Automatic Control

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Lecture – 24
Sketching of Root Locus – III

So, welcome to the lecture on root locus technique; we were discussing about sketching of root locus. So, we discussed several points and in continuation in that point we will discuss today one more point that is angles of departure and arrival. So, when we refining the sketch of root locus we found the real axis breakaway and break in points; then we found the j omega axis crossings. The third point that we are going to discuss is angles of departure and arrival.

So, what is angles of departure and arrival? So, we know that the root locus starts at some finite or infinite poles of G S; H S the open loop transfer function and the root locus ends to some 0s of finite and infinite of G S H S. So, when the root locus is starting from some complex poles; then what will be the angle through which it will depart from that pole that is the angle of departure. And when it is arriving to some complex 0s then at what angle, it will arrive to that complex 0s this is angle of arrival.

So, here we will discuss how to find that angle of departure and arrival for a root locus because this will help us to further refine the root locus and its path. So, here we can take one example.

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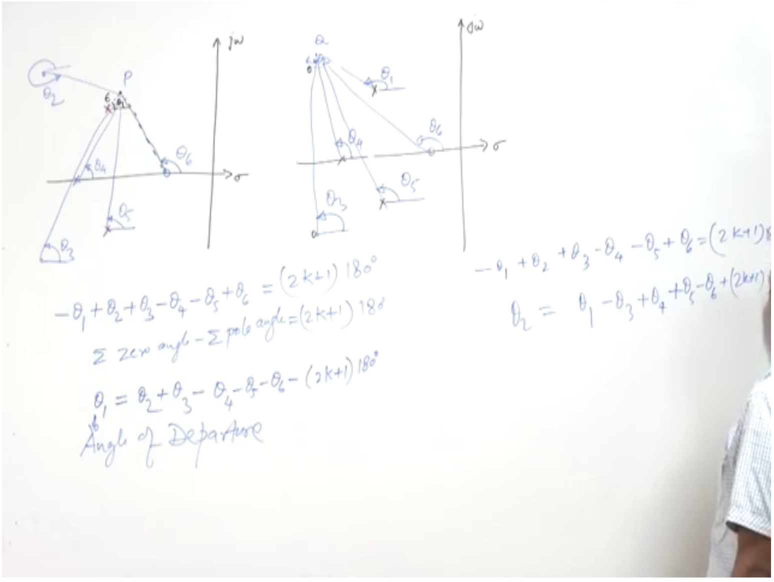

So, we have this S plane and we have here let us say 1 0 and then here is pole and then there are a pair of complex poles here and there are a complex pair of complex 0s here.

So, this is the complex pole and we have to find that the angle when. So, let us we take a point here epsilon distance away from this pole and we assume that this is the angle that let us call theta 1. So, we always take angle with respect to the real axis positive real axis and so, here this is a point where there is the root locus. So, let us say there are the different angles that is being made by this point on the root locus from this poles and 0s.

So, this angle which we it is making with theta 1 then this angle is theta 2, this is theta 3 then this is theta 4, theta 5 and this is theta 6. So, these are the different angles because we assume that we assume that there is a point here let us say P this point close to the complex pole where we want to see the angle of departure.

So, from this point we want to measure the angles for different poles and 0s including this pole. And we know that if these point P is on the root locus then it should satisfy this angle conditions that is angle of 0s minus angle of poles equal to odd multiple of 180 degree. And then we tend epsilon equal to 0 and we will find that what will be the theta 1 means what will be the angle of departure.

So, here this angle theta 1 because this is a pole; so, this is minus theta 1, this is the angle of 0. So, it is plus theta 2 then this is 0; so, plus theta 3 and this is pole. So, it is minus theta 4 this is pole minus theta 5 and this is a 0. So, it is plus theta 6. So, it is nothing, but summation of 0 angle minus summation of pole angles.

So, we are that is equal to 2 K plus 1 180 degree. So, this is 2 K plus 1 180 degree. So, here 0s, theta 2, theta 3 and theta 6 they are 0s and minus poles theta 1 theta 4 and theta 5. So, here we can write theta 1 equal to because we want to calculate theta 1, theta 2 plus theta 3 minus theta 4 minus theta 5 minus theta 6 and minus 2 K plus 1 180 degree.

So, this is the theta 1 is the angle of departure. So, this theta 1 is angle of departure ; so, this will we will get the angle through which that root locus will depart from. Now we want angles of arrival; so, angle of arrival let us we want in the same situation we want angle of arrival for this 0 that root locus is arriving at this point. So, here sigma j omega this is a 0 here the poles then there is a pole and there is a there are complex 0s and we want here that at what angle this will arrive.

