Automatic Control
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Lecture – 07
Rotational Mechanical system

So, welcome to the lecture on Mathematical Modeling. In this lecture, we will discuss the modeling of Rotational Mechanical System. So, in the previous lecture, we discussed about Translational Mechanical Systems. So, the system that had has translational degree of freedom. In this lecture, we discuss Rotational Mechanical System is the systems that they have the rotational degree of freedom.

So, these rotational systems will be solved in the similar manner as we did the translational systems. Only here the rotational degree of freedom will be used, so that the translational displacement will be replaced with the angular displacement and the torque will replace the force. We will to solve the systems; we will write the equation of motion by you by using the Newton's Law on the free body diagram of the system. And then, we will take the Laplace transform to find the transfer function of the systems.

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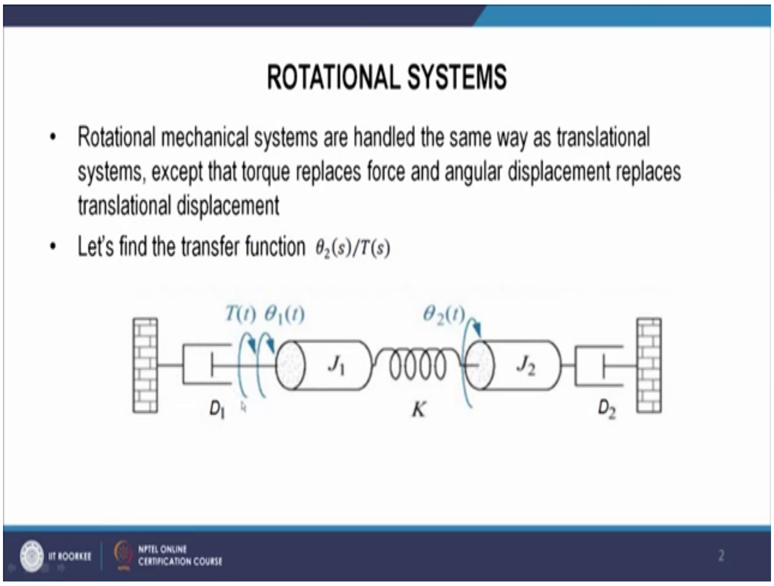

So, let us take one example of rotational system. So, we can see here, this is our system. So, here this Damper is a damper that response responds to angular velocity. And here is the inertia. Now, the in spite of mass, we use inertia that is subjected to some torque T and angular displacement is theta 1 for this first inertia. Then there is the spring, this is a spring is responds you to the angular displacement and this second inertia has the angular degree of freedom the rotational degree of freedom theta 2 t and there is the second damper attached to this second inertia.

So, this system, we can write the differential equation of motion and then we can solve to find the transfer function. So, here is the system.

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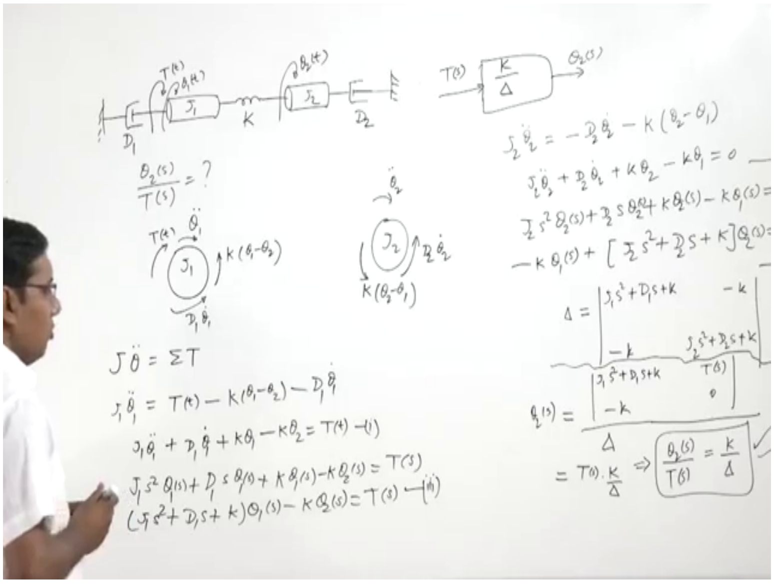

