Automatic Control
Dr. Anil Kumar
Department of Mechanical & Industrial Engineering
Indian Institute of Technology, Roorkee

Lecture – 15
Underdamped Second Order System – 1I

So, welcome to the lecture on transient response analysis, in this lecture we will continue our discussion on under damped second order system. So, in previous lecture we saw, that an under damped second order system has certain response characteristics and it is response overshoots the final value. So, we can see that, the response of an under damped system is. So, this is t this is Ct and if this is the C final.

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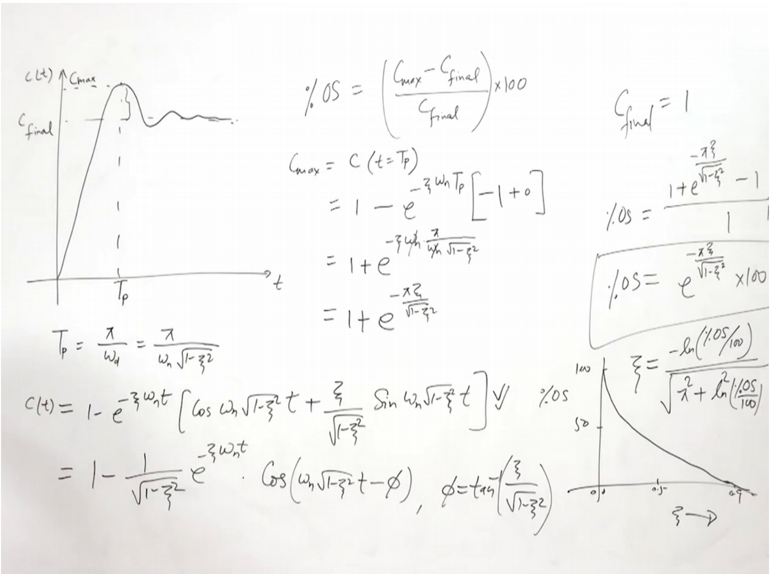

So, the final value robot shoots then, it comes like this. So, this is this time Tp is the peak time this is called peak time and this is C max that is C max maximum value of at this of this response at this peak time.

So, we see that the peak time we find equal to pi upon omega d or pi upon omega n root 1 minus zeta square where, zeta is damping ratio and omega n is natural frequency. So, the here response Ct was for this system we derived was 1 minus e power minus zeta omega n t into cos omega n root 1 minus zeta square t plus zeta upon 1 minus zeta square sin omega n root 1 minus zeta square t or this also we could write 1 minus 1 upon root 1 minus zeta square into e power minus zeta omega nt into cos omega n root 1 minus zeta square t minus phi.

Where, phi is tan inverse zeta upon root 1 minus zeta square. So, this was the response for this under damped second order system and we find peak peak time by taking the slope of this curve equal to 0, that is dct by dt equal to 0. Now, we have to find the percent overshoot of this system. So, percent overshoot is the value how much this maximum value go beyond this final value? That is how much it overshoots. So, percent overshoot defined as C max minus C final upon C final into 100.

So, maximum value minus this value; this quantity by dividing the final value, first we have to find this maximum value, and that C max will be equal to C t when t equal to Tp because, this C t equal to Tp because, at t equal to Tp the value we will get is the maximum value and so, if we use this formula this one this expression. So, we will get and Tp equal to pi upon omega n root 1 minus zeta square. So, we will get 1 minus. So, e power minus zeta omega n Tp and here, t equal to pi upon omega n root 1 minus zeta square, that will cancel out and so, we will get cos pi and cos pi is minus 1 here, plus when here it is 0 because, here sin pi is 0.

So, we will get here equal to 1 plus because, minus plus exponential minus zeta omega n and Tp, Tp is pi upon omega n root 1 minus zeta square. So, this equal to 1 minus e power minus zeta omega n into T p is pi upon omega n root 1 minus zeta square. So, here omega n will cancel out and we will get 1 minus sorry, this is plus 1 plus exponential minus pi zeta upon root 1 minus zeta square. Now, this final value is 1 because, we are considering the, this response for unit unit feedback unit input function a step function. So, here see final equal to 1. So, we will get percent overshoot equal to C max that is 1 plus e minus pi zeta by root 1 minus zeta square and minus 1 by 1 into 100, a in to 100.

