Modeling & Simulation of Discrete Event Systems
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Lecture – 08
Continuous Distribution Functions

Welcome to the lecture on continuous distribution functions. So, in the last lecture we discussed about discrete distribution functions. And we will very much frequently be dealing with such kind of distribution functions where the values are basically contained in certain intervals. You have certain values, but they are always represented in terms of certain interval values.

So, such systems are the continuous functions; continuous distribution functions, and we always find the probability associated for finding certain value between 2 intervals. That is why it is a continuous distribution functions contrary to the one which we discussed earlier that was discreet. So, where in that case, we wanted to find the probability of having certain discrete value. Certain fixed value whereas, in this we talk about the intervals. Now in this case we will discuss about few kinds of continuous distribution functions.

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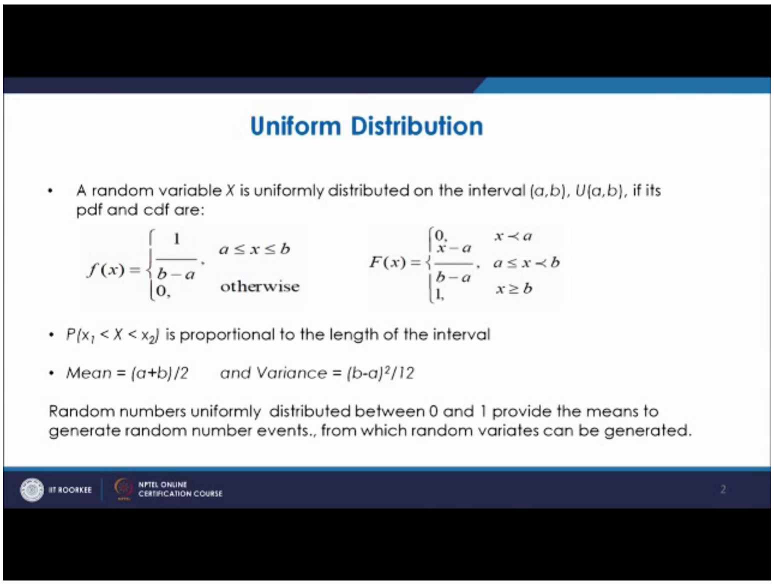

And the first one which will discuss is the uniform distribution. So, as the name indicates, the it is uniformly distributed in the interval a to b. You have the extreme values a and b, the probability of having any value between a and b will be there where as outside that the probability will be 0. So, that is why the f x is defined as 1 by b minus a in between when a x is from 1, I mean a to b. So, if x is in between a to b it will be f x equal to 1 by b minus a. And then if it will be 0 otherwise. So, if you have any value other than in between a and b, then it is probability value is 0.

So, similarly the cumulative distribution function is defined. So, as we see this value is telling about the probability value. That is 1 by b minus a. And then this value f x this is the cumulative distribution function value. And this is you have the value accepted only between a and b. So, if it is less than a the f x value is 0, and if it is more than b then it is once as we know that cumulative distribution function tells you, the probability up to that particular value. So, anyway before a anyway there is no probability. So, it is 0. Once we move from a to b the cumulative probability value will go on increasing. So, it will be value of x minus a upon b minus a. So, in that case a is the lower limit and b is the upper limit. So, x will be varying from a. So, it will be; I mean, a will be less than equal to x and then it will be less than b, and as we get b. So, the cumulative value cumulative probability value becomes equal to 1. So, which is more than equal to even b, when x is in that case it will be 1.

Now in that case. So, what we is probability is proportional to the length of interval. And then mean what we get here in the cases of uniform distribution, the means we have extreme values a and b. So, mean is a plus b by 2, and then you will have the standard deviation or that is variance which we calculate as the measurement for dispersion. So, that is b minus a square upon 12.

So, this is the properties for a uniform distribution and normally this distribution is used to generate the numbers between 0 and 1, or random numbers which are generated normally it is seen that it is uniformly distributed. So, for I mean, any number which is there between the 2 boundaries the probability of any number is same similar to that. So, on that basis we find the random number generation we find the random variates that is calculated for this uniform distribution.

