Automatic Control
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Lecture 14
Underdamped Second Order System – 1

So, welcome to the lecture on transient response analysis of second order system. So, in this lecture, we will discuss about the step response of under damped second order system. So, we discuss that a second order system in the transfer function, we saw that the highest power of S is 2 in the denominator of a transfer function and we defined the transfer function of second order system in terms of damping ratio and natural frequency and based on different limits of damping ratio. We divided the second order system in four parts.

That is over damped, under damped undamped and critically damped system. So, here we discussed more about under damped system, because under damped system has damping between 0 and 1 and they are the most useful system and the damping of the systems is found between this ah, range that is 0 to 1. So, this under damped system, we will have this.

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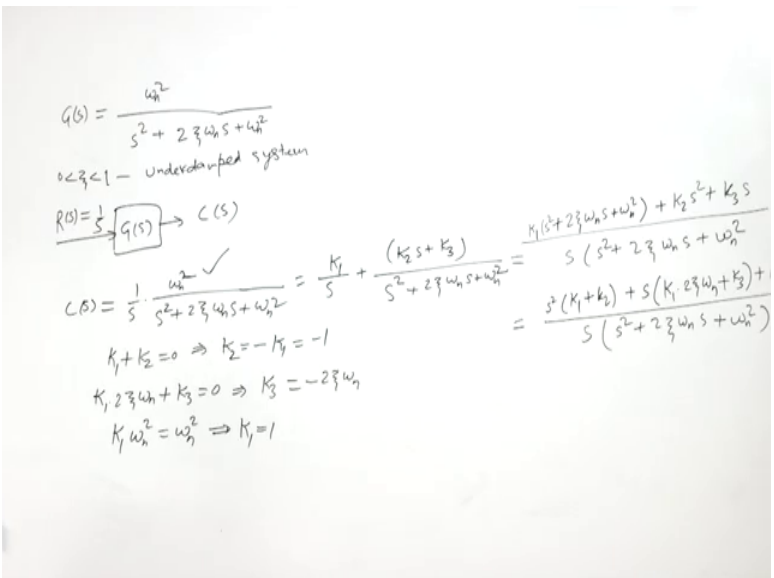

G S equal to omega n square upon S square plus 2 zeta omega nS plus omega n square and here 0 is less than zeta less than 1. So, this is under damped system. So, now, the objective of this lecture is to find the transient response of under damped, second order system in the step input condition and relate this response with the parameters of the system and the location of poles.

So, if the system is subjected to a step input and this is output. So, we have CS equal to 1 by S into GS. GS is omega n square upon S square plus 2 zeta omega nS plus omega n square. Now, we can write this as K 1 upon S plus.

So, here we have second order in this denominator. So, we have to take it K 2 S plus K 3 upon S square plus 2 zeta omega nS plus omega n square. So, now, we can solve it. So, here K 1 S square plus 2 zeta omega nS plus omega n square plus K 2 S square plus K 3 S upon S, S square plus 2 zeta omega nS plus omega n square.

So, we can write here, we collect the terms of coefficient of S square. So, S square we have K 1 plus K 2, then we collect the S coefficient. So, here we have k 1 into 2 zeta omega n plus K 3. Now, we have K 1 into omega n square and denominator is S, S square plus 2 zeta omega nS plus omega n square. Now, we compare this term with this numerator coefficients of S square S and constants.

So, the coefficient of S square is 0 cups, coefficient of S is 0 and there is constant term omega n square. So, here, K 1 plus K 2 equal to 0 and K 1 into 2 zeta omega n plus K 3 equal to 0 and here, K 1 omega n S square equal to omega n square. So, from here, we get that K 1 equal to 1 and from here, we get that K 1 equal to K 2 equal to minus K 1, given that is equal to minus 1.

Now, we put these values K 1 in this equation. So, we get K 3 equal to. So, K 1 equal to 1, we put K 3 equal to minus 2 zeta omega n.