So, this is inertia J 1, this is J 2, this is damper with damping coefficient D 1, this is D 2 and here is applied a torque to the first inertia T that is function of time and the degree of freedom theta 1 t and here is theta 2 t, this spring has stiffness K.

We have to find theta 2 s by T s. This is the transfer function we have to find. So, we have to find this theta 2 s and this T s. So, input to the system is the torque and output is theta 1 and theta 2, but here we are interested to find these transfer function, theta 2 s by T s. There is also possible to find the transfer function theta 1 s by T 1 s in the same manner.

So now, when we want to find this transfer function, again we have this 2 degree of freedom system. And we will make the free body diagram and we apply Newton's Second Law in the rotational degree of freedom and then we find a differential equation of motion of this system. So, let us say first this mass this inertia J 1. So, this is J 1. So, this J 1 is suppose it has this theta 1 double dot, this angular acceleration in this direction and the torque is applied here in clockwise direction.

Now, the damper will apply the force in opposite direction. So, in anticlockwise so, this is D 1 into theta 1 dot and this spring is between the 2 inertia. So, the difference of the two, theta 1 and theta 2. So, again it will apply in anti-clockwise K theta 1 minus theta 2. The second inertia J 2, this is theta 2 double dot, this acceleration angular acceleration, this damper will apply a force opposite. So, D 2 into theta 2 dot and this spring will have the force, We can say K times theta 2 minus theta 1 or this can be if we change the take the same this direction we can write K theta 1 minus theta 2.

So, here we are applying opposite, so, K theta 2 minus theta 1. Now, we have shown the free body diagram. Now, we will apply here the Newton's Law that will be J theta double dot equal to summation of torque. So, J is the inertia moment of inertia and theta double dot is the angular acceleration. So, this is equal to summation of the torques. So, for the first inertia J 1, we will have J 1 theta 1 double dot and this torques. So, we have the start T external torque that is input then these are the torque due to this K, minus K theta 1 minus theta 2 minus D 1 theta 1 dot.

So, we can write J 1 theta 1 dot double dot plus D 1 theta 1 dot and here minus K theta 1 this will go the other side that will be plus K theta 1 and here is minus minus plus, go other side minus K theta 2 equal to T, this is equation number 1. Now, for this system, we write J 2 theta 2 double dot equal to summation of torques.

So, minus D 2 theta 2 dot minus K theta 2 minus theta 1 so, we can write J 2 theta 2 double dot plus D 2 theta 2 dot plus K theta 2 minus k theta 1 equal to 0, this is equation 2. So, we have find the differential equation for both the inertias. Now, we have to find the transfer function.

So, we have to take the Laplace transform of both the equations. So, let us take first this Laplace transform of this equations and because objective is to find the transfer function let us put all the initial conditions 0. So, here J 1, this is second order. So, s square and theta 1 s plus D 1, this is first order differential equation differential term. So, this is s and theta 1 s plus K theta 1 s minus K theta 2 s equal to T s.

So, we can simplify this; we collect the terms. So, we can write J 1 s square plus D 1 s plus K theta 1 s minus K theta 2 s equal to T s, So, this is equation number 3. Now, we take the Laplace transform of this part. So, this is J 2 s square theta 2 s plus D 2 s theta 2 plus K. So, here s theta 2 s minus K theta 1 s equal to 0, this is equation number.

So, we can write this equation in this form, minus K theta 1 s plus J 2 s square plus T 2 s plus K theta 2 s equal to 0, so, this is equation number 4. Now, this equation 3 and 4, we can apply the Cramer's Rule to find the; these transfer function. So, Cramer's Rule as we saw we can apply. So, if our delta is the determinant.