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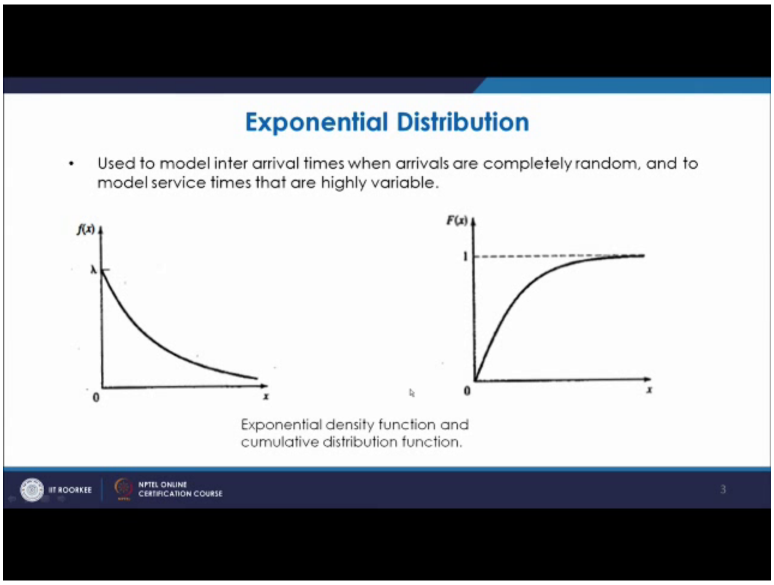

Next is coming the exponential distribution. Now this exponential distribution as we see, it is used to model arrival times, when arrivals are completely random. So, this is one of the very important kind of distribution function, which is used to normally in the case of arrivals we use them. And to model service time which are highly variable. So, normally in the case queues or the counters bank counters or so. So, in those cases the arrivals or the service times they are normally following this kind of distribution function. And what we see, the exponential distribution function this curve talks about the density function of the exponential distribution function, and this is the cumulative distribution function.

So, we will find the expression pdf, now pdf for these exponential distribution.

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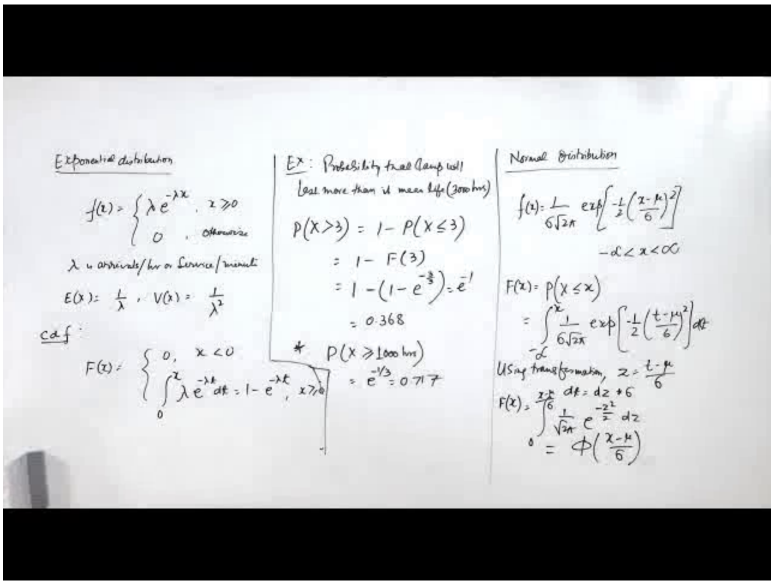

Now, in the case of exponential as we see this is exponential curve. So, in this case f x is defined as lambda e raise to the power minus lambda x, when x is less than equal to 0, I mean more than equal to 0, and it is 0 otherwise. So, lambda is the arrivals per hour or service per minute. Now in this case it is also used to model the lifetime of a component that fails catastrophically. So, in this case the mean value is 1 by lambda, and the variance which is computed is 1 by lambda square. The cumulative distribution function for such distribution is we can get the value, and cumulative distribution function will be 0 for x less than 0, and it will be 0 to x lambda e minus lambda t d t. So, that is 1 minus e raise to the power minus lambda t for x greater than equal to 0.

So, these are the pdf, and the cdf for exponential distribution function. We can solve problems for any particular value or by using the cumulative distribution function expression. So, let us take one example.

