Theory of Production Processes

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Lecture – 22
State of stress in two and three dimensions, Mohr's circle

Welcome to the lecture on state of stress in two and three dimensions and Mohr's circle. So, we will come to that lecture where we will discuss about the stress components how the stresses are defined in case of two dimension and three dimensions. So, in the previous lecture we had the concept of stress at a point and what we saw that you have one normal component that is perpendicular to the plane where you define the want to define the stress and then there will be one component in the plane of in that plane itself.

So, that will be shear component again it will have the shear component has two components. So, that will be in the two directions. So, you can again resolve them into two directions if there is x direction the normal stress is there then y and z you will have the shear components.

So, in that case, you will have at a point you define as one normal stress and then you have two shear components. So, the convention is that this normal stresses they are normally defined as either sigma 1 or sigma 1 like that you can define as or sometimes.

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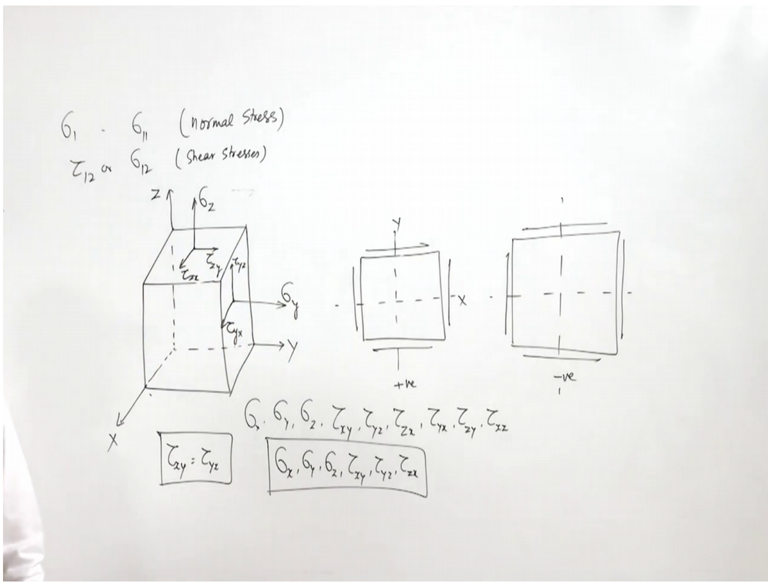

You also define as sigma x capital X. So, then these are basically positive normal there convention is that if it is tension then you take it as positive and if it is compression, then you take it as negative. Similarly, if the shear stress; in the case of shear stress the convention is that if it is you know positive, it will be positive if it points in the positive direction on positive phase of a unit cube or in negative direction on the negative phase of the unit of the unit cube. So, that way it is taken as positive, otherwise, if it is on the positive direction on the negative phase of the unit cube, then it as it in case that case it will be negative.

So, that way it is the sign convention is followed. So, let us see if you take a cube and if you try to see the state of stress. So, in that case; so, it will be like this. So, you can have these axis like that you can have this as z axis, this as y axis and this as x axis. Now the thing is that on any particular plane suppose on this plane if you define, then this on this particular plane the normal direction. This will be the normal stress and that you will define as sigma y. Similarly, for denoting the shear stress you either. So, this will be the normal stress. Now for the shear stress you may have something like sigma 1 2. So, this is the example of this is the convention for shear stresses we also denote it as sigma tau 1 2 or even sigma 1 2 there may be another notation also.

So, normally we define as tau 1 2 now tau 1 2 means this is the plane and this is the direction. So, basically if you take here in that case here you will have two kind of shear stresses and that will be basically now first will be the plane in which it is working and the second subscript will be the direction energy in which it is working. So, if you take this direction if you take the x direction that way. So, this is your x direction. So, this if you take that. So, this is a plane is. So, this will be tau plane is y and the direction is x similarly the shear stress which is in this plane itself, but which is acting in the z direction. So, it will be tau y z. So, this is the convention of sign which is followed for defining the stresses.