So, again we will take some point let us say at epsilon. So, here let us say a point Q at epsilon distance away and we measure this angle as theta 2 and the other angle like this is theta 1, this theta 4 then this theta 6 and this theta 5 and then this theta 3.

So, here again we can write the minus theta 1 plus theta 2 plus theta 3 minus theta 4 minus theta 5 plus theta 6 equal to 2 K plus 1 180 degree and so, here we have to find theta 2. So, theta 2 equal to theta 1 minus theta 3 plus theta 4 plus theta 5 minus theta 6 plus 2 K plus 1 180 degree.

So, from here we can get because this theta 2 will get angle of arrival. So, this is how we will compute the angle of departure and angle of arrival for the system given system. So, now, we relate let us take one example and we calculate this angle of arrival and departure.

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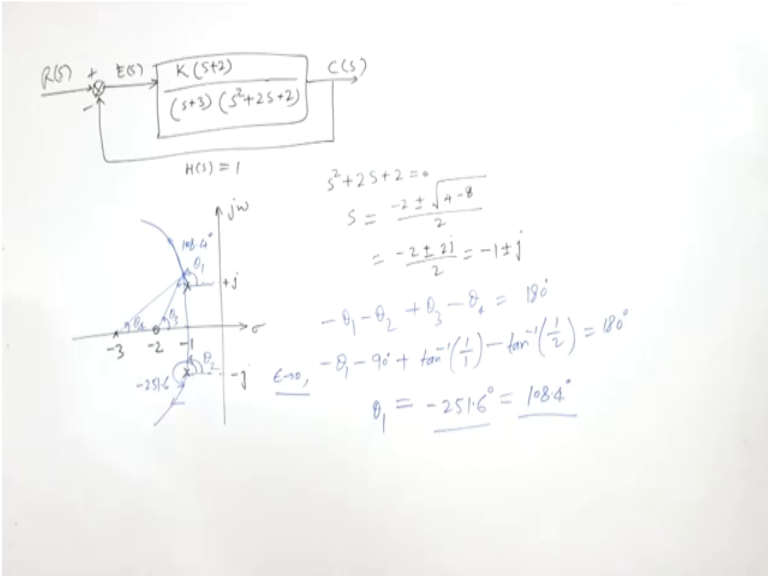

So, this is the system here H S equal to 1 and so, if we want to. So, this is our sigma and j omega axis here one pole is minus 3 0 is minus 2.

Now, we have to find these poles. So, we have to put S square plus 2 S plus 2 equal to 0 and we can get S equal to. So, we are getting minus 1 plus minus j; so, here is this pole. So, this is minus j plus j and this is minus j and this point is minus 1 now we have to obtain the because the these poles are in the complex a polar poles are complex. So, we have to get the angle of departure.

So, let us assume that here this angle of departure is this is theta 1. So, we take some point here at epsilon and we take from here we take epsilon theta 1 this point. So, we get theta 2 this point and these points. So, theta 3 and theta 4 ; so, now, here we can write minus theta 1. So, here we can write minus theta 1 minus theta 2 plus theta 3 minus theta 4 equal to let us say 180 degree.

So, K equal to 1; so, we can write minus theta 1 when epsilon stands to 0 now let us say epsilon tends to 0. So, minus theta 1 minus theta 2; so, theta 2 is when epsilon will tend to 0 this point will come to this point and the angle will be here theta to 90 degree. So, this is 90 degree plus theta 3. So, theta 3 is equal to tan inverse 1 by 1 and therefore,. So, tan inverse 1 by 1 here and minus theta 4 theta 4 will be tan inverse 1 by 2 and that is equal to 180 degree.

So, from here we can get theta 1 and theta 1 equal to minus 251.6 degree or if we add 360 we will get 108.4. So, we can see that the one angle we will get for this pole and other one will be get because they are the complex pair. So, this minus 251; so, this is 108.4. So, it is going like this and this is going like this ; so, this angle is minus 251.6 and this angle is 108.4 degree.

So, that is how we have obtained the we have obtained the angle of departure from a complex root locus. Now let us say we have almost discussed almost all the rules that can help us to sketch a root locus. Now we take one numerical example to apply all these rules and plot the root locus.