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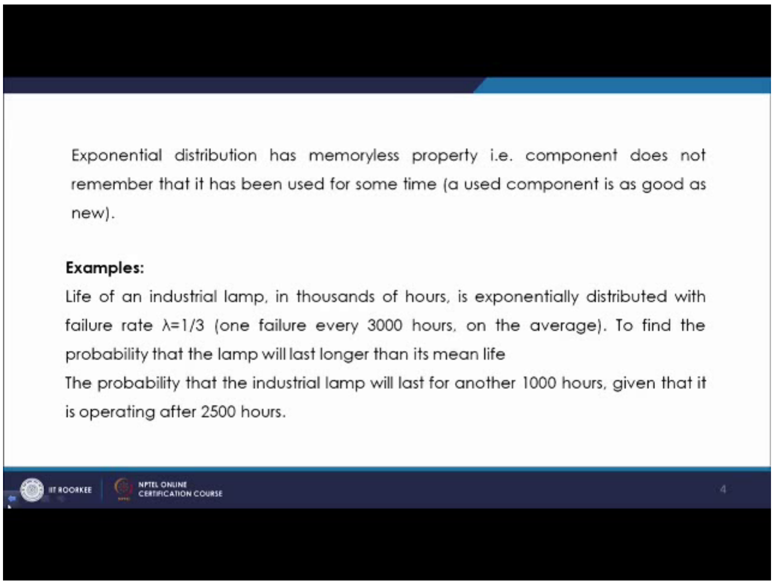

Before that we should be knowing that this exponential distribution, it is said to have a memoryless property. Because if we have suppose there is certain rate which is there mean rate. And once you have used any product for certain time, and it is stopped and then further. So, it forgets in the past how much, it has been used if you have suppose it has been in the machine has been used for certain time.

And then further it is told that how much it will be is the probability that it will work for another some hours. So, from at that point of time it forgets how much it has been used. So, this is known as a memoryless property. So, this is the trait of this exponential distribution just like in bank or so. As you see that does not depend upon the time how much time has passed. So, anything can be the arrival rate or so.

Now, in such cases, what we see? You have certain example. Life of an industrial lamp is exponentially distributed with failure rate lambda is 1 by 3. That is one failure every 3,000 hours on the average. So, that is given lambda this is exponentially distributed. To find the probability that the lamp will last longer than it is mean life. So, we will find the probability that it will not last longer than it is mean life, and then it will be subtracted from 1. That will tell us that the lamp will last longer than it is mean life. So, what we see in this case? If it will last for more than it is you know mean life.

So, probability that; so, if you talk about the example, so probability that lamp will last more than it is mean life that is 3,000 hours. So, for that; so, P e will be X more than 3 that is 1 minus P X less than equal to 3. So, this is again calculated using the cumulative distribution function. It will be 0 1 2 3 and it will be; so that why it will be f P minus f of 3 and you can get this cumulative distribution function from here. So, it will be 1 minus cumulative function is 1 minus e to the power minus lambda t. So, 1 minus lambda is 1 by 3. So, e raise to the power minus 3 upon 3. So, it will be ultimately coming as e raise to the power minus 1, and it will be 0.368. So, this way the probability that lamp will last more than it is mean life of 3,000 hours, you can compute by using these exponential distribution function.

Again, the second part is that probability that the industrial lamp will last for another 1,000 hours, given that it is operating for 2500 hours. So, as we discussed that in that case we discussed that it has a memory less property. It forgets component does not remember; that how much it has been used. So, that is known as the memory less property. Now in that case the probability that it will be used for another 1,000 hours means, the you have to find the probability that it will last for 3500 hours and before that it has already been I mean used for 2500 hours.

So, it means it is nothing but the probability that it will be used for more than 1,000 hours. So, it will be nothing but in that case we are getting probability of X more than 1,000 hours. And in that case, it will be f 1,000 hours. So, it will be 1 minus e raise to the power 1 minus 1 minus e raise to the power minus 1 by 3. So, it will be e raise to the power minus 1 by 3, and it will be 0.717. So, this way this is the probability that the industrial lamp will last for another 1,000 hours.