So, here J 1 s square plus D 1 s plus K and minus K and here minus K and this term J 2 s square plus T 2 s plus K ; this is delta. Now, we have to find the theta 2 s. So, theta 2 s equal to this in this is delta in the denominator, this term will be this will be replaced with the column of the this side the right side that is t s and 0.

So here, this J 1 s square plus T 1 s plus K, here minus K and this is T s and here is 0; so, 0. So, we should understand here, we should differentiate. So, this column is same and because we are finding theta 2 is this column will be replaced with the column to the right hand side term that is T s and 0. So, from here, we will find this equal to now this is 0 minus T s into K. So, minus minus plus, so, T s into K K by delta.

So, from here, we can find that theta 2 s by T s equal to K by delta. So, this is the transfer function theta 2 s by T s. So here, it is K by delta and delta is given with this determinant. Now, you see that this transfer function contains the parameters of the system J 1, D 1, K, J 2, D 2 and so, this transfer function so, is the characteristic of the systems. So, this is how, we can find the transfer function of rotational systems.

Now, we have some discussion on Impedance. We can represent the impedance of a mechanical system. So, the impedance of a mechanical system is represented as the force upon displacement that is in the s domain. So, for example, for the Inertia, we can see from this in this slide this inertia.

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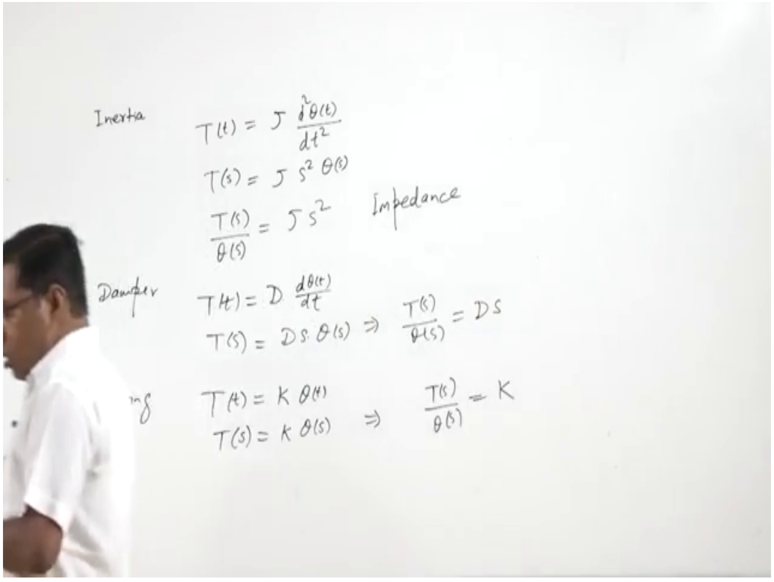

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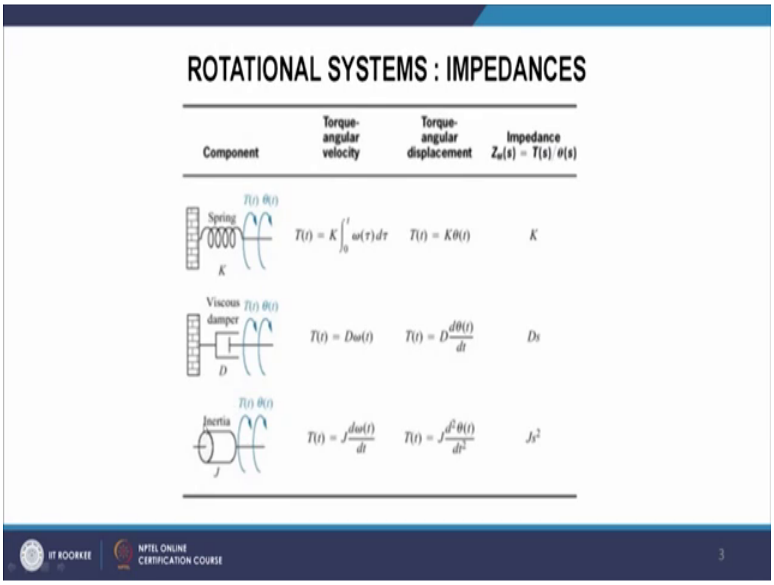

