**Modeling & Simulation of Discrete Event Systems**
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**Lecture - 24**
**Input Modeling: Multivariate Input Models**

Welcome to the lecture on Multivariate Input Models. So, in this lecture we will talk about the joint distribution functions then the covariance correlation among the random variables. So, as we know that in the simulation many a times we have to deal with more number of random variables.

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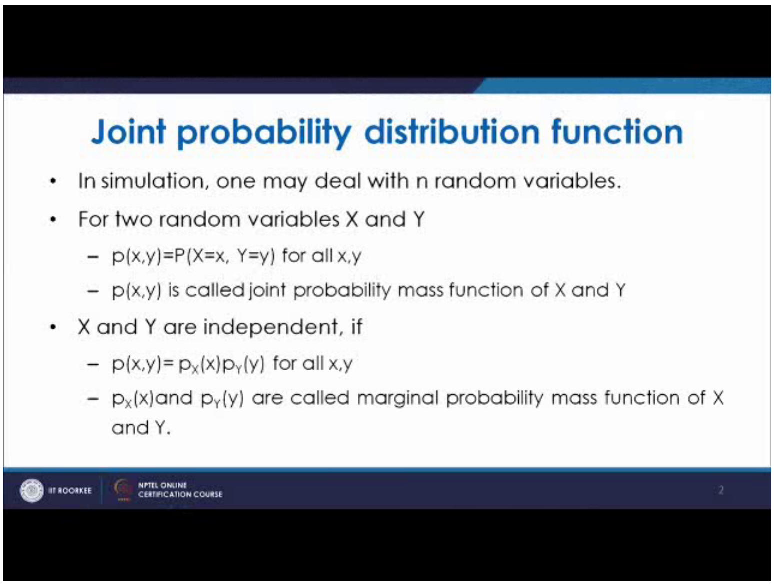


So, in that case where you have 2 random variables: that is when n equal to 2 in those cases p x y means the probability of x you have X and Y as random variable. So, if the X equal to small x and capital Y, equal to small y. So, if that is. So, that is basically known as the joint probability mass function of x and y. So, if p x y is equal to p x into p y, then they are said to be independent and if they are not so if the p x y is not equal to p x into p y in that case they are said to be dependent. And p x x and p x y p y y they are called the marginal probability mass function of x and y.

So, this random variables may be either discrete or it may be you know continuous and we must know how to see whether the 2 random variables which have been defined whether they are independent or not. So, for that certainly if the p x y will be p x x into p x p y y then you can say that they are independent otherwise they are dependent.

Now, before that before we go to that now even this works well even for the continuous distribution functions. Now the so we must know certain definitions for that as we know that.

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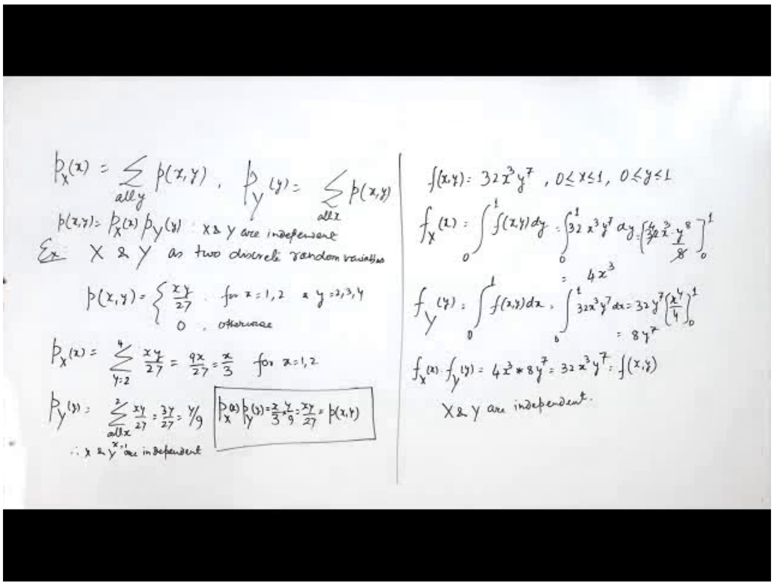