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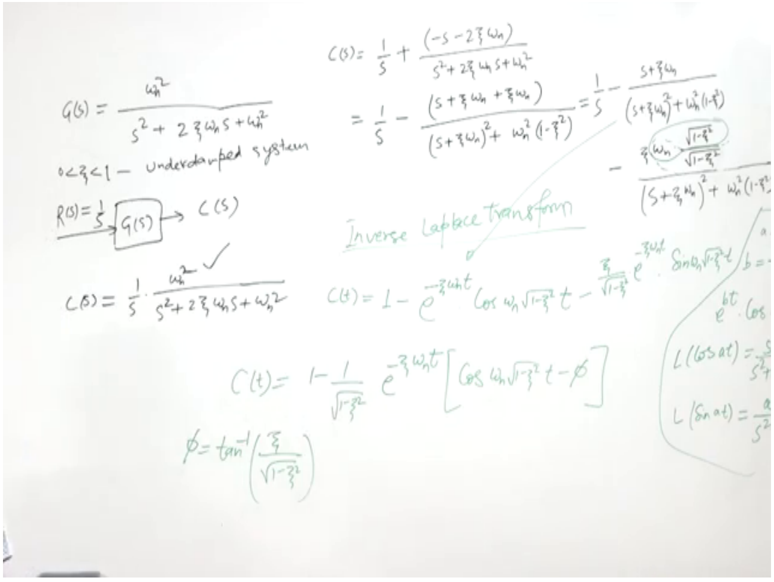

So, we can write CS equal to 1 by S plus. So, K 2 K t is minus 1. So, minus S plus K 3, K 3 is minus 2 zeta omega n upon S square plus 2 zeta omega n S plus omega n square. So, we can write here 1 by S minus here, S plus zeta omega n plus zeta omega n, because here, we take minus out and this is S plus 2 zeta omega and we can write like this. This we can write S plus zeta omega n whole square minus plus omega n square 1 minus zeta square.

Here, S square plus zeta omega n is whole square plus 2 zeta omega nS and this is omega n square. So, minus omega n square zeta square. So, this will be cancel out. Now, we can rewrite this expression as 1 by S minus S plus zeta omega n upon S plus zeta omega n whole square plus omega n square 1 minus zeta square. So, we can write here minus zeta omega n upon. So, here root 1 minus zeta square. So, we multiply and we divide. So, here power root 1 minus zeta square upon S plus zeta omega n whole square plus omega n square 1 minus zeta square.

Now, we can take this together. So, this is omega n by root. We can take this together, omega n into root 1 minus zeta square that is, square of that it is, square is equal to this term. So, if we say a equal to omega n root 1 minus zeta square. So, we can say, this is a square and this is S plus zeta omega n whole square. So, this formula we know that, if we take the inverse Laplace, Laplace transform.

So, we can write Ct equal to, so, 1 by S. We have 1, because this is unity function minus this is this term. This is S upon S square plus a square. Here, a is this, S upon s square plus a square is universe. It is inverse, Laplace is cos at and there is with S, there is minus G S minus minus zeta omega n. So, it will bring another term that is, e power minus bt.

So, if we say b equal to minus zeta omega n. So, it will be cos a into t into e power b t. So, this will be the. So, 1 minus e power minus b e power minus. So, b is minus zeta omega n. So, minus zeta omega nt into cos at a is omega n root 1 minus zeta is square t. So, this is this part, because Laplace transform of cos a t equal to S upon S square plus a square and if we say S minus b. So, it will be multiplied with e power bt. Now, Laplace transform of sin at equal to a upon S square plus a square.

So, here a is this omega n root 1 minus zeta square and we have this term minus zeta upon root 1 minus zeta square into e power bt. So, minus zeta omega nt into sin at. So, sin omega n root 1 minus eta square t. So, this is the trans, this is the response of this system moreover. So, we can write is as 1 minus e power minus zeta omega nt cos omega n root 1 minus zeta square t minus phi.

So, we can write this cos omega t plus a cos omega t plus b sin omega t as just 1 harmonic cos or sig with some phase difference and here, will be. So, 1 minus 1 upon root 1 minus zeta square here, phi is equal to tan inverse zeta upon root 1 minus zeta square.

So, here we have a cos theta plus b sin theta and we have written it in this form, cos theta minus phi and phi is tan inverse zeta power root 1 minus zeta square. So, here we have the response of the system. So, we can see that, this in the response, there is coming damping and natural.

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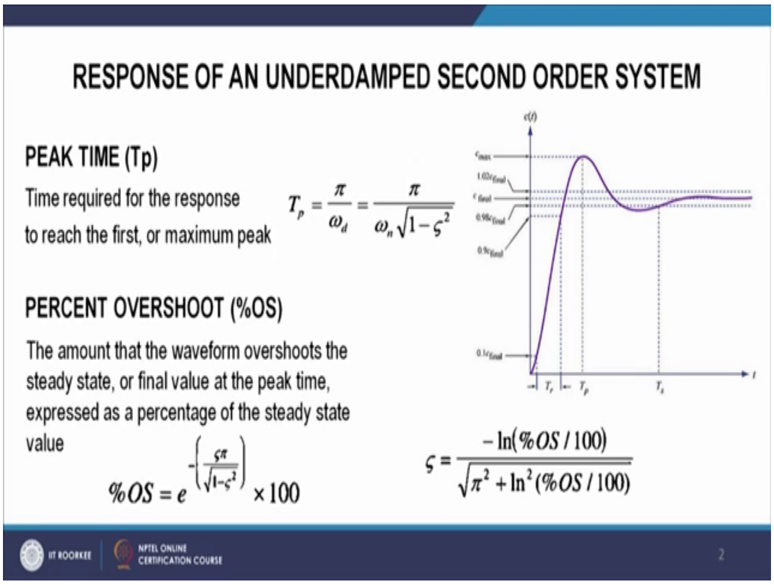