That is subjected to torque and theta. There is the theta degree of freedom. So, we can write torque equal to J into D square theta by D T square and so, T s equal to J into s square. So, here torque equal to J into D square theta by D T square.

So here, T s, we take the Laplace transform. So, equal to J into s square into theta s. So, T s by theta s equal to J s square, this is called Impedance of the inertia. Similarly, for the rotational damper, we have torque equal to D into d theta by d t. So, here T s we take the Laplace so, D into s into theta s.

So, this implies that t s by theta s equal to d s. For rotationally spring, we have torque equal to stiffness of the spring into theta. So, we take Laplace T s equal to K times theta s. So, here T s by theta s equal to K. So, these are the impedance of a mechanical system and defined as that torque upon the displacement or force upon displacement for the translational system in the s domain.

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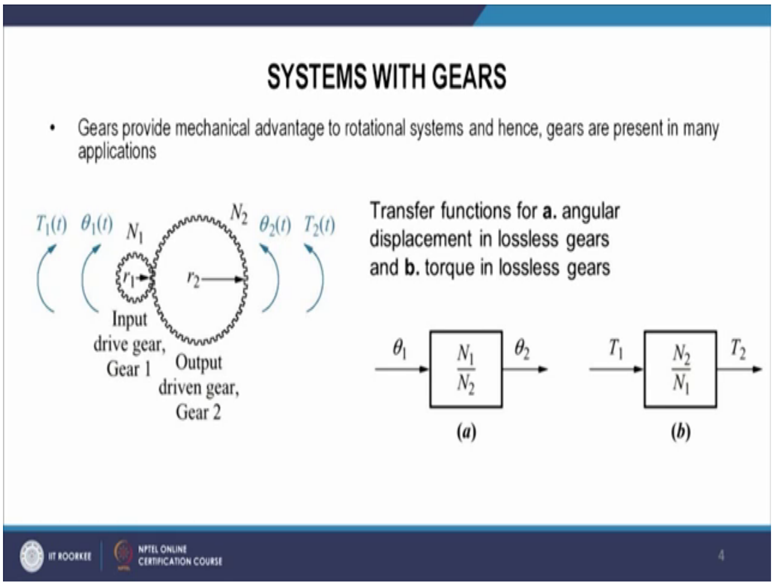

Now, we come to the another mechanical systems. They are more applicable in mechanical components that are the gear trains. So, because gears provide mechanical advantage to rotational systems and so, they find frequent applications in several applications. So, we can understand how we can represent the impedance of gears, how can we find the transfer function that contains gears. So, if we have 2 gears here.

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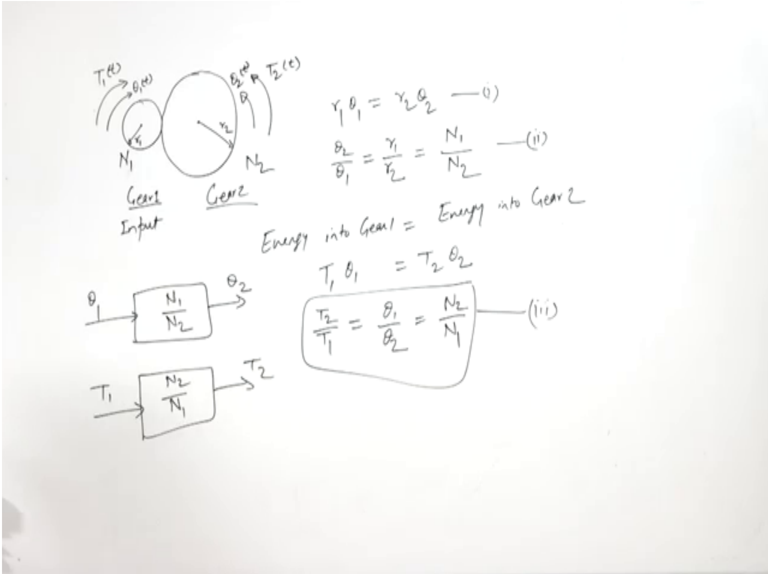