So, in that case p x x is nothing but for all y this is p x y, similarly p y y is for all x p x y. So, this way you find the p x x and p y y and if p x y is equal to p x x multiplied by p y y then we say that x and y are independent. Now in the case of continuous random variables this sign will be you know replaced with the integral sign and in that case d x and d y will come whatever the random variable we choose. So, for example, we can have certain examples suppose we have an example the which is given like suppose we have X and Y as 2 discrete random variables. So, they are defined like this p x y they are defend as x into y by 27 and for x equal to 1 and 2 and y equal to 2 3 and 4 and this is 0 otherwise.

Now, we have to see whether these 2 random variables x and y they are independent or they are dependent. So, for that as we know if the p x y so for that as we know if p x y is equal to p x of x and p y of y then in that case x and y are said to be independent. So, this is the condition and it is already defined what is p x x and what is p y y. So, let us find the p x x. So, p x x will be all the summation of all the values. So, you have p x y.

So, that is x y by 27 over all y. So, y is from 2 to 4. So, y varying from 2 to 4 so, for all the values these are discrete values. So, we are taking every value you know y equal to 2 y equal to 3 and y equal to 4. So, if you do that. So, in that case what we see. So, y will be varying. So, it will be once 2 x by 27, plus 3 x by 27, plus 4 x by 27. So, that is 9 x by 27. So, it will be x by 3. So, we have x value as 1 and 2. So, we get p x x equal to x by 3, similarly we can get p y y p y y will be again summation of all x. So, x is varying basically from for 1 and 2. So, that is x equal to 1 to 2 and in that case you have x y by 27. So, it will be once we take 1 so y by 27 plus 2 y by 27 so 3 y by 27 that is y by 9.

Now, what we see is if we multiply now what we get whether we are getting p x y equal to p x x into p x p y y or not. So, if we multiply these 2 p x x into p y y it will be we have got p x x as x by 3 and this as y by 9. So, it is x y by 27 and it is nothing but it is equal to p x y. So, it means x and y can be said to be independent. So, x and y are independent.

So, this is how for a discrete random variable for the joint probability distribution function we can say and we can find whether they are dependent or not. Let us check it check 1 for, we can also check this you know this type of independence or dependence, for even the continuous distribution function and that we can do once we solve the problems in the next lectures or so.

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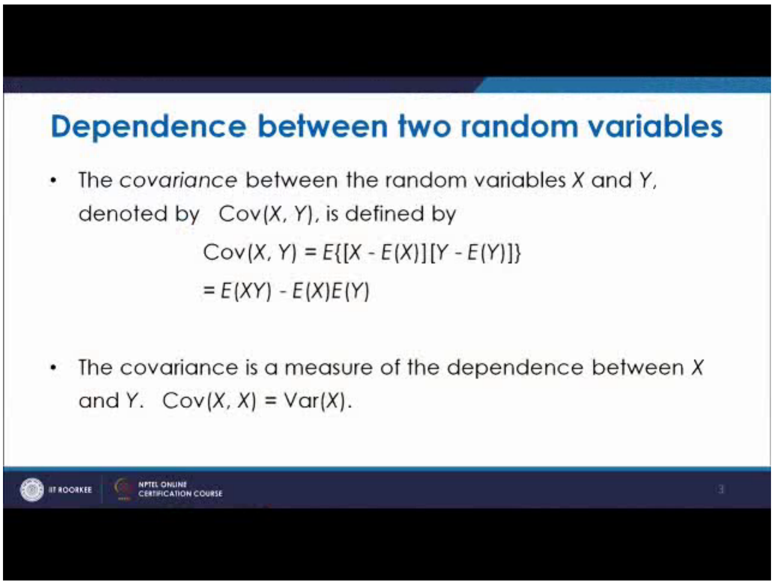


Now, further we must know that, what is the dependence between 2 random variables? So, how the dependence is to be calculated? So, let us see how we do it for a continuous random variable. Now let us take another example where x and y are continuous random variables and the f x y is defined asd 32 x cube y 7. So, in that the range for x is 0 to 1 and y is again 0 to 1. Now in this case again because here we see that the x and y is defined in intervals. So, x and y are basically the continuous random variables, now in this case again we have to see we have to check whether the x and y are independent or not. So, we will find f x x and this will be integral of f x y d y for all y. So y is between 0 to 1.