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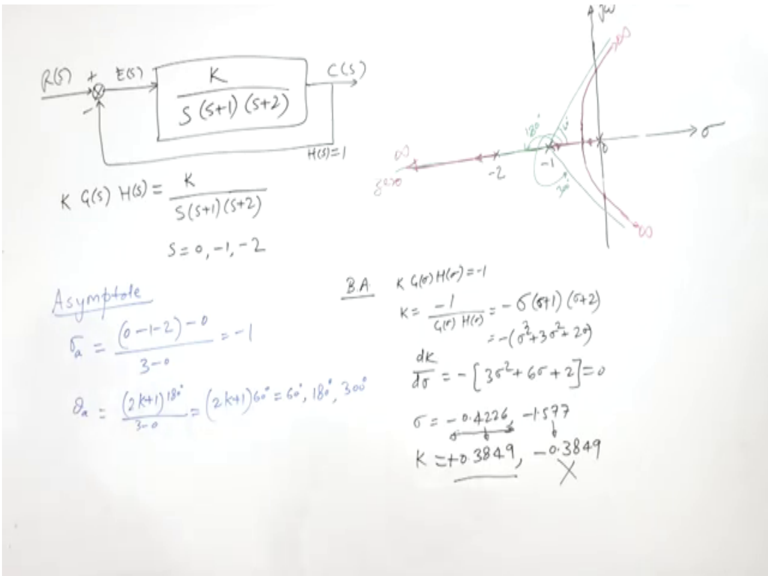

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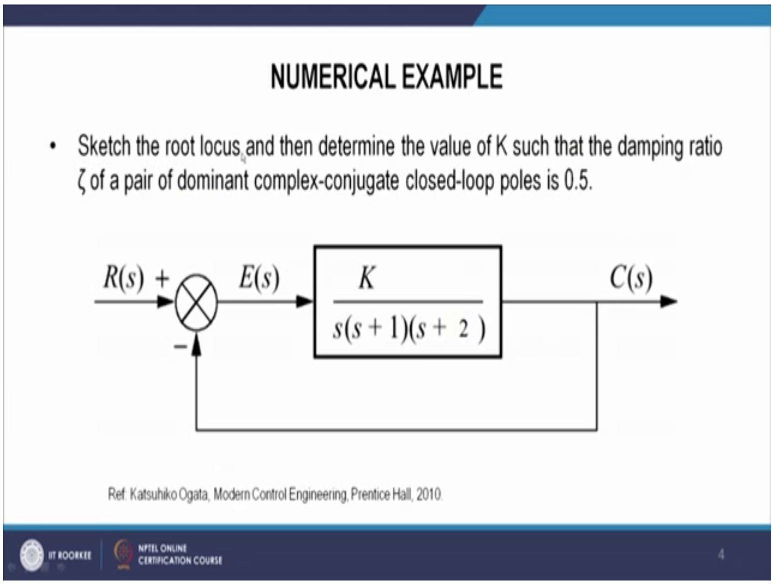

So, here this is the numerical example we can see a sketch the root locus and then determine the value of K; such that the damping ratio of a pair of dominant complex conjugate closed loop poles is 0.5.

So, here we can see this is a system and this example is taken from the reference book K Ogata Modern Control Engineering. So, we can see here we have to sketch the root locus first and we have to find that value of K when the damping; the dominant complex conjugate poles are on the line of damping equal to 0.5. So, here let us first plot the root locus; so, when we have to plot this root locus first of all we have to make this complex plane and we have to find G K G S; H S. So, we have K S S plus 1 S plus 2 because H S is 1 here H S is 1; so, K G S; H S H K G S.

Now, we absorb this; we locate this is called open this G S; H S is called the open loop transfer function. So, open loop transfer function has 3 poles that is S equal to 0 minus 1 and minus 2. So, we plot this here the pole here is 0 then minus 1 and then minus 2. Now we see that here we do not have any 0s and we know that the number of finite plus in finite poles equal to number of finite plus infinite 0s.

So, here if we have 3 finite poles; I know any finite 0 it means there are 3 infinite 0s. So, the 0s are at infinite they are 3 0s. So, we know that root locus will start from finite and infinite poles of G S H S. So, root locus will start at these 3 points and it will end at finite and infinite 0s of G S H S. And we have no any finite 0s, but we have 3 infinite 0s; so, the root locus will start at these points and they will end at in finite.

Now, so, rule number one will say that root locus will start. So, they will start from here now whether at this point the root locus will start and go to what side. So, we know that root locus on the real axis segment; they lie on the left side left of the odd number of poles or 0s. So, here this is one pole; so, it will lie to the left of this one. So, this is the region where root locus will lie then here it will not lie because there are two left of these two poles. So, here it will not lie, but it will lie to this region.