So, that is equal to exponential minus pi zeta by root 1 minus zeta square into 100. So, this is the percent overshoot. Now here, we can also get from here we see that, percent overshoot is only the function of damping. And so, if we want to see the and when damping is more this quantity is less. So, percent overshoot will decrease when the damping will increase and that is likely because, the role of damping should be to decrease the response amplitude of the system. So, therefore, more damping less overshoot, less a maximum amplitude of the system. So, here we will get something here let us say this is 100, this is 50, this is 0 and this is damping this is damping and this is percent overshoot and damping is varying from 0.

Let's say 0.5, 0.9 something. So, here percent overshoot at when damping is 0 this is 100 percent and then, it will start decreasing exponentially negative exponents and it will meet somewhere here. So, we can see the exponential decaying this quantity. Now here, we can also write the from we can take log and we can write damping here, and that is minus ln percent overshoot by 100 upon pi square plus ln square ln percent over overshoot by 100.

So, we can also express damping in terms of person to about shoot. So, we take the natural log and we solve this to find the damping and we will get this value. So, we have discussed this percent overshoot now, the next parameter. So, here we have discussed the percent overshoot the amount that the waveform overshoots a steady state or final value at the peak time expressed as a percentage of the steady state value. So, this is percentage overshoot and this is the damping in terms of percentage overshoot.

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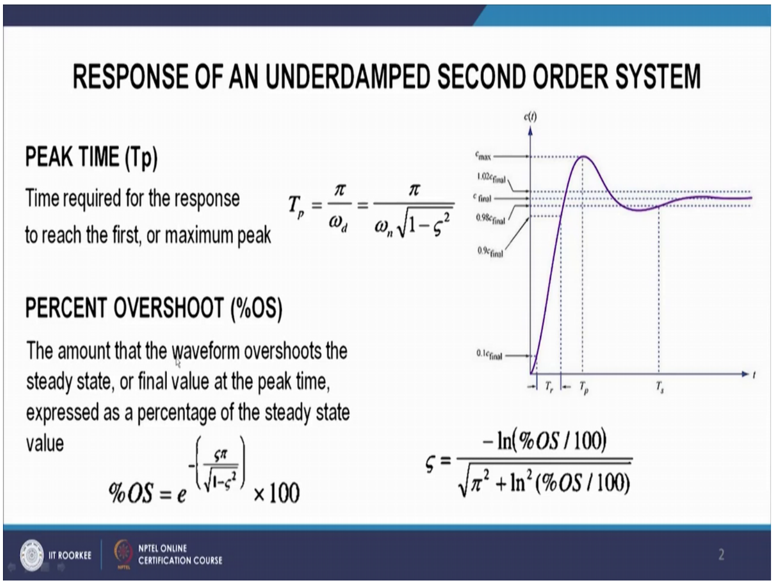

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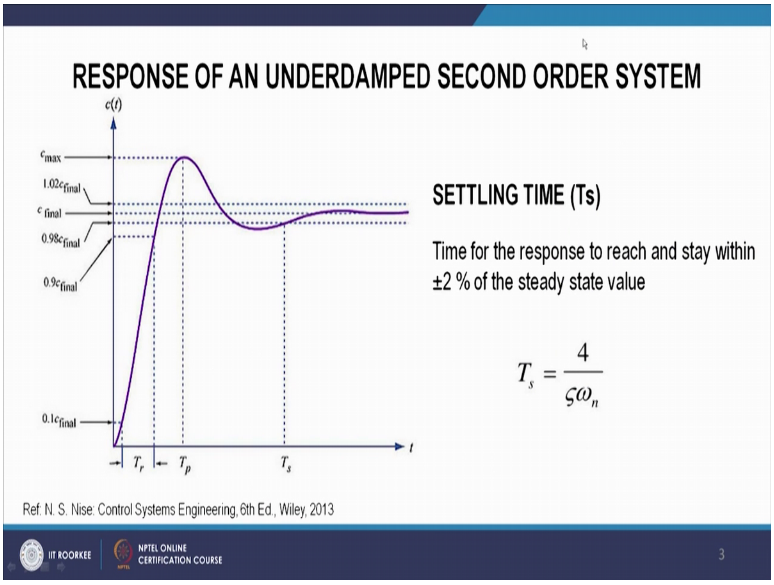

Now, we come to other parameter like settling time. So, I define the settling time is the time for the response to reach and stay within plus minus 2 percent of the steady state value. So, here we can see the T s here. So, here it is reaching at value 0.98 of the final value and it stays between 0.98 to 1.02 of the final value that is 1; here 0.98 to 1.02. So, here response so, this the time when it reaches 2.98 of the final value is we call it settling time and the settling time we can find. So, we can find settling time C Ts 0.98. So, it is 0.98 if our and then we can write here, let us use this formula.