So, it will be 32 x cube y 7 integral d y 0 to 1. So, it will be 32 x cube into y 7 d y will be y 8 by 8. So, it will be 4 x cube y 8 and the y value is changing from 0 to 1. So, it will be 4 x cube and the next time it will be 0. So, it will be 4 x cube, then f y y this will be again it will be f x y this time for all x. So, it will be d x and x is also varying from 0 to 1. So, this will be coming as integral 0 to 1 32 x cube y 7 d x. So, it will be 32 y 7 into x cube will be x 4 by 4 and this will be between 0 and 1. So, it will be 8 into y 7. So, what we see that we get f x x and f y y, now if we multiply them f x x f y y it will be equal to 4 x cube multiplied by 8 y 7. So, it is 32 x cube y 7 what we see is that in this way we see that this is nothing but same as f x y.

So, since f x y is same as f x x into f x f y y it means x and y are independent. So, this is how you check if the random variable either discrete or continuous their dependence or independence can be checked. Now we have to see the dependence between the 2 random variables so the dependence between 2 random variables for that basically there are certain parameters which are computed, and we compute the covariance and covariance between 2 random variables is defined as Cov X Y we write it as and it will be E X Y minus E X into E Y.

So, as we know that we have the joint probability distribution function or probability mass function. So, we have the E X Y we have to compute E X Y and then we have to compute E X and both E Y and then we have to get the difference of that. So, this way you get the covariance. So, covariance is basically a measure of dependence between X and Y. So, if we try to see what is covariance X X. So, if Y is taken as X it is coming same as the variance of X. So, variance X we find as E of X square minus n times E X or E X whole square. So, this way we get the variance X. So, that is what we get when we get covariance X X.

Now, what do you mean by covariance X Y. So, when the covariance X Y is 0 it means the X and Y are uncorrelated.

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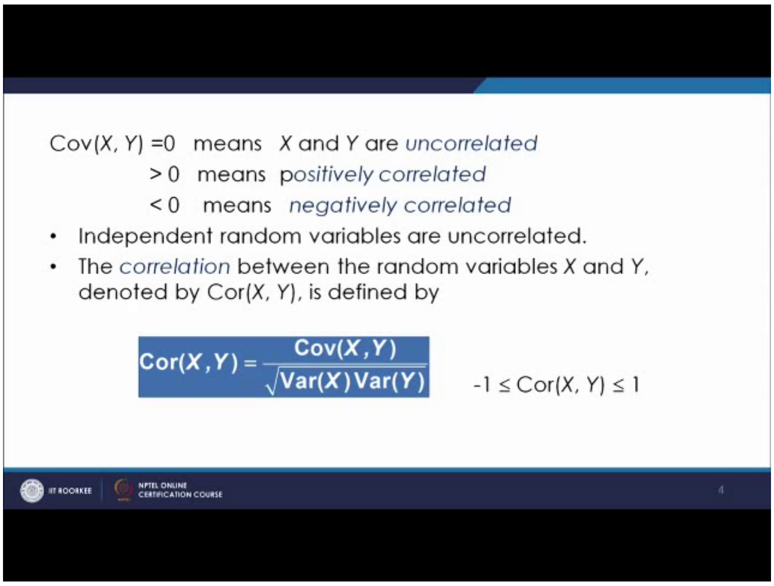


And if it is 0 I mean it is more than 0 it means they are positively correlated and if it is less than 0 means it is negatively correlated. Basically covariance is nothing but you have X 1 minus mu 1 into X 1 minus mu 2 X 2 minus mu 2. So, if it is more than 0 means if X 1 is more than mu 1, in that case X 2 will also be more than mu 2. So, that is known as positively correlated if 1 is 1 value is larger than the mean values than for another random variable also it is values will be larger than the mean values.