So, it means that the root locus is starting here starting here and they are starting and to this side. So, because root locus are going to start at these 3 points; so, starting it cannot go this side. So, it will come to this side it cannot go this side; it will come this side and we know that there will be 3 branches of the root locus. Because one branch for there are here 3 we will have 3 poles of this closed loop transfer function. So, we will have 3 branches; one branch for each transfer for each closed loop pole.

So, we know that this root locus are starting and going to infinite. So, rule number 5 we have to apply and we find that if they are going to infinite; what will be the asymptotes that they will follow as a straight line and that line will cut on the real axis and axis at what point? So, we know that to find the asymptotes we have sigma a and sigma a we can find. So, some of the poles that is 0 minus 1, minus 2 finite poles minus finite 0s; so, there is no 0 upon number of finite pole that is 3 and minus number of finite 0 that is 0.

So, we are going to get it as minus 1 and then theta a equal to 2 K plus 1 180 degree by number of finite pole minus number of finite 0; so, 3 minus 0. So, we are going to get to K plus 1 into 60 degree because 3 we divide 180 by 3. So, when K equal to 1; we will have 60 degree, K equal to 0 we have 60 degree K equal to 1 we will get 180 degree and when K equal to 2; we will get 300 degree.

So, because we need 3 asymptotes because 3 branches and 3 poles will move to 3 different in different asymptotes; so, here we see that we the asymptotes line will cut at minus 1. So, the asymptotes line is going to cut here and it is going 60 degree. So, here we can plot 60 degree this is the asymptotes at 60 degree and then 180 degree. So, this is 180 degree and other is 300 degree.

So, R minus 60 degree; so, this is. So, this is 60 degree and this is 300 degree and this is 180 degree. So, we have; so, from here we can understand that. So, this is going to infinite, this is going to infinite and this is going to infinite. So, one thing is clear that this root locus branches started at this pole and going towards a 0 at this infinite. So, here is a 0 at infinite and it is moving towards that 0 here the other poles two poles; they started here it started here and they have to move towards these asymptotes.

So, here they have to go in the complex plane. So, they have to break away and so, we have to find the breakaway point that at what point they will break away. So, that breakaway point we can find break away point. So, we know that from here K G sigma H sigma equal to we can write minus 1. So, K equal to minus 1 by G sigma H sigma and so, minus 1 upon. Here so, we will get minus S S plus; so, sigma plus 1 and sigma plus 2.

So, we have sigma Q plus 3 sigma square plus 2 sigma. So, d K by d sigma we can find; so, that is minus 3 sigma square plus 6 sigma plus 2 and this is equal to 0. And from here we can get sigma equal to minus 0.4226 and minus 1.577. So, we know that here we should consider this point between 0 and minus 1.

So, this is the point that is likely to be because this point is going in this range and root locus branch that is moving this side. So, this is a likely point; however, we will take calculate the gain where it will occur. So, we can find when we put this value in this to get here to get in. So, K equal to we will get 0.3849 and here we will get minus 0.3849. So, for this value we are going getting a positive gain and for this value we are getting a negative gain.

So, we discard this value because we are taking positive gains. So, this is the value of gain for which and at 0.423 or 0.4226 here this root locus will break away. So, this will at this point; they will break away and they are going to follow this asymptote. So, this is the root locus and this and then this now when it is following this asymptote it is going to cut this point axis j omega. So, we have to find the j omega axis crossing and to get the j omega axis crossing.

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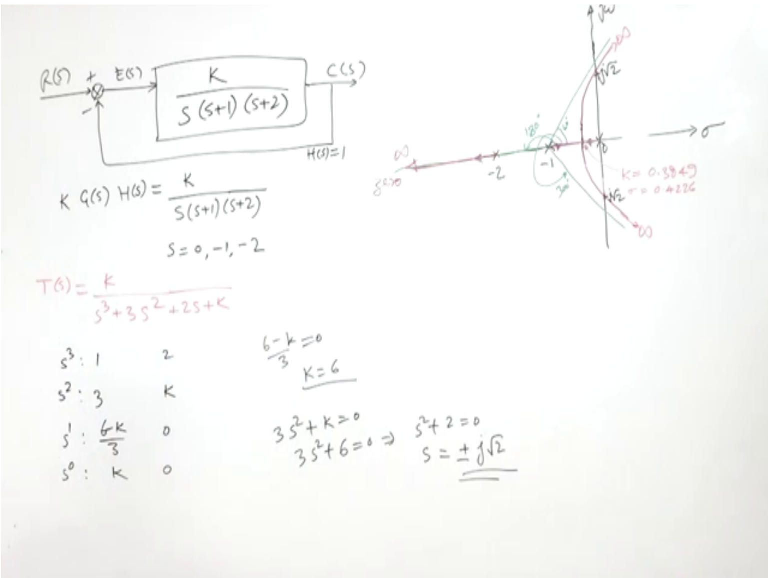

So, this is going to occur at K equal to 0.3849 and at sigma equal to 0.4226.