So, 1 minus 1 by root 1 minus zeta square exponential minus zeta omega n Ts cos omega n root 1 minus zeta square Ts minus 5 and from here we will get that, 0.02 equal to 1 by root 1 minus zeta square exponential Ts cos omega n root 1 minus zeta square Ts minus phi see this the value of this term is maximum value is 1. So, we can have e power minus zeta omega n T s by root 1 minus zeta square equal to 0.02 and from here we can find Ts, that is equal to minus ln 0.02 into root 1 minus zeta square upon minus zeta omega n.

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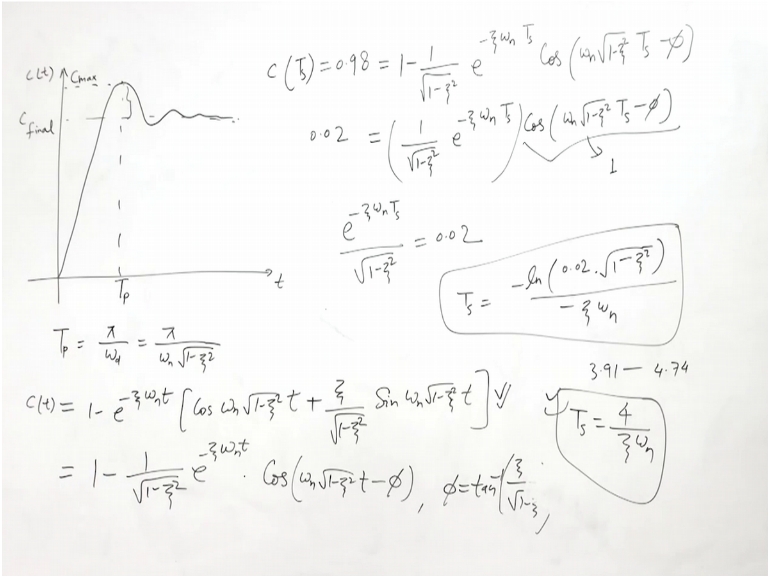

So, this is the settling time and so, this value for different values of damping zeta when we vary zeta from 0 to 0.9 this value this term the numerator term varies from 3.91 to 4.74 and therefore, some approximate value we can take as Ts equal to. So, this value is equal to 4. So, 4 upon zeta omega n, we take approximate value 4 between these values when because, here the this term is very small. So, it is effect is less and so, we this term we approximate as 4 and this term is zeta omega n. So, settling time is 4 upon zeta omega n. So, we will have. So, settling time is we can define and we find at Ts equal to 4 upon zeta omega n.

Now, we come to the rise time and rise time is defined as time for the response to go from 10 percent to 90 percent of it is final value. So, here we can see the response here is this 10 percent here. So, 0.1 of C final and it reaches here, 0.9 that is 90 percent of C final. So, this time Tr is called the rise time.

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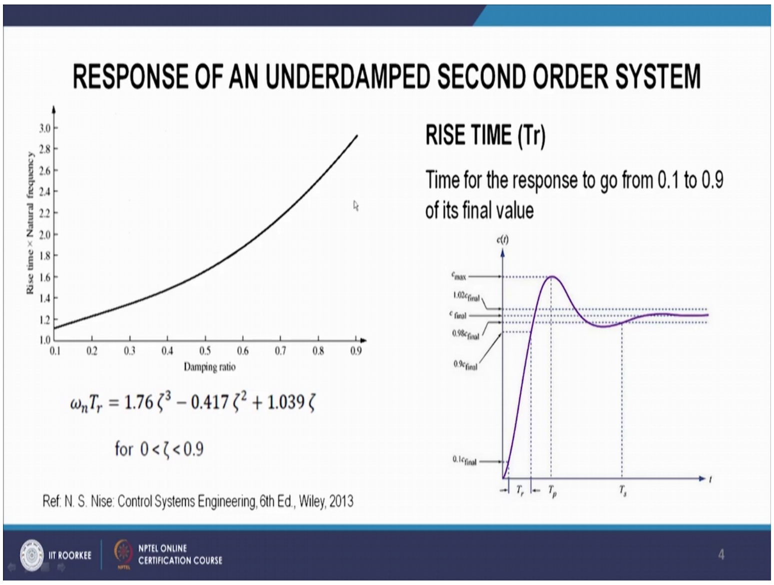