This is basically also the probability. So, already it has been used for 2500 hours, that is immaterial it is I mean it is of no use, we are further going. So, this is a new fresh case that it will tell you the probability of going for another 1,000 hours. So, that is why it is known as the memory less memory less property of such distribution, and among the distribution discrete distribution normally the geometric distribution has the similar property. The next distribution curve which is very popular among the continuous distribution is the normal distribution.

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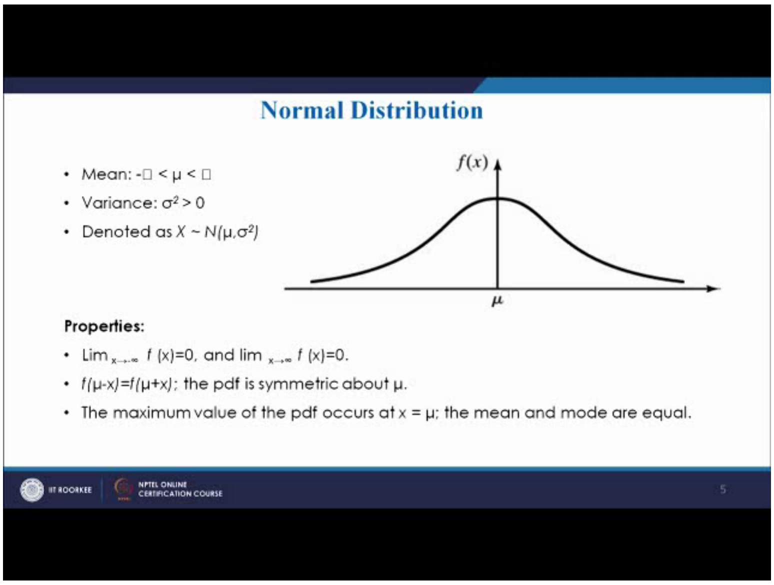

Now what we see here in the normal distribution? Mean value will take a value from minus infinity to plus infinity. So, that is value is usually here you have minus infinity to plus infinity. So, for any value between them the mu a mean value can take. And the variance is known as sigma square. So, normal it is denoted by n mu sigma square. So, whenever we talk about any random variable X as normal distributed it is also denoted as mu, I mean n mu sigma square. So, it is normally distributed n represents the normal distribution with mu as mean, and this is as the variance. Sigma is the standard deviation. So, sigma square is the variance. So, mu and variance. So, this is the typical probability density function for the normal distribution. Where mu is the mean and symmetrically it is distributed on both the sides.

Now in this case for normal distribution the pdf will be like this. So, let us write the normal distribution.

Normal distribution the f x takes of this form 1 by sigma root 2 pi exponential minus 1 by 2 x minus mu upon sigma square, when x is between minus infinity to plus infinity. And then; so, this is the probability density function, and you calculate the probability value for any value of x when you know the mu and sigma. So, the maximum probability is for the value mu. As we see here the maximum probability will be for mu, and then depending upon the sigma square the shape will vary a little bit.

So, that way it goes on both the sides. Now the properties of the random this normal distribution is that as the x value takes to minus infinity side means the extreme left side, as we see this value will be 0. Similarly, as x takes to the extreme right side that is plus infinity the value will be 0. Then this pdf if it is a standard normal distribution, then it will be symmetric about the mu. So, the probability value that f x at any point equidistant from mu on either left side or on the right side the value will be same. So, as we see f mu minus x will be f mu plus x.

So, standard normal distribution is the symmetric about mu. And maximum value of pdf occurs at x equal to mu. So, as we see you have the maximum value of f x is occurring at this point. So, this is mean as well as the mode. So, in in this case mean and mode both are equal. Now in this case what we get? So, we get the cumulative distribution function as P, when x will be less than equal to x. And it is expression is coming as minus infinity to x 1 by sigma root 2 pi exponential minus half t minus mu upon sigma square d t.

So, this talks about the cumulative distribution, cumulative value of the probability. So, what will be? So, if you look at this distribution; if you find, want to find the probability that the variable takes any value less than any point, the area under this curve up to this point that will be basically the probability. If you want to have the probability between any 2 values; so, area in between that curve basically talks about that. So, whole area under the curve taken as unity. So, the fraction of that will be the probability that will tell that the value will lie between these 2 intervals.

Now, this is the expression for the cumulative distribution, cumulative probability and we can do the transformation.