So, that is what in the meaning of positively correlated. Similarly if it is negatively correlated means the value is less than 0 in those cases the if 1 of the random variable has values, less than the less than their mean in that case for the other random variable it is value will be more than it is it is mean. So, in those cases what will happen that their product will be negative? So, that is why if it is less than or more than 0 it has certain significance, whenever we have independent random variables they are said to be uncorrelated. So, normally you have the random independent random variables these are uncorrelated

Now, for the correlation there is another parameter which is defined that is rho and that is correlation X Y. So, this is nothing but covariance X Y by under root variance X into variance Y. So, this is how you find the correlation and this correlation will be varying from minus 1 to plus 1. So, if it is closed to minus 1 it is highly negatively correlated and if it is closed to plus 1 it is highly positively correlated the 2 random variables and if it is near to 0 it means they are uncorrelated they are independent random numbers.

So, this is how you find the different you know correlation parameters by which you calculate and you come to the conclusion whether it is dependent or not dependent. Let us see how you compute the correlation and you come to this conclusion whether the variable is independent or not.

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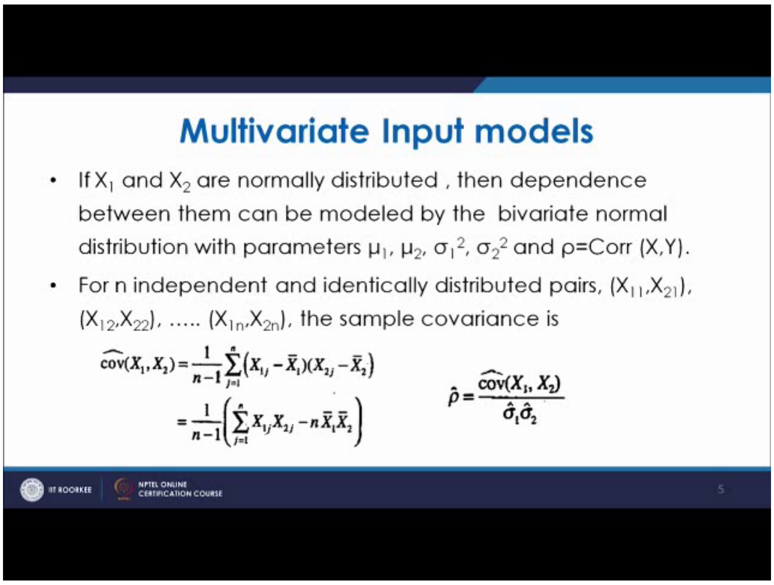


So, before that before we go to further see with the problem where we try to find the correlation let us see that further if you have X 1 and X 2 as the normally distributed 2 random variables. So, their dependence can be modelled by the variate, bi-variate normal distribution.

So, as we know that once you have the data’s. So, in the normal distribution case you have the parameters like mu and sigma that is mu is mean sigma is the standard deviation. So, sigma square will be your variance. So, in those cases you will have mu 1 mu 2 sigma, 1 square sigma, 2 square and then you have to find the correlation X Y. So, correlation X Y as we discussed, it will be rho and if you have n independent and identically distributed pairs like X 1 1, X 2 1, X 1 2, X 2 2, X 1 and X 2 n all that. So, in those cases how you find the sample covariance that is given by this formula. So, as we see you have the covariance of x 1 and x 2. So, we are finding this by x 1 j minus this is x 1 bar. So, this this x 1 bar. So, you have 2 random variables given and they are going like this. So, you have x 1 j and this is j 1 2 n. So, x 1 j minus x 1 bar. So, that is your mu similarly x 2 j minus x 2 bar and then it will divided by n minus 1. So, this will be basically covariance. So, it will be nothing but 1 by n minus 1 j equal to 1 to n x 1 j x 2 j minus n times x 1 bar x 2 bar.