Frequency of this ah, this system. So, damping and natural frequency they decides the response of the system, especially the transient response of the system.

So, we can see here a typical response of ah, second order system and we see here, C final is the a step input and the response is rising and it is going to overshoot this final value and then it is coming down and then trying to follow the input or final value. So, here we can see that there are the maximum peak, peak. This is called peak and this is maximum value of the response.

Then there is the overshoot value that how much this maximum response rises with respect to the final value. So, here from seeing the response.

So, we plot the, if we plot the response of the system of under damped, second order system we can see that.

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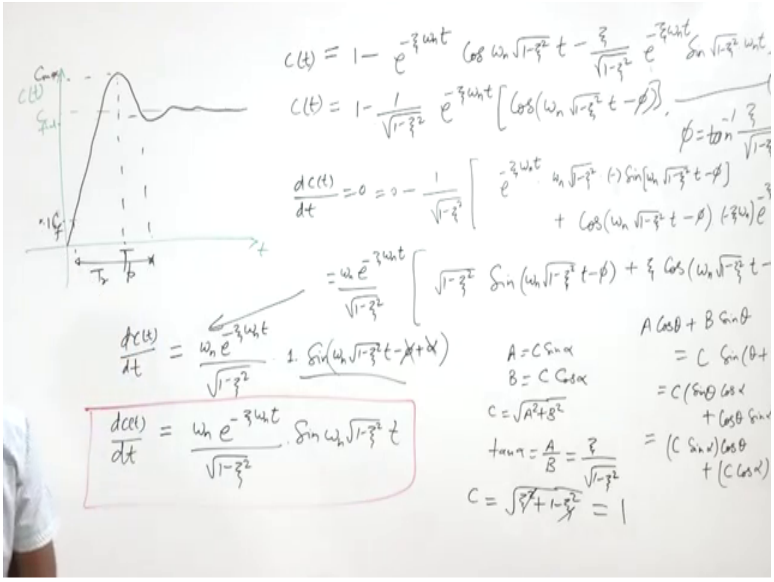

If this is our C final or final value, the response will rise, will overshoot and then it will come back down and then it will so,. So, this value is called C max or peak value and the time at which this peak value occurs is called the peak time, then this value C max, how much it rises with respect to this, is called overshoot, then here again this 0.1 of the final value C. Final is to 0.9 of, this C final.

This is rise time tR and then 0.98 percent that is settling time when the response rate to 2 percent and A stay within that limit of the final value. So, here peak time. Peak time is time required for the response to reach the foreshore maximum peak. Here, is the peak time the time when this first or peak or maximum occurs that is peak time. So, let us find the peak time what will be the peak time. So, let us collect this expression. So, that we will use them to derive this, time, the response characteristics peak time, and other time.

So, C t equal to 1 minus e power minus zeta omega nt cos omega n root 1 minus zeta square t minus zeta upon root 1 minus zeta square e power minus zeta omega nt sin root 1 minus zeta square into omega n t and this is equal to 1 minus 1 by root 1 minus zeta square e power minus zeta omega nt cos omega n root 1 minus zeta square t minus phi.

And where phi is tan inverse zeta upon root 1 minus zeta square. So, this is the expression and we will use this. So, if we, let us say, this is equation number 1 and this is equation number 2. So, if we want to know the peak time of this response, because this is response and it is represented by this curve.

And if we want to get the peak time, we must know that peak time is the time when the response is maximum, the first maximum occurs. So, if we take the first derivative of this response and equal, equate to 0 that will give the idea of peak time. So, means dc t by dt equal to 0. So, let us take the first derivative of this. So, here 0 minus, let us take here 1 by root 1 minus zeta square here, e power minus zeta omega nt into let us take this function.