So, you if you use the transformation we can have this value as z. So, you take z as t minus mu by sigma. So, this is a z variate we call it later on basically we were taking this t minus mu by sigma as the z. So, once we take t minus mu by sigma as z. Then d z will be d t upon sigma. So, d t will be d z times sigma. So, in that case d t will be d z times sigma, if we differentiate both side. So, it will be d z into sigma. And in that case, f x comes out as; so, it will be t minus now this will be x minus mu by sigma. So, 0 to x minus mu by sigma, because this t is expressed by x. So, and then it will be 1 by root 2 pi e raise to the power minus z square by 2. So, this is z. So, z square by 2 d z. Because d t will be d z into sigma this sigma and this sigma is cut. So, that comes as e to the power minus z square by 2 into d z.

And we can further do the computation into it. And we can write it as a function phi x minus mu upon sigma. Basically, this x minus mu upon sigma that is basically the value of z which we look at you can get these values from the table of this normal distribution. So, the table looks like this. Now you can take this function also as if phi z. So, it will be phi z d z; now that finally, has been expressed like x minus mu by sigma and this will be basically z. So, depending upon the value of z you can directly find the cumulative probability value in such cases.

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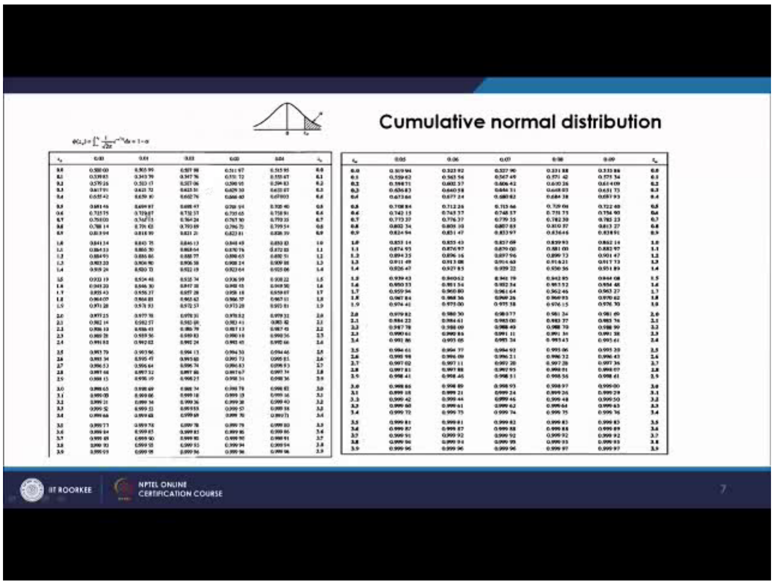

So, this is the table which talks about it this z value that is at any point that will be x minus mu by sigma based on that you can find these cumulative probability values. Using so that is cumulative normal distribution it is written. You can get the value once you know the mean and the variance you can; so, upon that if you know the mu and the sigma that is standard deviation which is the square root of variance. Then for any value you can calculate this amount. So, this was basically z variate you can get this value from this particular table. So, this table will let us know about this probabilities.

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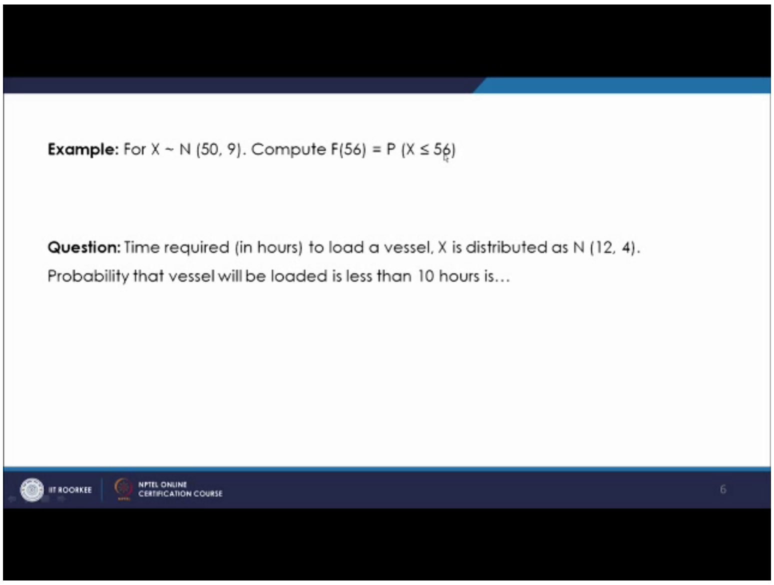