So, we have to find j omega axis crossing. So, for this we need to know T S the equivalent closed loop transfer function and if we compute we will get K upon S cube plus 3 sigma square or S square plus 2 S plus K. So, we can make the root table; so, root table we make S cube S square 1 2 0 and then 3 and K. So, here S 1 we will get 6 minus K by 3 and 0 and S 0 we will get as K and 0.

So, we can see that if 6 minus K by 3 equal to 0 this means K equal to 6 this row will be 0 and so, we have this polynomial 3 S square plus K equal to 0 and so, 3 S square plus 6 equal to 0; this implies that S square plus 2 equal to 0. So, S equal to minus; so, plus minus j root 2; so, we are seeing that or plus minus j 1.41.

So, let us see here j root 2 and this is j root 2 minus. So, it is at 1.41 root 2; it is cutting here this j omega axis. So, we have plotted this root locus by using the rules that we discussed. Now the question is here that we have to find the value of K such that the damping ratio of a pair of dominant complex conjugate closed loop poles is 0.5.

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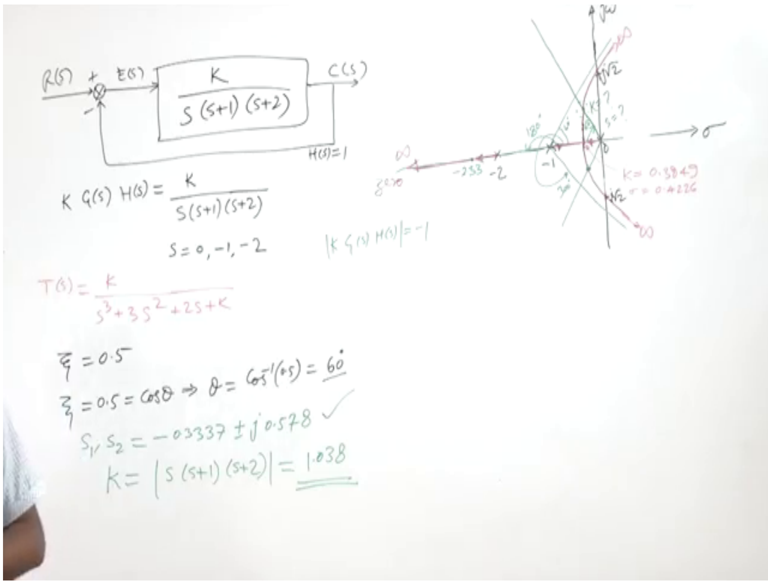

So, zeta equal to 0.5; damping is here; so, now, we have damping that is equal to 0.5.

So, it means damping equal to 0.5 equal to cos theta where theta is the line. So, we can get cos is inverse 0.5 and we get 60 degree that is equal to plus minus 60 degree. So, we can get that theta equal to 60 degree is this line ; this is 60 degree. Because we remember that we when we discussed this pole plot we discussed that this is the cos theta and zeta damping equal to cos theta. So, this is 60 and we have to find this point and the gain for which and the point that is S for which this line is cutting this root locus.

So, along this line we can take some points in this region, we can check and we can satisfy the condition that when we take a point here and that should satisfy the condition of odd multiple of 180 degree. And the we can take to 3 test point on this line nearby in this area. So, we assume some point on this line and we check the angle is odd multiple of 180 degree and the point that satisfied this criteria will be the point that cuts the root locus by this line. And when we do this we will get that point S 1 equal to S 1 S 2 because we will get a complex point pair. So, that is minus 0.337 plus minus j 0.578 and we can find gain for this.

So, gain we can find S S plus 1 S plus 2. So, we put one point here and we can find the gain for any one point maybe this plus 0.578 and we get 1.0383. So, 1.038 ; so, we get this gain and when we put this value of gain here , this equal to K G S H S equal to minus 1 ; we put this value of gain we will get all these in the third pole also and that pole will be on the real axis here and that is at minus 2.33.

So, it is here that point corresponding minus 2.33 for the same gain because at this gain all these 3 poles we should get. So, two poles here these are two poles and the third pole is this minus 2.33. So, that is how we obtain the poles that are cutting this damping line. So, I thank you for attending this lecture and let us see in the next lecture.

Thank you.