Automatic Control
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Lecture – 09
Linearization of Nonlinear System

So, welcome to the lecture on mathematical modeling. Today, we will discuss about linearization of non-linear systems. So, in previous lectures, we have performed mathematical modeling using transfer functions. And to derive the transfer functions for a system, there was an assumption that transfer function can be derived only for linear time invariant systems; however, if we have non-linear systems, we cannot develop the transfer function; however, we know that there are several systems they are in non-linear range, their behavior is mainly non-linear. They are linear only in certain operating range.

For example, if we have an amplifier. So, initially it is linear, but then it saturates for certain voltage. If there is some electrical system like motor; this motor will not perform until certain input, it will perform only after certain threshold input. Because it will take that to counter the friction in the motor; so, we can see that the for an amplifier, if this is input and this is output.

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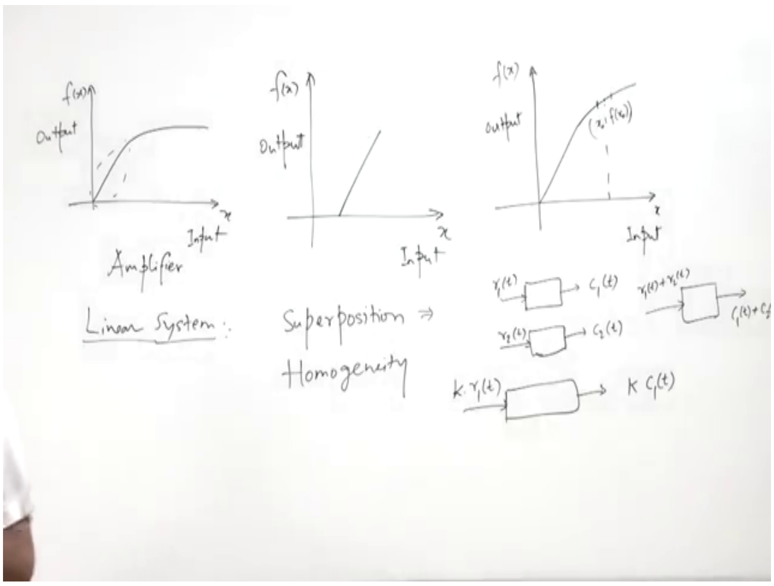

So, it is relation is linear, but then it saturates after certain input. If this is mortar; so, and this is output and this is input. So, this may we have like this. So, initially there could be some dead zone. So, no response, and then it will respond. If there are some dampers; so, here is input, this is output. So, damper behavior can be linear and then it can become non-linear.

So, we can see that there are several systems, they we have as a non-linear system. But if we consider certain range like this part, here we can approximate it as linear, this part we can approximate as linear, and this part we can approximate it. So, in this range we can apply the concept of transfer function but if the system is working in this range or this range; so, this is a non-linear range of the system, and we cannot apply the transfer function concept; however, if we linearize this non-linear range we can apply the transfer function.

So, this non-linear range, suppose this is some point f x 0 x 0, and the system is operating about this point, and in certain range, this range is small. So, in this range we can approximate this non-linear system as a linear system. So, therefore, a linear system what is a difference between linear system and non-linear system? That linear system contain 2 properties superposition and homogeneity.

So, linear system has this property that is superposition, and homogeneity. So, what is superposition? So, if I have a system linear system I give an input and I am getting some output, I am giving some different input and I get corresponding output. Now I give another input r 2 t, and I get certain output c 2 t. And now, for the same system I give my input r 1 t plus r 2 t, I should get an output that will be the sum of the individual outputs.

So, this is c 1 t plus c 2 t. So, that is superposition. Because the system is subjected to 2 different inputs and we are getting 2 different outputs. Now this sum of the inputs is applied we should get the sum of the outputs. So, this is superposition. Homogeneity is if we have this system r 1 t giving c 1 t, and I apply I input k times r 1 t I should get an output k times c 1 t, where k is some scalar constant.

So, if amplify the input we should get the same amplification in the output, that is homogeneity. So, there are several systems and they this system, we have as linear system in certain range. So, for the non-linear system, if there is some operating point, that we call equilibrium point and the system is operating in certain close range of that operating point. We can approximate that system non-linear system as a linear system.

So, how we can do? That we will see.