So, you will have here if you look at you will have this as sigma z, then you will have one is this is sigma. So, this will be tau z and this is in the direction x. So, tau z x and again this will be tau z y. So, this way you can define the normal and shear stresses and try to see what they look like as indicated that sigma x sigma y and sigma z if they are in tension or in compression depending upon that if they are in tension it will be positive and if they are in compression it will be negative. Similarly, if you try to see that how the shear stresses will look like. So, you may have the unit phase of the cube and in that now what we have discussed that on the positive phase of the cube if it is in positive direction and on the negative phase it is in negative direction then it is a positive value of the shear stress.

So, if it is x and if it is y now in this case you have here and this is this. So, if this is positive and this is negative this is the positive phase and this is the negative phase. So, on the positive phase you will have. So, this is a positive shear stress similarly this is the positive phase. So, you will have this and this is a negative phase. So, you will have this. So, that is basically the negative phase. So, you have in the negative direction this is positive x this is negative x; this is positive y this is negative y.

So, depending upon that this is a sign of the positive shear stress and otherwise you will have negative shear stress in that case you will have again this one. So, if the positive on the positive side if you have the negative and the on the negative side you have positive. So, that is the negative convention similarly on the positive side you have a negative and this is positive.

So, in that case, these are the conventions this is positive and this is negative. So, this way shear stress and normal stresses their values are taken as positive or negative now what we see that if you take the small element now what we see here that in such cases you will have basically for describing the stress at the point ultimately you need the 9 quantities. So, you will have sigma x or sigma 1, sigma 2, sigma 3 normal stresses and 6 shear stresses.

So, these are the; so, 9 complete 9 quantities are required 9 stress components are required to define the stress at a point in that case and that is why you also call it as a stress tensor stress as a tensor of second order because as the order is 2, you will need three raised to power 2 that is 9 components to specify the value of stress in that case. So, these 9 components are basically you have sigma x sigma y sigma z, then you have tau xy tau y z tau zx tau yx tau zy tau x z.

Now, that way you have the 9 components; however, if the element is considered to be very small and if you do the forces analysis then you come to the conclusion that tau xy can we prove to be equal to tau yx. So, in that case because if you look at and do the analysis of the element area is considered to be very small in that case this tau xy can be considered to be equal to tau yx and in that case you are ultimately you require the six you know components. So, six components which are required is sigma x sigma y sigma z tau xy tau y z and tau zx. So, ultimately you require these six components for describing the stress of state of stress at a point.

So, next we will discuss about the state of stress in two dimension; so, many a times in many of the situations.

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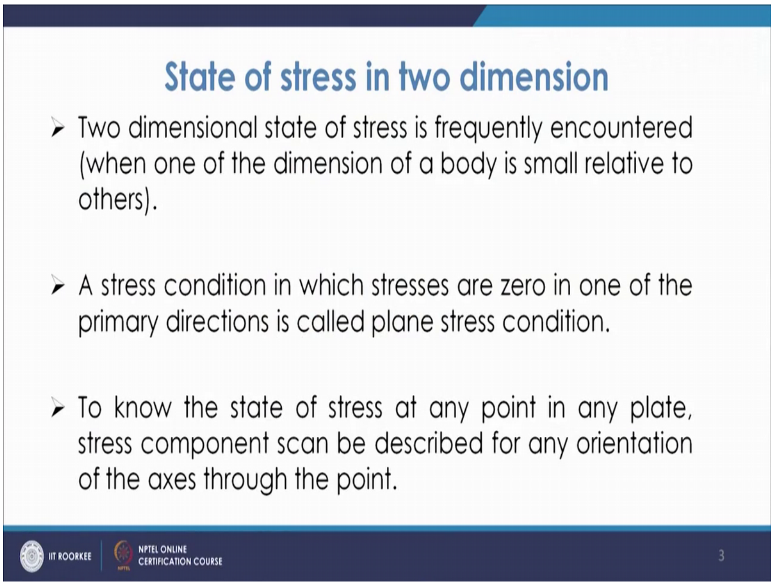

We frequently have to encounter a state of stress in which predominantly you have a stresses acting in two directions and in one of the dimension of the body the value of the stress is quite small as compared to the other dimensions. So, these stress conditions in which the stresses are 0 in one of the primary direction that is called the plane stress condition.