Now, let us see how that can be solved. So, let us see why there is one example that if you have x which is a random variable, which is normal distributed with 50 as the mean and 9 as the variance that is 3 is the standard deviation. In that case you have to compute f 56. So, once you get com compute f 56 that is probability, x will be less than equal to 56. So, what we see is in that case phi will be. So, the value which we want to see from the table, for that x will be 56 and you have mu is 50. So, it is shown that the in this normal distribution this is the mean 50 is the mean. So, 56 minus 50 divided by the standard deviation that is square root of 9. So, it comes 3. So, 56 minus 50, that is 6 by 3 phi of 2. So, your that from the table you can get this value. So, if you see the value 2 for that it is 0.97725.

So, from here the probability cumulative probability value can be taken as 0.9772. So, that is basically taken from this table you can directly get from this table. Then similarly time required in hours to load a vessel is distributed as n 12, 4. So, probability that vessel will be loaded in less than 10 hours. So, in that case the value that is 10, 10 minus 12 divided by under root 4. So, it will be minus 2 divided by 2, so minus 1. So, you have to see the minus 1 value. And minus 1 value will be the complement of this value. So, one value is 0.8134. So, it will be phi minus 1 will be 0.1587 something like. So, it will be 0.1587. So, this way you calculate the cumulative probability values in the case of normal distribution.

Next distribution is the gamma distribution.

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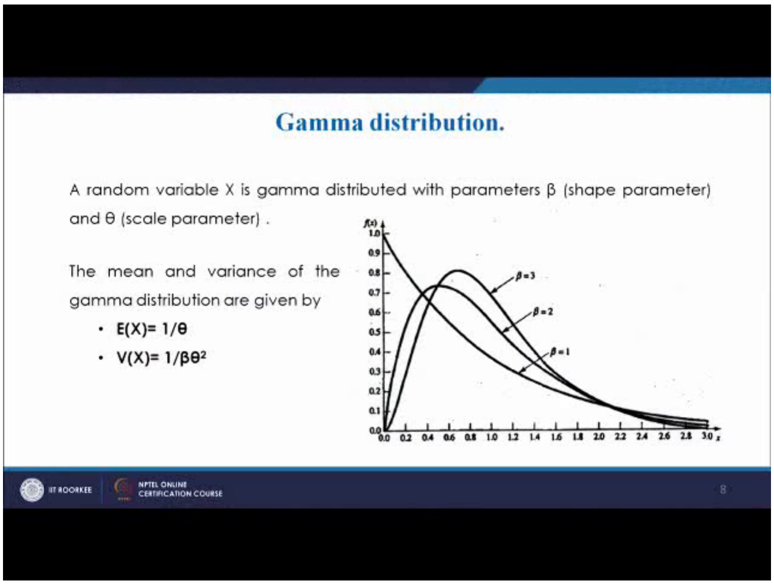

Now we will discuss about this distribution. So, the gamma distribution; this is random variable x is gamma distributed. So, you have the parameters beta and theta you have 2 parameters here beta, which is known as the shape parameter, and theta is the scale parameter. So, the gamma distribution for that before that we must be knowing what is gamma function.

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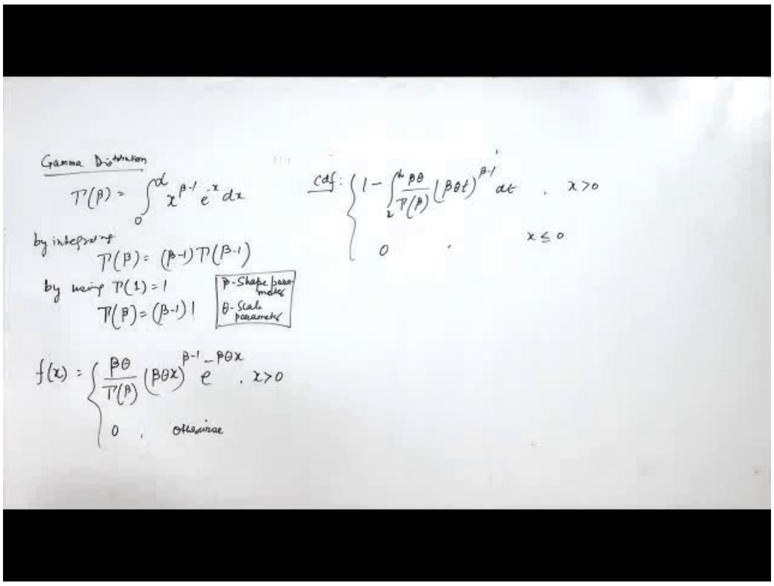