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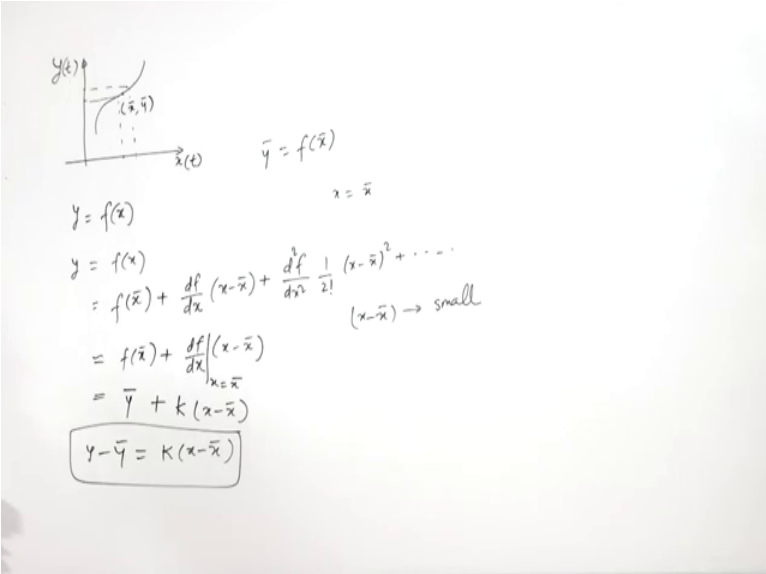

So, let us take this non-linear system. Here input is x t output is y t. And this is the non-linear function y equal to f x. And this is the equilibrium point. The system is operating about this point. So, we can write y equal to f x, we can expand as a Taylor series about this point. So, the Taylor series we can write here. So, here we can write this Taylor series as the higher order derivatives of this function. And these derivatives d f by d x d square f by d x square and higher order should be evaluated at x equal to x bar.

So, at this equilibrium point at x equal to x bar, these values should be evaluated. Now suppose this because we assume that x minus x bar is small, because we say that the system is operating very close to this point. So, the input what input we are giving is not very far away from this. So, we are in a very small range. So, our x is varying about this point maybe from this point to this point. So, it is close to this. And so, our output will also vary in this range; so, small range. So, we assume that x minus x bar is. So, small that the higher order like a square power 3 power 4, and these terms we can neglect. Because there will be even more smaller.

So, we can approximate this as f x bar plus d f by d x x minus x bar. So, we can approximate these. So, y bar equal to f x bar, because the value of function at x bar is y bar. So, here we can write y bar, plus this d f by d x is calculated at x equal to x bar. So, let us write this k and x minus x bar. Or we can write y minus y bar equal to k x minus x bar.

So, this is a linear function. So, this non-linear function, this is now approximated as a linear function it close to this point. So now, if in any differential equation is formed that contain some non-linear function f x, we can approximate these 2-linear function. And then we can apply the Laplace transform and find the transfer function for this non-linear system. Here we can see that there is only one variable x. Now if there are more than one variable or 2 variables x 1 and x 2. Then how to write this function? So, let us assume y is function of 2 input variable x 1 and x 2.

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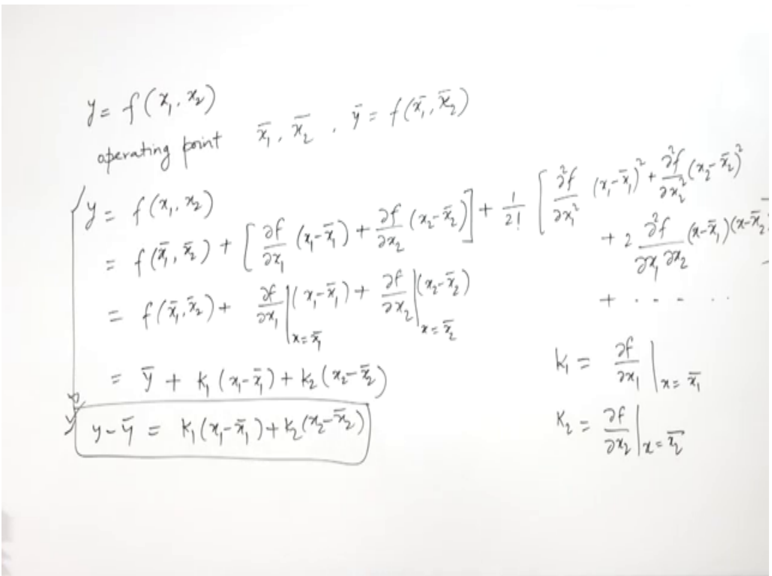