So, rise time here is plotted rise time into the natural frequency for certain data refer to the book of nise N S control systems engineering. So, here we can we find this relationship further rise time omega n Tr equal to 1.76 zeta cube minus 0.417 zeta square plus 1.039 zeta and this is fine found for damping between point and between 0.1 to 0.9.

So, here we have this relationship. So, if we know the damping and frequency we can calculate the rise time for the system. So, we understand that, for a under damped system and under damped second order system we have 4 characteristics that, characterize the response of the system. So, there is peak time percent overshoot settling time and rise time.

So, these are the a specifications that, we can demand when we design the this second order system that, how what value we want for these specifications? And we can design our system. Now, let us show the on the S plane this under damped system the poles and we see how what will affect of the movement of the poles on of the second order system?

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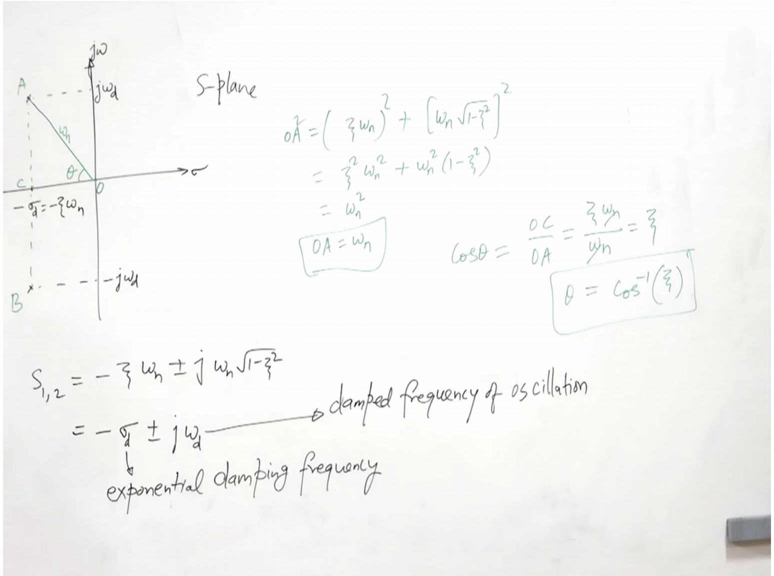

So, we can represent on the pole this S plane these poles. So, it is sigma and j omega this is S plane and we have under damped system. So, we get the poles. So, S1, 2 equal to minus zeta omega n plus minus j omega n root 1 minus zeta square. So, these are the poles of an under damped second order system and we also call them minus let us say sigma d plus minus j omega d. So, here omega d is the imaginary part of this pole and sigma d is the magnitude of the real part of this pole and so, sigma d is called exponential damping frequency, exponential damping frequency because, it decides the exponential decay of the amplitude of this of the response and omega d is called the damped frequency of oscillation damped frequency of oscillation.

So, this is the frequency, with which the damped system oscillates and you can see that this is less than, the natural frequency because, this damping is greater than 0. So, this value will be less than 1 and therefore, the damped frequency is less than, the natural frequency of the system. Now, if you want to plot these so here, is minus sigma d or minus zeta omega n and we have here. So, these are the 2 poles and this is j omega d and here is minus j omega d. So, we can see that, this is a origin and so, if we plot this line what will be the this radius or the distance from origin to this pole? So, let us say this is O, this is A, this is B, this is C.