So, you have gamma distribution. So, a function is there gamma. So, gamma beta is defined as integral 0 to infinity x raise to the power beta minus 1 e raised to the power minus x d x. So, this is a function that is known as gamma function and this function by integrating we get gamma beta as beta minus 1 gamma beta minus 1. And further by using gamma one as one, we can write gamma beta can be written as beta minus 1 factorial. So, this is the gamma function and because of that it is known as a gamma distribution.

Now, in this case a random variable X is said to be gamma distributed, when with parameter beta and theta. For that the pdf is given as f x equal to beta theta upon gamma beta beta theta x raised to the power beta minus 1 and e raised to the power minus beta theta x. So, this is for x more than equal to 0, and it is 0 otherwise. So, this is a gamma distribution. So, this is the pdf for a gamma distribution. Now for the gamma distribution the beta beta is known as shape parameter. So, here the beta is known as shape parameter. Beta is shape parameter, and theta is scale parameter.

Now, with the different beta values, you have the scale parameter of one you have with different values of beta such is the curve of pdf in the case of gamma distribution. So, as you see here you have theta as 1 and for beta with value as 1 2 and 3. You have different graphs which is shown here. If you look at this you see the different curves. So, if you look at beta and theta both has 1, what you see is it will be exponential type of curve one you see and it will be this this curve with beta and theta both has one. And once you have beta as 2 and 3, then your curve looks different. So, in this case as you see your mean is defined as 1 by theta and variance is defined as 1 by beta theta square. This is the mean and variance in the case of gamma distribution function. For gamma distribution function the cdf is defined as 1 minus x to infinity beta theta upon gamma beta beta theta t beta minus 1 and d t x more than 0, and 0 for x less than equal to 0.

So, this is the cumulative distribution function for the gamma distribution.

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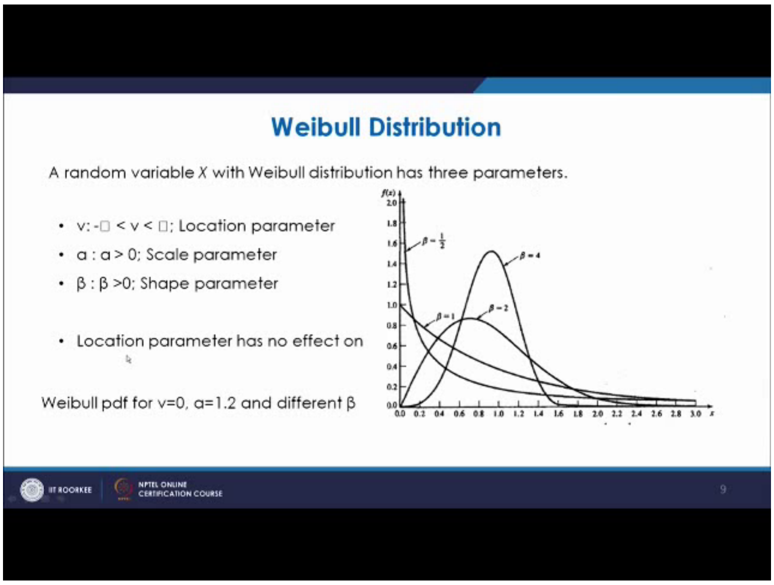

Now we can say we will have a; so, we can see that how with different you know values of beta and theta you get the different type of curves will discuss more about more kind of distribution functions; that is continuous distribution functions and later on about the empirical distribution functions in our next lecture.

Thank you.