So, here x 1 and x 2 there are 2 inputs and we are getting output y. And operating point operating input point is x 1 bar and x 2 bar, and y bar equal to f x 1 bar and x 2 bar. So, y equal to f x 1 x 2. Now here are 2 inputs. We have to use the partial derivatives. Because there are 2 parameters x 1 and x 2. So, we can write this equal to f x 1 x 2 plus del f by del x 1, x 1 minus x 1 bar plus del f by del x 2 x 2 minus x 2 bar. This is the first derivative, plus 1 by factorial 2 del e square f by del x 1 square x 1 minus x 1 bar square plus del square f by del x 2 square x 1 x 2 minus x 2 bar square. Plus some higher even more higher order terms.

So, this term is part of this first derivative second derivative and plus the higher order term. Now we if we neglect these terms and higher order terms, assuming that this difference is very small. X 1 minus x 1 bar and x 2 minus x 2 bar, we will get f x 1 x 2 plus del f by del x 1 x 1 minus x 1 bar plus del f by del x 2 x 2 minus x 2 bar.

Now, these del f by del x 1 should be able to evaluate it at x equal to x 1. And del f by del x 2 should be evaluated at x equal to x 2 bar. And so, we can write let us write this y bar plus. Del f by del x 1 we say k 1, x 1 minus x 1 bar plus let us say this is k 2 x 2 minus x 2 bar. And we can write it y minus y bar equal to k 1 n x 1 minus x 1 bar plus k 2 x 2 minus x 2 bar. Here k 1 equal to del f by del x 1 at x equal to x 1 bar, and k 2 equal to del f by del x 2 at x equal to x 2 bar.

So, we see that this non-linear function y is converted into a linear function about these operating points. And we calculate the partial derivative of this function with respect to x 1, and we find the value at x equal to x 1 bar, and then partial derivative k 2 that that is del f by del x 2 at x equal to x 2 bar. And we can put these values here, and we can find for any input values x 1 and x 2 we can find the output y.

So, let us take one example to solve it. How to apply this? So, we have if we take one example.

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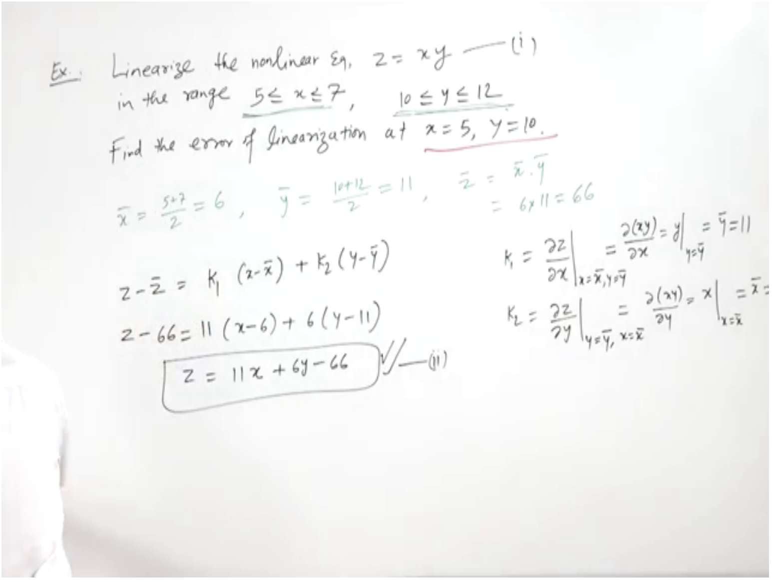

So, we want to linearize the non-linear equation or function z equal to x y, in the range 5 less or equal to x less or equal to 7. And y, find the error of linearization at x equal to 5 and y equal to 10.

So, here we have this non-linear equation or non-linear function z equal to x y. So, we have 2 variables input variable x and y. And it is sent said that they have this range 5 to 7 for x and 10 to 12 for y. And we have to find the error of the linearization at x equal to 5 and y equal to 10. So now, here this range from this range we should find that equilibrium point. So, let us take this equilibrium point, at as the mean of these 2 extreme values.