So, this is how you are basically finding the covariance and once you find the covariance the covariance value will be divided with this estimated value of so this is unbiased estimator of standard deviation 1 and 2. So, we had seen in the earlier formula this was here it was under root variance 1 and variance 2. So, this is sigma 1 bar sigma 2 bar. So, this is how you compute the correlation now let us see how we solve if we face certain problems of that type.

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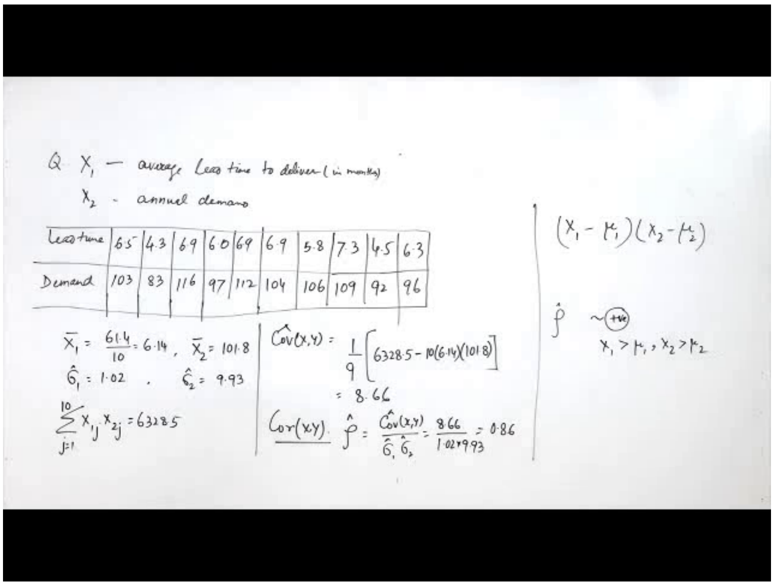


Now, suppose a problem is there like this that you have x 1 which is average lead time to deliver and x 2 is the annual demand. So, you have x 2 as annual demand and x 1 is average lead time to deliver in months. So, these are 2 random variables which have you have the data and this is lead time and demand values are given. So, for the, first value is 6.5 and this is 103, then you have 4.3 83 further you have 6.9 116 you have 6 and 97, you have 6.9 11 2 then further 6.9 and this is 104. So, you have further 5.8 and this is 106. So, for the last 10 years you have 10 data further you have 7.3 and this is 109, then you have 4.5 and 92 and then you have 6.3 and 96. So, these are the data which you have for the 2 random variables 1 is talking about the lead time another is talking about the annual demand.

Now, in this case you have to find whether there is any correlation between them for correlation you need to calculate covariance for covariance you need to calculate summation x 1 j x 2 j and then you need to compute the mean sample means for the 2 random variables and also you need to calculate the variance for them. So, that you get the standard deviation and further you can get the value of covariance.

So, let us if we do the analysis over this what we get is. So, x 1 bar for that we have to add these values and these values once added it has to be divided by 10. So, this value comes, but to be 61.4 and then you divide it by 10. So, it will be 6.14 then similarly you have to find x 2 bar mean value of x 2. So, x 2 also again we are going to add these 10 values and divide it by 10. So, this is coming out to be 101.8. Now we need to compute the sigma 1 and sigma 2.

So, sigma 1 is calculated as. So, this is estimator value basically. So, this is computed to be 0.021.02, similarly you compute the sigma 2 value that is estimated value of sigma 2. So, that comes out to be 9.93 this can be computed using the standard formulas for that you need to have these square and then further we do the necessary computation. So, you can get this sigma 1 sigma 2 x 1 bar and x 2 bar and ever thing what you need is summation x 1 j x 2 j.