So, these 2 gears, they have radius because these 2 gears are attached on different shaft and they have the different radius. Suppose the shaft applied with the torque T 1 t and there is the angular displacement angular displacement theta 1 t and it has a number of teeth N 1. So, N 1 is the number of teeth on this first gear and. So, this is gear 1. So, here we are giving input.

Now, this is gear 2. So, gear 2 it has number of teeth N 2 and torque on this shaft is coming as T 2 and angular displacement is theta 2. So, as we know that on the circumference, the same displacement will be troubled by these both gears because they are in contact, so, r 1 theta 1 will be equal to r 2 theta 2. And so, here we will get theta 2 on theta 1 equal to r 1 by r 2, here r 1 is proportional to the number of teeth and r 2 similarly proportional to the number of teeth.

So, r 1 by r 2 equal to N 1 by N 2, so, this is expression second expression. Now, if we assume that these gears, there is no any energy dissipation, so, this is an assumption. So, the energy coming to the input will be same as we are getting to the output. So, we assume loss no loss in the gear. So, energy into gear 1 equal to energy into gear 2 and energy can be represented as torque into the angular displacement theta.

So, T 1 theta 1 equal to T 2 theta 2, so, we will find T 2 by T 1 equal to theta 1 by theta 2. And theta 1 by theta 2 equal to N 2 by N 1, so, this is equal to N 2 by N 1. So, we can represent it like this. So, input torque T 1 and output torque T 2 and this is the transfer function that is N 2 by N 1. Here is theta 2 and theta 1.

So, output this angular displacement theta 2 and this is theta 1. So, c theta 2 by theta 1 is N 1 by N 2. Now, we take another example here. So, let us take one example, we can see here the example.

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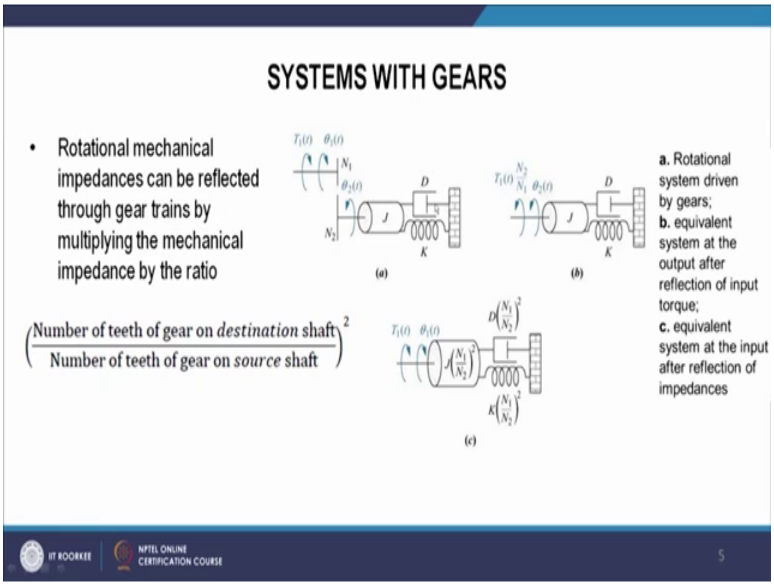

So, we have one damper which is spring and inertia. And here is input shaft, there is a gear that contain N 1 number of teeth. And input torque T 1 is applied here and there is the input angular displacement theta 1. Then here is gear 2 that is on the second shaft, there is Inertia, Damping and Stiffness.