So, here x 1 x equal to x bar equal to 5 plus 7 by 2, that is 6, and this y equal to 10 plus 12 by 2. So, that is 11. So, this is the equilibrium point. Because at this point about 6 is the equilibrium point and we are going to range varying from 5 to 7. And here 11 is the equilibrium point and we have this range 10 to 12.

Now, corresponding to this point z equal to x bar into y bar. So, it is 6 into 11 that is 66. So, we are getting z bar as 66. So now, we just developed the equation for linearization, containing 2 variables. And that is we can apply here. And so, here is z minus z bar equal to k 1 x minus x bar plus k 2 y minus y bar.

So, this is the equation here output is z and 2 inputs x and y, and here k 1 is derivative del f by del x, or we can say del z, because here f is z. So, del z by del x at x equal to x bar. And k 2 equal to del z by del y at here y equal to y bar. So, we can find these values. So, here z minus z bar is 66 equal to k 1. So, del z by del x. So, del z is x y by del x of course, here y equal to y bar and here x equal to x bar. So, here we will get del x y by del x is nothing but y. And so, y at y equal to y bar equal to y bar, and that y bar is 11.

Similarly, we can find this. So, del z by del y. So, del x y by del y, and that is x at x equal to x bar. And that is x bar and this is equal to 6. So, we can write here k 1 is 11. And then x minus x bar x bar is 6 plus k 2 is 6 y minus y bar is 11. So, we can simplify this equation. And we can find z equal to 11 x plus 6 y minus 66.

So, this is the linearized form of the equation one. So, this one is the non-linear equation or non-linear function. And in this range, we have linearized this as this equation number 2. Now you can see if we have this function in when we are in our differential equations, and we know that these values are in this operating range. We can replace these at this equation. And we can take easily the Laplace transform and we can obtain the transfer function because now this is linear.

Now, the second question is find the error of linearization at x equal to 5, and y equal to 10. We have to find the lowest range at lowest range of x and y, what is the error? How much error we are getting off when we are linearizing? So, we can find this. So, equation one we get z at x equal to 5 and y equal to 10, that is equal to x into y.

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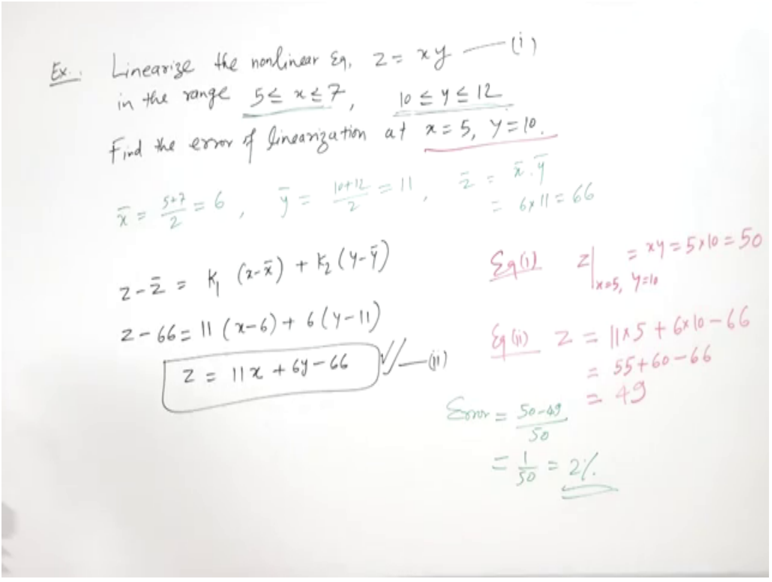

So, x is 5 and y is 10. So, we are getting 50. So, this is the correct value according to equation one following a non-linear equation.

Now, from equation 2, we get z equal to at the same x equal to 5 y equal to 10 11 into 5 plus 6 into 10 minus 66. So, we are getting 55 plus 60 minus 66. And we get here 49. So, the error we are getting it 50 minus 49 upon 50. That is 1 by 50 or 2 percent. So, we can see that when we are linearizing this at the lowest point extreme point of the range. We are getting a maximum error of 2 percent.

So, here we learn that we learned that how to linearize a non-linear system when the operating range is about some very close to some equilibrium point. So, we can linearize that non-linear system non-linear function about that range. And once we are able to linearize, we can be able to write the transfer function for that non-linear system in that range. And that will help us to study the system using the transfer function. So, here we would like to stop, and let us see in some next texture.

Thank you.