated at the same nodes okay. So this converts this integral into an algebraic equation which is very easy to solve.

Okay once we know the weights and we know the function values, we can solve for this equation. Now we can write down this like this.

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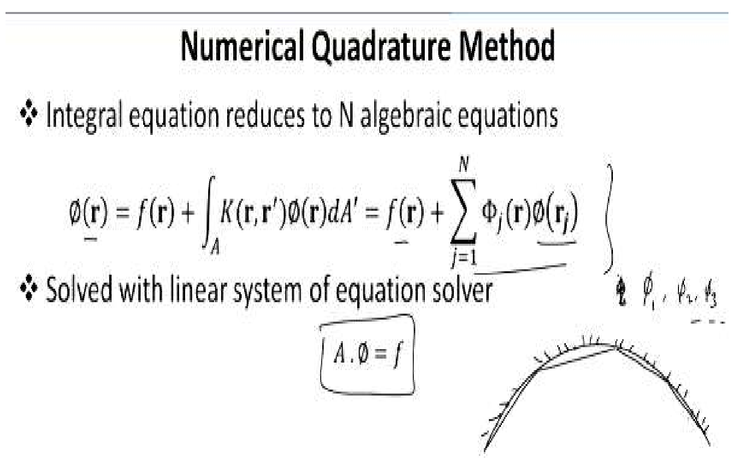

$\phi(r)$= $f(r) + \sum_{j=1}^{N} \boldsymbol{\phi}_j(r)\phi(r_j)$ okay. So we have defined this term as including b –a /2 as phi okay. So our equation now reduces to this simple equation which is a system of linear equation and the unknown are phi (ϕ) okay, at the respective quadrature points okay. So based on this, so we have ϕ_1, ϕ_2, $\phi_3$and so on as the unknown okay and we can solve the system of linear equation very easily and the method works very fine.

So just a quick look how basically we are doing it suppose we have an enclosure like this how the problem is different from finite number of surfaces and finite number of quadrature points. In finite number of surfaces, we divide the enclosure into finite number of flat surfaces like this okay and we calculate it average heat flux over the entire surfaces. Now in this case quadrature we are not selecting flat surfaces we are just taking the quadrature points okay.

And evaluating our function at those points okay and we assume that if we evaluate the quadrature in this way we can approximate the heat flux over the entire enclosure. So this method is much more accurate than the method used by approximating the enclosure with flat surfaces. So I will show you one graph explaining the accuracy of this method. So on this graph we are plotting the non-dimensional heat flux for the same problem that we have just solved.

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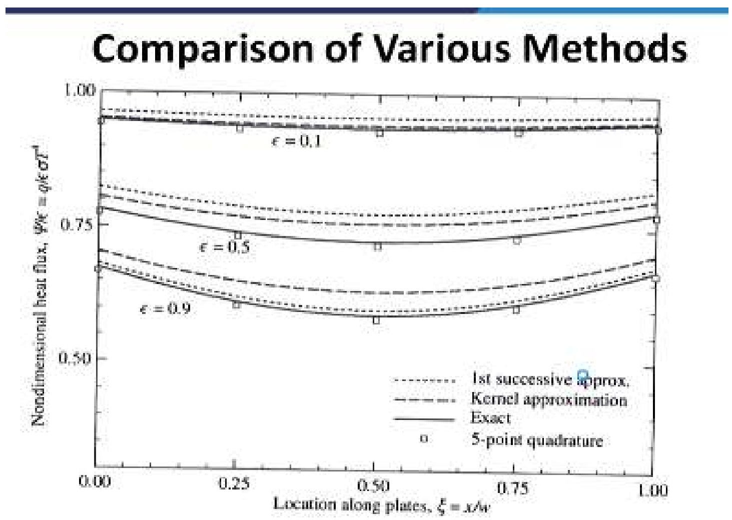

Two infinite parallel plates, the x axis is non-dimensional x, and y axis is non-dimensional heat flux and we see the solid line is the exact result and the exact result is in good agreement with the quadrature result, numerical quadrature result with 5 points taken, 5 nodes taken on the other hand the successive approximation method and the kernel approximation methods are not that accurate okay.

So the two methods are quite involved mathematically, difficult to solve and they are not in good agreement with the exact results. On the other hand, the quadrature is the powerful technique that gives you good results and it is easy to solve on computers. So thank you for your kind attention. In this lecture we learnt an accurate technique to find heat flux without approximating the enclosure with flat surfaces. In the next lecture we will learn something about specular reflection, thank you.