So, OA OA equal to this is square plus this is square. So, zeta omega n square plus omega d square so, that is omega n root 1 minus zeta square whole square. So, here we get zeta square omega n square plus omega n square into 1 minus zeta square and so, we will get here omega n square minus omega n square zeta square will cancel out. So, we will get omega n square here OA square. So, OA equal to omega n. So, this point this is omega n now, come to this theta. So, if we calculate cos theta. So, cos theta equal to this quantity zeta omega n upon OA, OC upon OA, this is OC upon OA and OC is zeta omega n and OA is omega n.

So, this cancel out that is equal to damping ratio. So, here theta equal to cos inverse damping ratio. So, it means this theta shows the damping of the system. So, here we can see now, these things are clear. So, theta equal to cos universe damping or damping equal to cos theta now, we know that Tp the peak time equal to pi upon omega d and settling time is equal to 4 upon zeta omega n. So, that is equal to 4 upon sigma d because, zeta omega n is sigma d. So, we can see that, settling time each function of the real part of the pole and peak time is function of the imaginary part of the pole.

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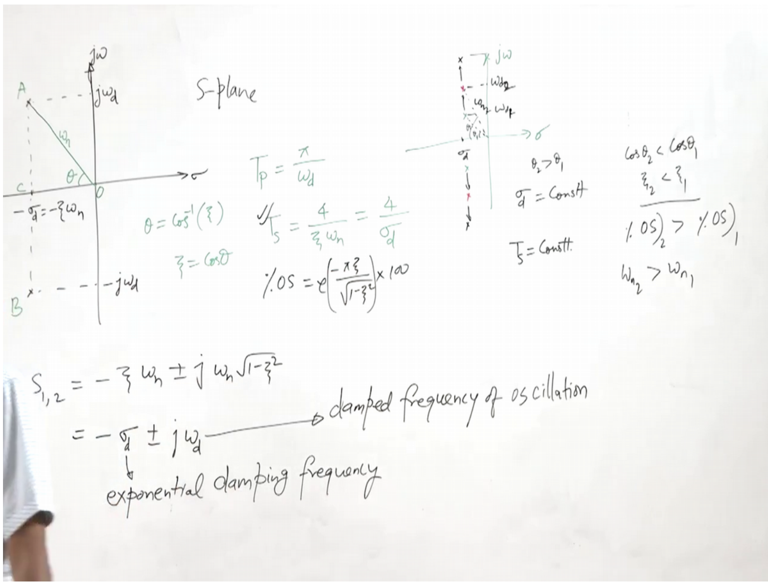

So now, we let us see what happens when thus these poles there are some poles this pole sigma j omega and we have these complex poles and these poles start moving in this direction. So,. So, if these poles start moving in vertical direction or parallel to the imaginary axis what will be the response of the system? So, when these poles are moving we can see that. So, this is theta 1, this is theta 2. So, theta 2 is greater than theta 1 and here one more quantity we can write percentage over soot, that is. So, exponential minus pi zeta by root 1. So, we can see that, theta 2 is greater than theta 1 here, and sigma d is constant because, this real part that is sigma d is constant it is not changing.

So, sigma d is constant means settling time is constant and theta 2 is greater than theta 1 means, we have an cos theta 2 is less than cos theta 1. And so, damping 2 is less than damping 1 because here, damping is equal to cos theta. So, theta 2 is greater than theta 1. So, cos theta 2 is less than cos theta 1, damping, we have less damping here, when we are moving we have less damping and so, your percentage overshoot is more. So, percentage overshoot for 2 is greater than percentage overshoot of 1. And when we are moving from here to here omega n is increasing. So, omega n 2 omega n 1; omega n 2 is greater than omega n 1. So, we have natural frequency that is increasing. So, settling time is constant Ts is constant same, that is same.

So, here we can see when we are moving in the vertical direction 1, 2, 3 and this is a response of the system here, 1, 2 and 3. So, we see that we have the same envelope because, envelop is decided envelope is decided from the sigma d, that is exponential damping frequency because, it comes as e power minus zeta omega n, that is e power minus sigma d and sigma d is constant therefore, that envelope will be constant. So, here envelope is the same; however, we can see that, the for the peak time of the third pole po position is more than second than first. So, peak time is depending on the omega d and we can see that, omega d is increasing here is omega d 3, omega d 2 and omega d. So, this is 1, 2 here 3 this is 2.