So, just like you take a very thin sheet of metal in that case the value of stress in the direction I mean if it is flying flat in that normal to this flat direction the in that direction the stresses value are considered to be very small. So, that way you do not consider that if it is the; if we very thin you know seat. So, that is basically. So, once you neglect that in z direction. So, you have the analysis in two dimension. So, that is known as the plane stress condition now we have to know the state of stress at any point in the plane and you have to see that how these stress components can be described for any orientation of the access through the point.

So, if suppose you have a two dimensional domain in that you have the x and y directions shown as shown in the figure.

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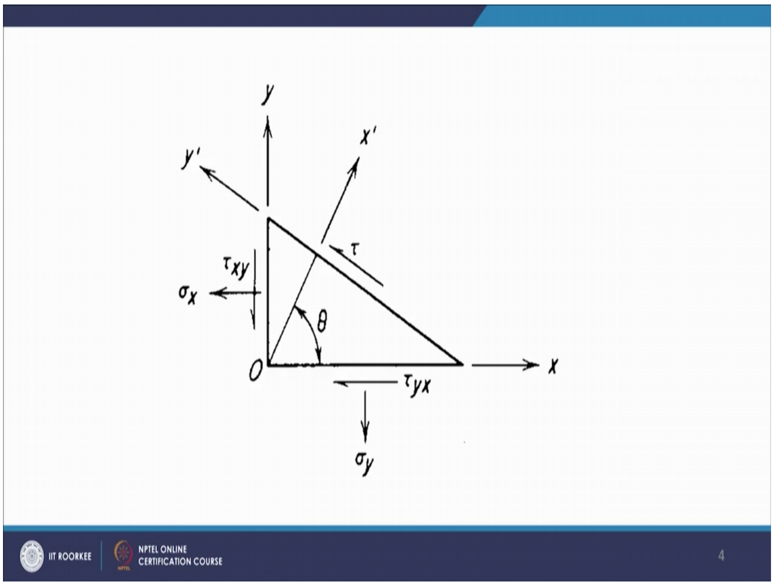

In this figure, if suppose here you have an element and you have the sigma x sigma y acting and you have tau xy. So, in that if you want to find the component of the stress on a plane which is at an angle theta. So, this plane which is whose normal. So, on this plane the outward normal is direction x prime. So, this plane is making angle theta with this x axis.

So, this is the plane and we want to find for this plane what will be the. So, if any plane is inclined at certain angle with some axis and in this case the axis is this x and this normal to this plane this plane is here. So, this planes normal this is x prime it is making angle theta with this x axis. So, now, what is the normal and shear stress component which will be working on this element? So, in that case what you have to do is now for this basically the direction cosine is defined as l m and n.

So, basically this normal will be making angle that is theta cos theta. So, it has the direction cosines element that is you have l is cos theta and. So, that is with the x y and z. So, this plane this outward normal x prime. So, that is it has the direction cosines l and m because it is being the two dimensional case. So, l is cos theta and m is sin theta and then if we do the analysis of the forces. So, after the analysis what we see we can have the expression for sigma x prime sigma y prime and tau xy; x prime y prime. So, we have to have the expression for the normal stresses as well as the shear stresses on this angle. So, if you do the derive derivation and the analysis can be shown out to be like sigma. So, you will get sigma x prime, it will be sigma x plus sigma y by 2. So, for this in; this direction you will have sigma x prime.

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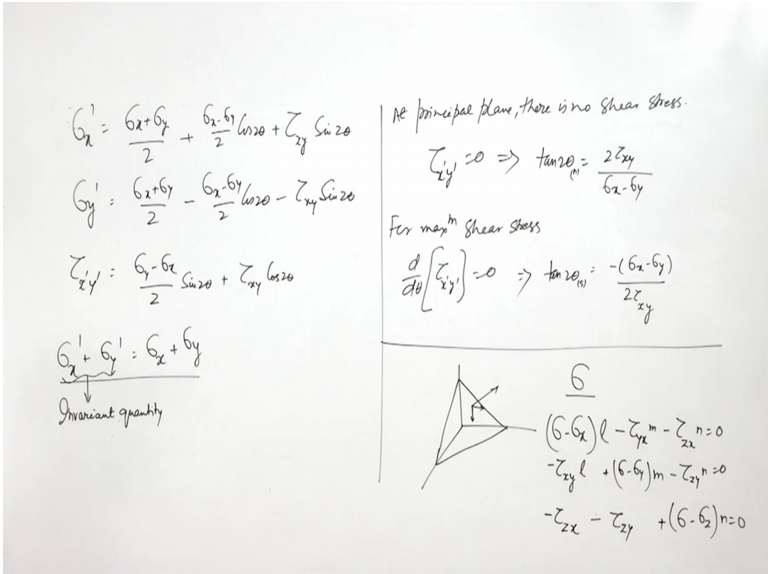