So, derivative of the this function first. So, this is omega n root 1 minus zeta square into and this is minus sin omega n root 1 minus zeta square t minus phi plus cos omega n root 1 minus zeta square t minus phi here, this is, this minus zeta omega n into e power minus zeta omega nt. So, now, we have to, we can write it equal to e power minus zeta omega nt into omega n by 1 minus zeta square. So, we are minus, we can take out as well as this omega n, we can take out. So, what we will remain is root 1 minus zeta square into sin omega n root 1 minus zeta square t minus phi plus zeta cos omega n root 1 minus zeta square t minus phi.

So, here a cos theta plus b sin theta, let us say equal to C sin theta plus alpha and. So, we can write equal to C sin theta cos alpha plus cos theta sin alpha and. So, we can write C sin alpha into cos theta plus C cos alpha into sin theta.

So, it means we want to express this term just with the replace, with single term. So, we will, we can get C equal to, we can get C. So, here we are getting that A equal to C sin alpha and B equal to C cos alpha. So, C is equal to root a square plus B square what is here, A square plus B square equal to C square and tan alpha equal to A upon B.

So, we can get A upon B equal to tan alpha. So, here we see A equal to zeta. In this case, if we compare this A equal to zeta and B equal to root 1 minus zeta square. So, we will get alpha equal to tan inverse zeta power root 1 minus zeta square and C equal to A square plus B square. So, a square is zeta square under root zeta square plus B square that is One minus zeta square. So, equal to One. So, C is u equal to 1.

So, we can write this dc t by dt equal to omega n e power minus j 2 omega nt upon root 1 minus zeta square and he here 1 into sin omega n root 1 minus zeta square t minus phi and we know that phi is tan your zeta by root 1 minus zeta square and. So, this is equal to phi is alpha phi is tan inverse zeta by root 1 minus zeta square and alpha is also tan inverse zeta by 1 minus zeta square.

So, here minus phi minus here plus alpha here. So, this will cancel out. So, we will get only sin omega and root 1 minus zeta square t. So, we are getting here omega n e power minus zeta omega nt upon root 1 minus zeta square into sin omega n root 1 minus zeta square t. So, now, to get the maximum. So, we have to make dc t by dt equal to 0.

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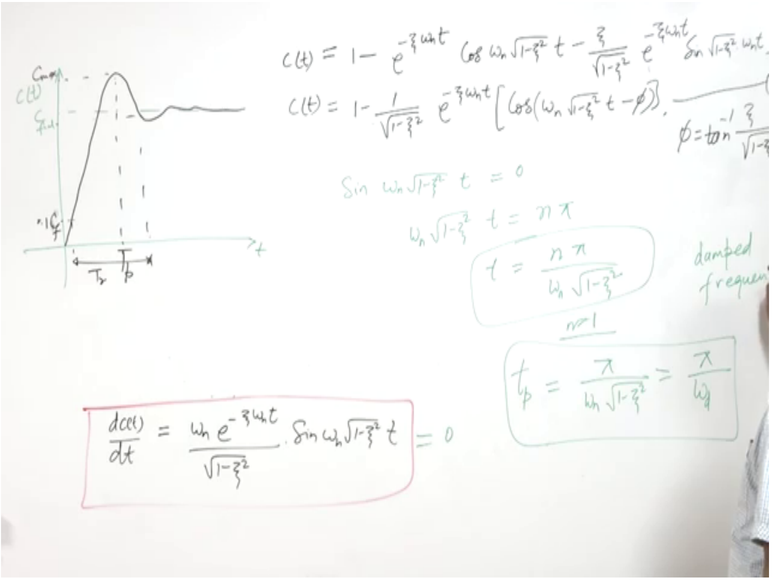

To get the maxima and to make it 0, we will have of course, this is exponential. So, this can not be 0. So, we have to make this sin omega n root 1 minus zeta square t equal to 0 and. So, omega n root 1 minus zeta square t equal to n pi.

And therefore, t equal to n pi upon omega n root 1 minus zeta square. Now, for n equal to 0 for n equal to 1, we will get the first maxima and therefore, that t equal to tp and that is equal to pi upon omega n root 1 minus zeta square and this we can also write pi upon omega d. So, here peak time is equal to pi upon omega d omega, d is called the damped frequency and this is equal to omega n root 1 minus zeta square.

So, here, we can see that from the step response of the under damped system, second order system, we found the ah, response of the system and then we find the peak time by taking the first derivative of that response with respect to time equal to 0 and we get this peak time. So, this is the tp, is the peak time, when this maxima, first maxima occur and that is equal to pi upon omega d. So, we will continue other parameters in the next lecture. So, I thank you for attending this lecture and see you in the next lecture.

Thank you.