So, summation of x 1 j x 2 j and as we know j is varying from 1 to 10. So, what we will do we have do the multiplication of this and this plus this and this plus this and this plus this and this like that you have to multiply and then we have to add it. So, j equal to 1 to 10 this summation this summation will come out to be 63 28.5. So, this is nothing but the summation suppose here 103 into 6.5 plus 83 into 4.3, plus 106 into 6.9, plus 97 into 6, plus 112 into 6.9, plus 104 into 6.9, plus 106 into 5.8, plus 109 into 7.3, plus 92 into 4.5, plus 96 into 6.3. So, this will come out to be 632 8.5 that is x 1 j x 2 j not formula is shown here. So, this value is coming out to be 63 28.5.

Now we have all the values, we have this value known, we have n as 10 x 1 bar we know x 1 bar is known to be 6.1 4 x 2 bar we know. So, and we know also n n is 10. So, we have to get the covariance. So, covariance we will be finding covariance will be actually it will be 1 by n minus 1 so 1 by 9 and then summation of x 1 j x 2 j. So, that is 63 28.5

and further minus n times x 1 x 2. So, n is 10 10 times x 1 bar and x 2 bar x 1 bar is 6.1 4 and x 2 bar is 101.8.

So, this is coming out we have to do this computation and this will come out to be 8.66. So, this is coming out to be 8.66, now we have to find the correlation how they are correlated. So, for finding the correlation you have to divide this covariance x y with sigma 1 estimator sigma 2 estimator. So, sigma 1 estimator and sigma 2 estimator we know. So, correlation x y that is estimator rho this will be covariance x y divided by sigma 1 estimator sigma 2 estimator.

So, it will be 8.66 divided by x 1 bar is 6.14. So, this is sigma 1 bar is 1.02 and then it will be 9.93. So, this is coming out to be 0.86 now what we see is in this case the rho value the correlation coefficient value this value is coming out to be 0.86 which is very much close to 1. It means they are very strongly correlated, it means that whenever x is more than mu in that case the x 1 is more than mu x 2 is also more than it is mean by you can look at here by that. Now in this case if you see the x 1 mean mean x 1 is 6.14 now you see and the x 2 mean is 101.8. So, whenever it is more than that 6.5 in that case it is more than it is mean. So, these both are higher.

This is in this case if you look at this value is less than it is mean it is mean is 6.14 and this value is less than it is mean. So, so basically when you have x 1 minus mu 1 and you have x 2 minus mu 2. So, if the rho value is positive is nearly plus 1 in those case if x 1 is more than mu 1 then x 2 will also be more than mu 2 in that case only it will be positive and in those cases. So, it will be positive. So, you can see that it is positive when x 1 is more than mu 1 in that case x 2 will also be more than mu 2.

So, what we see whenever x 1 is more than mu 1 the x 2 is also more than mu 2 it is mean demand mean and lead time mean. So, these values if you look at this is the lead time mean here it is 6.9 it is more than it is mean at the same time it is also more than it is mean 101.8. So, in most of the cases we see that in most of the cases we may not see in every case, but in most of the cases we see by looking at this statistics also it becomes clear, but then statistically if you find you will see that how they are correlated, if it becomes 0 it means that data is quite random they are independent data, that is what you mean to have it and whenever it will be negatively correlated in those cases if 1 value is larger than it is mean in that case it is value will be smaller than it is mean.

So, in that case if you find the correlation it will be coming in an in the negative direction it will be having the value towards less than 0 certainly the correlation coefficient value will be in between 0 and 1, but it will be closed to minus 0.5.6 or minus 0.7 So, in those cases we say that it is negatively correlated and whenever it comes 0. So, basically what we need to study later on that we will have to compare these values.

Now the thing is that we have already studied the significance of that and we have seen in the earlier lecture that what is the significance of this rho and we plotted these diagrams. And we have seen that by looking at the curve you can say that this rho value how it is (Refer Time: 30:23) closed to 0 then we say that that data is independent.

Thank you very much.