So, that will be sigma x plus sigma y by 2 plus sigma x minus sigma y by 2 cos t 2 theta plus tau xy sin 2 theta. So, that will be the normal stress for that element, then you have sigma y prime that will be again sigma x plus sigma y by 2, then you will have the minus sign. So, that will be minus sigma x minus sigma y by 2 cos theta 2 theta minus again tau xy sin 2 theta. So, it will be sigma x minus sigma y by 2 cos 2 theta and minus tau xy sin 2 theta, then the shear stresses which are acting on this plane that is tau x prime y prime. So, that will be sigma y minus sigma x by 2 that is sin 2 theta plus tau xy cos 2 theta.

So, tau x prime y prime will be sigma y minus sigma x by 2 sin 2 theta plus tau xy cos 2 theta. So, this is the you know expression for sigma x prime sigma y prime and the tau x prime y prime which is which can be derived and which can be seen. So, if you fall if you have the sigma x and sigma y known you are told a problem in which the sigma x sigma y and tau xy value is known to you then if you are told to find the value of normal and shear stresses components at a plane which is having this angle theta with x axis. So, whose the plane on whose outward normal is making angle theta this the plane; this plane has a outward normal, it is making angle theta with this axis x axis on that plane the normal stress and the shear stress component they are like this.

Now, in this what is seen is that if you add this sigma x prime plus sigma y prime? So, what you see here is that sigma x this comes out to be sigma x plus sigma y. So, what we see that summation of these normal a stress components it is not depending upon the orientation angle. So, that is why this quantity is known as. So, this quantity is known as invariant quantity this is when we will do the analysis, we will study about the three dimensional state of stress in that case, we will see that you get sigma x prime plus sigma y prime plus sigma z prime will begin that will be set as the first invariant of the stress. So, that is this is the environment quantity this is known as first invariant and this is one of the trait of such stress states.

Now, if you try to draw the graph for any problem for any value of sigma x sigma y and also tau xy or for normal cases then few of the facts are becoming clear and these are that the maximum and minimum value of the normal stresses you know on any oblique plane through a point it will be basically occurring when the shear stress value will be 0. So, this is one of the finding that is seen when you try to draw the plot for that the second what you get in this case here in this is that the maximum and minimum value of the normal stresses and shear stresses which occur they are normally ninety degree apart. So, at from ninety degree apart these maximum and minimum values of normal and shear stresses occur now the maximum value of the shear stress that occur at angle halfway between the maximum and minimum shear stress.

So, this is a third finding what you get and then also the variation of normal and shear stresses which occur here in this case that basically is forming a sine wave with a period of 180 degree so that again; so, that way because it is going 2 theta. So, that encompasses the 2 theta at 360 degree. So, after 180 degree again it changes. So, it repeats in itself. So, that has a period of 180 degree.

So, this is how if you we have the state of stress in 2 dimension you have any value of stress given then in that case, for any plane which is at certain angle orientations you can have the calculation of the normal and shear stresses that way. Now in this case, what you can see is there are stresses now we call some principal planes the principal planes are the ones where there is no shear stress and the value of. So, only the normal stresses are acting on that and. So, that is principal stresses are acting and the shear stress value is 0 at that point.

So, if you see if you we try to see the principal planes where that there is no shear stress. So, at principal plane there is no shear stress. So, if you try to see that what is the orientation of the principal plane in that case you have to equate this to 0 and then once you equate that to 0 you get tan 2 theta. So, that will be equated to 0 and tan 2 theta will be basically tau x 2 tau xy divided by sigma x minus sigma y. So, this if tau x prime y prime is equated to 0, then this sin 2 theta by cos 2 theta.