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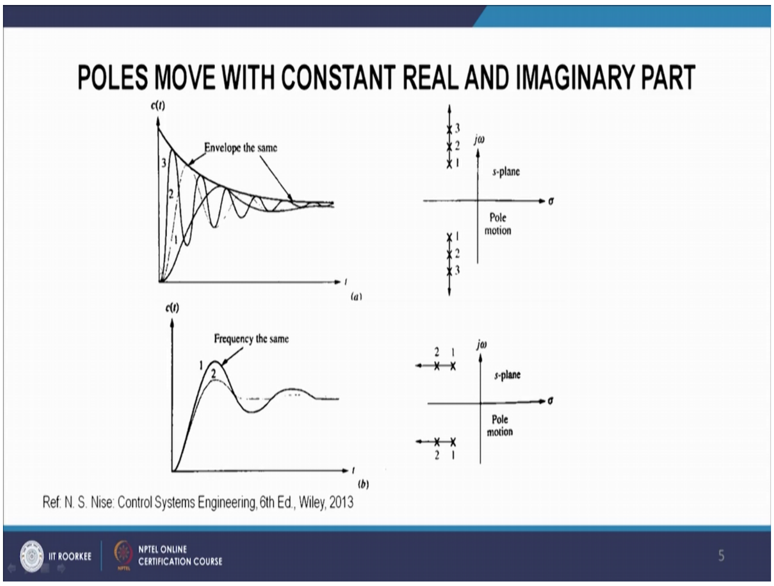

So, omega d 2 is greater than omega d 1. So, peak time will decrease for the last one because, the omega d 3 is even greater than omega d 2. So, peak time is inversely proportional to omega d. So, your peak time is decreasing. Now, come when the poles move parallel to the real axis further from the origin. So, we have omega d constant. So, we will have peak time constants. So, this e we can see here, this peak time is for constant; however, we see that less overshoot because, when we are going further the cos theta the theta is decreasing and so cos theta is increasing therefore, damping is increasing.

So, due to increased damping there will be the less overshoot. So, therefore, overshoot of second one is less than overshoot of first one. Now, we come to the case when poles move in radial direction. So, when poles will move in the radial direction we can see that, the these this is the constant this damping line because theta is constant. So, your damping will be constant and therefore, percent overshoot will be constant.

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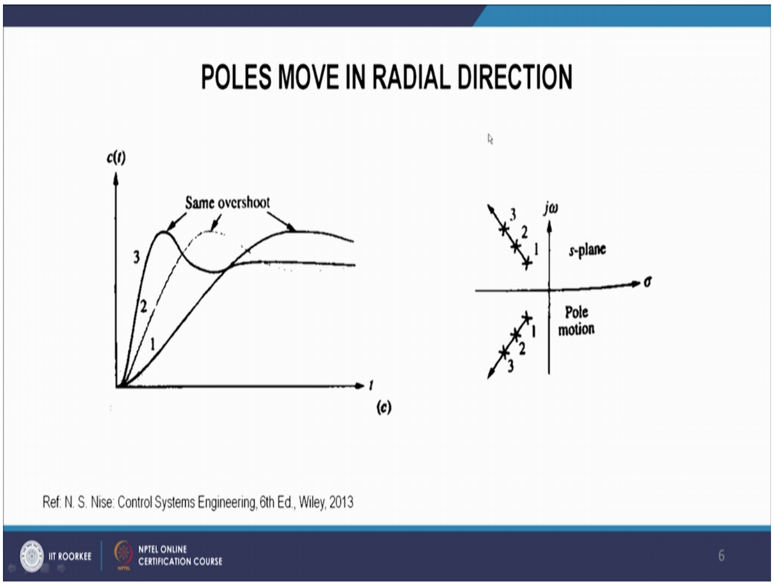

So, we can see from this curve we have the same percentage overshoot; however, the peak time is also decreasing because, we are moving in the radial direction. So, we are also increasing the omega d that is the imaginary component. And so, your peak time is inversely proportional to omega t and therefore, your peak your peak, peak time will also decrease.

So, here we stop our discussion on under damped system. So, we saw that, the under damped system they have response that oscillates, but there is the amplitude is decaying with time as an exponential function and we derived this formula for the peak time settling time overshoot rise time and we saw what will be the effect of the movement of the poles in 3 different directions? And how the response will change? So, this we related the pole position with the response of the under damped system. So, I stop here and see you in the next lecture.

Thank you.