So, here we can express. So, we can express this system. So, because we know that T 2 T 2 by T 1 equal to N 2 by N 1, so, here we know that T 2 by T 1 equal to N 2 by N 1.

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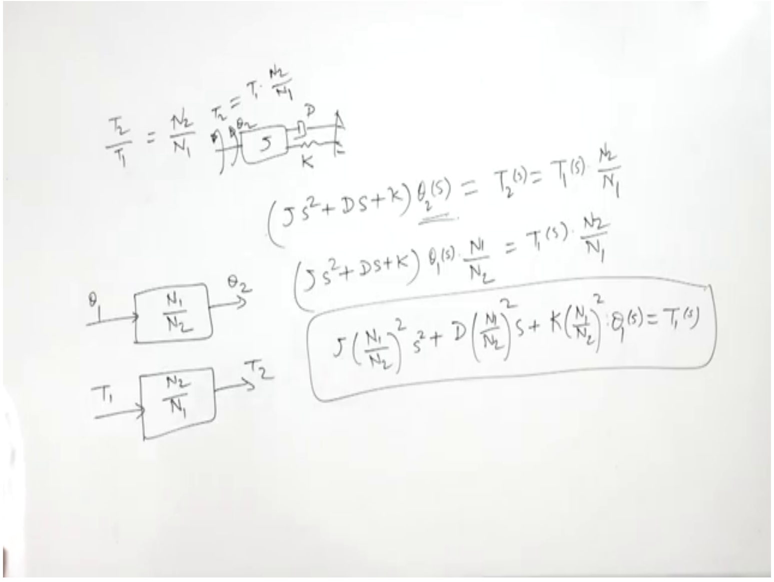

So, the torque, on torque can be represented here on this shaft, this T 2 and theta 2. So, T 2 equal to T 1 into N 2 by N 1 and this is J, this is D, this is K .

So, we can write the equation like J s square plus D s plus K theta 2 s equal to T 2 s and that is equal to T 1 s into N 2 by N 1, this theta 2 s we can write here theta 2 s equal to theta 1 into N 1 by N 2. So, J s square plus D s plus K theta 1 s into N 1 by N 2 and that is equal to T 1 s into N 2 by N 1.

So, here we can take this here. So, this will be N 1 by N 2 square, so, J N 1 by N 2 square s square plus plus D N 1 by N 2 square s plus K N 1 by N 2 square equal to theta 1 s equal to T 1 s. So, here the theta 2 by T 1 s, we can find, N 2 by N 1 upon this term J s square by plus D s plus K. We can also find here this transfer function theta 1 s by T 1 s and that will be this term.

So, we see that N 1 by N 2 square is multiplied with these impedances J s square D s and K. So, we can shift the impedance of gear train to find the transfer function, whether we are interested to find the transfer function theta 2 s by T 1 s or theta 1 s by T 1 s. And we can represent here, we can see that we can represent this system as we transferred the impedance of the gear train as well as this system on the input shaft because here this input shaft subjected to torque T 1 and the angular displacement theta 1 t and the impedance of these components on the second shaft as well as this gear train, all these impedances are transferred to the input shaft and this is the system we can represent like this.

So, here we this input shaft is your destination shaft. And here we see the rotational mechanical impedances can be reflected through gear trains by multiplying the mechanical impedance by the ratio number of teeth of gear on destination shaft upon number of teeth of gear on source shaft square. So, here we are representing to the input shaft and. So, the input shaft is the destination shaft therefore, we are doing N 1 by N 2 square here, we have multiplied and we have shifted this input impedance to the input shaft. So, that is how we can handle the rotational systems and gears and we can write the equations and we can write the transfer function for these systems. So, I thank you for attending this lecture and see you in the next lecture.

Thanks.