So, this right it will be 2 tau xy and this sign will be changed. So, it will be tan 2 theta equal to 2 tau xy divided by sigma x minus sigma y. So, this is how you can have the orientation the location of that value of this principal plane where there is no shear stress now in this case you will have 2 values of. So, this tan 2 theta once you have that. So, as you know it will be n. So, theta for that you will have theta has 2 roots. So, theta 2 will be theta one plus a n pi by 2. So, like that you will have 2 roots.

So, there will be defining 2 mutually perpendicular planes which are basically free of shear; so, you will be getting the 2 planes which are free of the shear, now in that case, you can also have the planes where there is maximum shear stress. So, you can have the plane at which there is maximum shear stress. In that case, you will differentiate this expression with respect to theta and equate it to 0. So, for getting the plane for maximum shear stress you will have to have derivative of tau x prime y prime that has to be equated to 0. So, if you do that further. So, in that again you will get the similar kind of this, but that will be basically the inverse reciprocal of I mean negative sign also of this. So, if you look at that value it will be again that will be minus of sigma x minus sigma y by 2 x y.

So, it will be basically. So, in that tan 2 theta. So, this is basically for the there is no shear. So, this is only for normal and this is for shear and this is basically minus of sigma x minus sigma y upon 2 tau xy. So, what you see is that slope of these and this, they if you combine them that becomes minus 1. So, they are perpendicular to each other. So, this way you see that how they are interrelated. So, this is how you can find the state of stress in case of two dimension.

Now we will discuss about the three dimensional state of stress. Now in case of three dimensional state of stress, you will have the three components sigma x sigma y and sigma z. Similarly you have the sigma tau xy tau yz and tau zx. So, in that case if you think of an element in that if suppose you are taking an area and this area you will have the; you have the three directions. So, if suppose you have. So, as we have discussed you have the values of the normal stresses.

So, this is basically a principal plane and you will have again the different stress components. Now in this case, if you consider this as a principal plane and if if sigma is the principal stress in that case if sigma is the principal stress in that case. So, in those cases if you do the summation of forces analysis and if for this principal stress the element n be the direction cosines. So, means that is making angle i mean that is cost it l m and n that is with this xy and z axis.

So, that if l and m with the axial cosines of these three directions in that case you can have the three equations by summation of forces and the three equations lead to the equations like this. So, you will have sigma minus sigma x that is l then you have minus tau yx m minus tau zx n that will be equal to 0. So, you will have another set of equation that will be minus of tau xy l plus sigma minus sigma y m minus tau x.

So, that is y y z zy into n equal to 0 similarly you will have further minus of tau z x plus minus of tau zy and plus sigma minus sigma z into n that is equal to 0. So, this is the these are the 3 equations which you get when you do the summation of forces in those cases.

Now these three equations if they are to be solved for getting the non trivial equation the coefficient of this lm and n they are the; you know homogenous linear equations. So, these coefficients forming the determinant, sigma minus sigma x minus tau yx minus tau zx, then similarly you have minus tau xy then sigma minus sigma y and minus tau zy further minus tau that is x z y z and this sigma minus sigma z. So, that way because here you can have the value x z and y z they are same basically because you have xz equal to z x or y z equal to z y.

So, for that the determinant has to be equal to 0. So, if you do that if you set this determinant equal to 0 that comes out to be sigma minus sigma x then minus tau yx.

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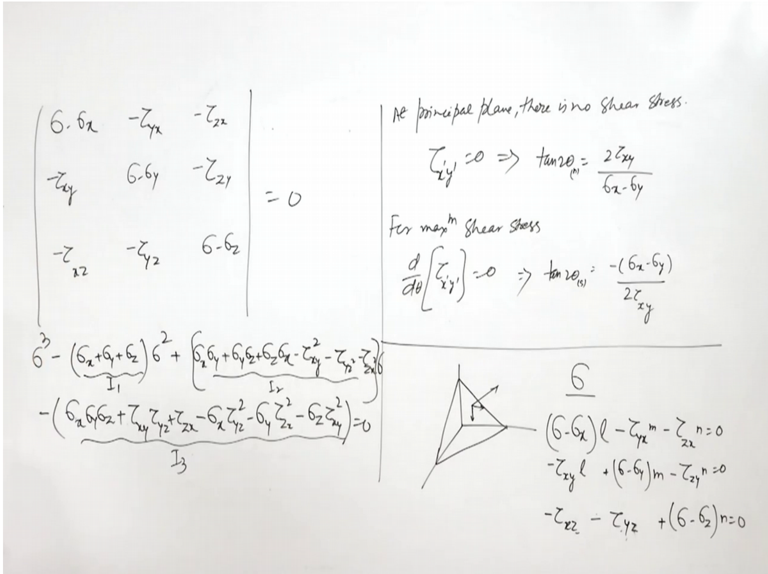

And further we have minus tau zx you have minus tau xy sigma minus sigma y and minus tau zy and then further you have minus tau x z minus tau y z and sigma minus sigma z. This determinant basically has to be equal to 0. So, once you equate that to be 0. In that case, you get 3 roots that will be sigma 1 sigma 2 and sigma 3. So, that will be given as the 3 roots for this equation. So, that will be three there will be three principal stresses which will be acting and then you in that you get basically the solution for that what you get equation is. So, you will be getting sigma cube minus. So, once you expand this it will be sigma cube minus then sigma x plus sigma y plus sigma z into sigma square.

So, that will be basically in that sigma square. So, that will be the second term then further you have plus sigma x sigma y plus sigma y sigma z plus sigma z sigma x and then minus tau xy square minus tau y z square. So, it will be further sigma x sigma y sigma y sigma z plus sigma z sigma x minus tau xy square. So, minus tau y z square minus tau zx square into sigma into that is sigma further you have the next term as minus.

So, this will be plus term say again will be minus and then that last term will be sigma x sigma y sigma z plus tau xy tau y z tau z x, then z minus sigma x tau y z square minus sigma y tau xz square minus sigma z tau xy square. So, like that. So, it will be sigma x sigma y sigma z plus tau xy tau y z plus tau z x these are the 2 terms which we get and then further you will have minus of sigma x tau y z square minus sigma y tau z x square minus sigma z tau xy square.

So, this has to be equal to that is 0. So, that that is how you solve these three equations and you get the, this equation and you get the three roots. Now all these three quantities which is there in the bracket these are the three invariants of the stress. So, they are known as the three invariants i 1, i 2 and i 3. So, these are the stress invariants which basically will be used maybe when we discuss about the stresses in the later lectures. So, and then further if you try to further resolve for the plane which is had at certain angle then there can be further the analysis and you can see that how you find for any plane which has certain direction cosines what will be the maximum shear stress.

So, for the particular value of l m and n you will see that the maximum value of shear stress suppose for l equal to 0 m as plus minus one by root two and n as plus minus one by root two it will be sigma 2 minus sigma 3 by two. So, similarly you will have four lm and n values you will have the value of maximum shear stress you can find it by doing the force analysis you can go through the book of strength of material for studying such correlations and also see that what will be the maximum shear stress in that case basically that will be used when we study the theory of yielding further.

Now, let us know something about the Mohr circle of stress. So, we studied about the two dimensional state of stress and we saw that how you find the expression for finding the normal and shear stresses at a plane which is inclined at certain angle with the x axis or. So, whose outward normal is making certain angle? So, we got the expression also that sigma x prime will be sigma x plus sigma y by 2 plus again further you have sigma x minus sigma y by 2 cos theta 2 theta plus tau xy sin 2 theta. So, like that you had the expression for the principal stress that is normal stress then you have the expression for the shear stress. So, if you try to further see. So, you what you see is that if you find sigma x prime minus sigma x plus sigma y by 2.

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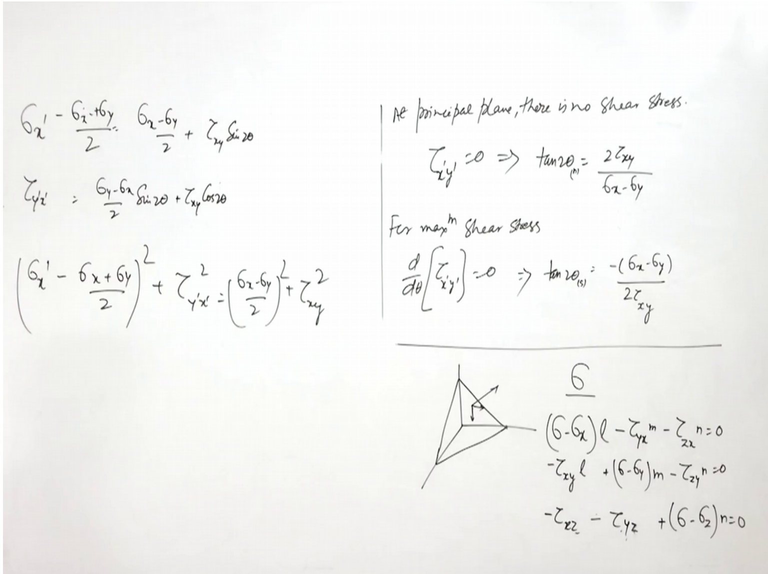

If you take the sigma x plus sigma y by 2 on this side; so, what you see is that on that side you will have sigma x minus sigma y by 2 cos 2 theta and plus tau sin 2 theta. So, it will be sigma x minus sigma y by 2 plus tau xy sin 2 theta.

Similarly, you have another expression that is tau y prime x prime. So, that is again sigma x minus sigma. So, sigma y minus sigma x by 2 sin 2 theta and further you have. So, that is tau xy cos two theta. So, that will be; so, what you can do is further you try to solve these equations for sigma x prime and tau y prime x prime now for that you do the square of the left hand side and also right hand side and then you sum it and then equate it.

So, what you see is that if you do that side sigma x prime minus sigma x plus sigma y by 2 that square plus tau y prime x prime square if you do that on the opposite side you get sigma x minus sigma y by 2 whole square plus tau xy square. So, this is what you get now if you look at this; this is the equation of a circle with the origin as sigma x plus sigma y by 2 and 0. So, origin is shifted to distance sigma x plus sigma y by 2 to the right of the origin and then this is the radius.

Now, that is can be re interpreted by that can be understood by looking at this Mohr circle where you see that this.

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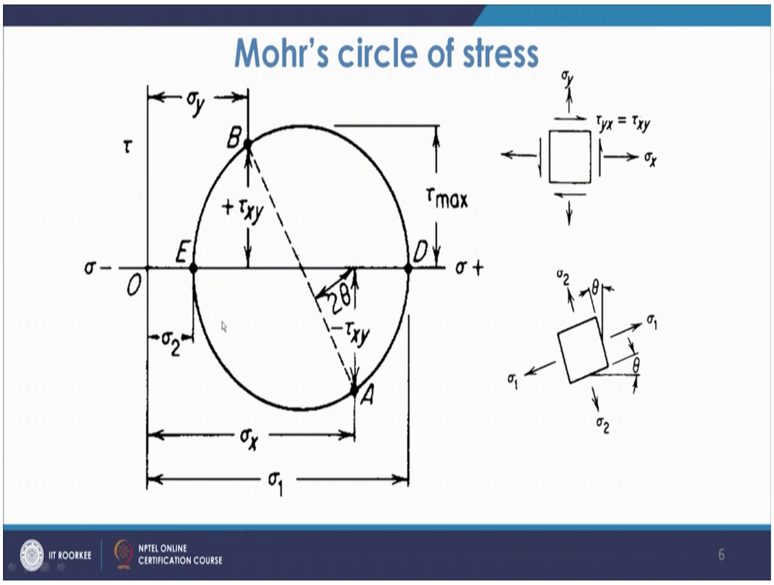

If you see that sigma x and then sigma y; so, basically what you see is this is what the origin is here from. So, origin will be at sigma x plus sigma y by 2 and 0. So, that will be your origin. So, what you see is sigma x plus sigma y by 2. So, that will be your origin. So, this way you get the expression for sigma 1 and sigma 2. So, basically this side what you get that will be sigma 1 and this is getting you sigma 2 and the point on this periphery that talks about the shear stress. So, as you see that maximum shear stress is the radius of this Mohr circle. So, this value will be the sigma 1 and this value will be sigma 2.

So, this way the Mohr circle of a stress can be drawn, but once you know the values and also one thing is there that in this case theta is taken as 2 theta here. So, that has to be seen. So, any angle theta on the physical element a will be it will be represented by angle two theta on this Mohr circle. So, this way you can have the values of these sigma a normal stresses and also you can find these angles. So, so that that is how you find this Mohr circle of a stresses.

Thank you very much.