Modelling and Simulation of Dynamic System
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Lecture – 32
Simulation of Simple and Compound Pendulums

I welcome you all in this lecture on simulation of simple pendulum, so which is a sub-module for modelling and simulation of dynamic system course through, which you are going through, here essentially we want to see the simulation examples and in this lecture, we will see that how can we use the Simulink as a simulation tool to simulate a dynamic system.

That dynamic system we are going to take out the simple pendulum as the example and of course, we will be taking many more external effects on the simple pendulum such as we can consider or will be taking the effect drag or effect of friction and the pivot and so on, so here you can see this figure illustrate the simple pendulum okay.

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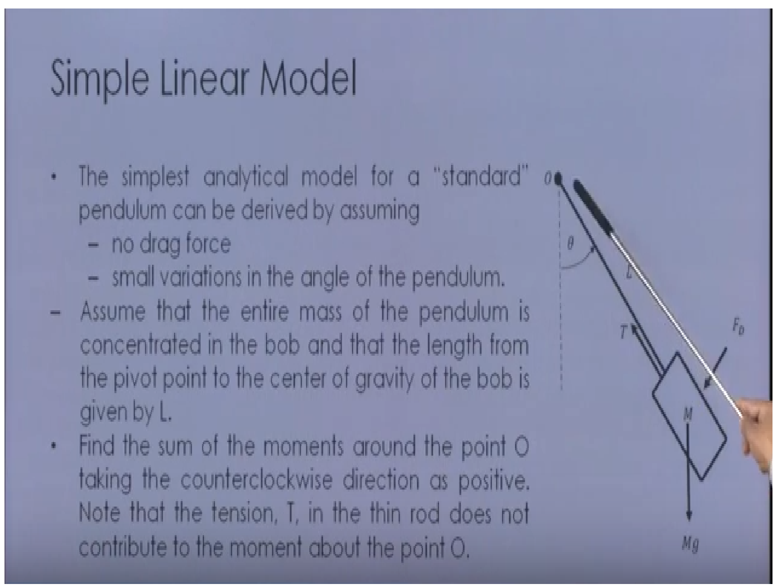

Where M is the mass of the bob and there is a string of length L from the centroid of this mass and of course, there are certain drag forces are assumed and say this is my access through which I measure this angle, now the simple analytical model for a standard pendulum can be derived by assuming no drag force and small variation in the angle of the pendulum.

So we will begin with the simplest possible case and then we will go on making our model more and more involved okay, so here initially as I said we will assume that there is no drag force acting on the bob and we will be taking this small variation in theta okay, and we can assume that the entire mass of the pendulum is concentrated in the bob and that the length from the pivot point to centre of the gravity of the bob is given by L okay.

So first we can say write the equation of motion for the bob and that we can do by taking the moment around the pure point O and of course the counter clockwise, we take positive and of course, you know that this tension is not going to cause any moment because this T is going to pass through this pivot point.

So actually what we are going to do is that we are going to write the equation of motion for this one, now you see in this example, say this is our bob and this length is actually the length L here, this is weight Mg, this mass is M and this is my reference axis with respect to which say I measure this angle theta, so for writing the equation of motion what we do is essentially, we write the torque acting say here about this pivot point okay.

So to do that what we can do is that we can resolve this force that is weight into a component along the length of the rod or a string and another say perpendicular to this one, so since this angle is theta, this angle is going to be theta, so this component will be actually MG sine theta fine, so now this MG sine theta force will be causing a moment here okay.

So that moment will be given by MgL cos theta okay and since this moment direction is clockwise that is opposite to our convention of taking the positive value of theta, so I put a minus sign here and I equate this to that is Ix theta double dot okay, so this is the inertial torque and of course, this is the unbalanced torque, which is acting on the bob okay.

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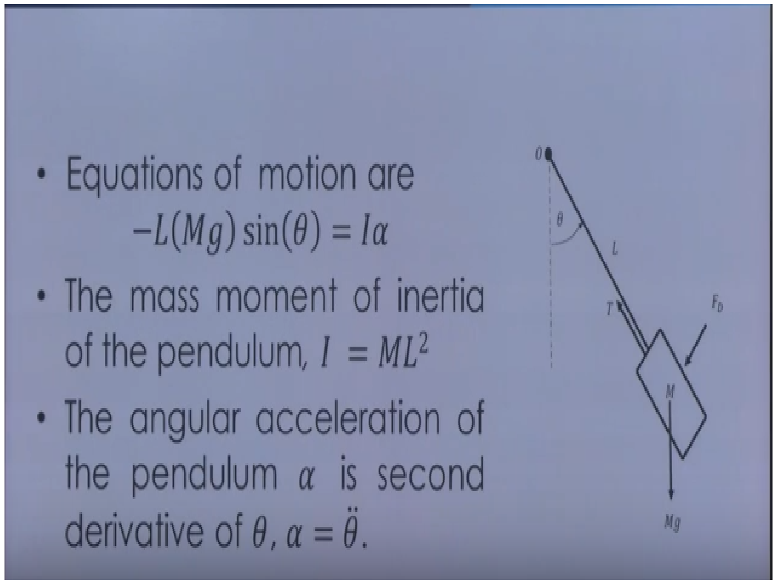

So this alpha is basically the theta double dot and I of course, we know that the mass moment of inertia of the pendulum I is given by ML square and the angular acceleration of the pendulum alpha is second derivative of theta, so alpha=theta, so alpha=theta double dot, as I have written here.

So I can simplify this dynamic equation okay, as this one I can take this to that side, so what I had is ML square theta double dot+Mg sine theta here, so this is not cos theta, this is MgL sine theta, so ML square theta double dot+MgL sine theta, this is equal to 0 or I can write it as theta double dot+I divide it by ML square, so I get G/L sine theta=0, so this is my dynamic equation for the pendulum okay.

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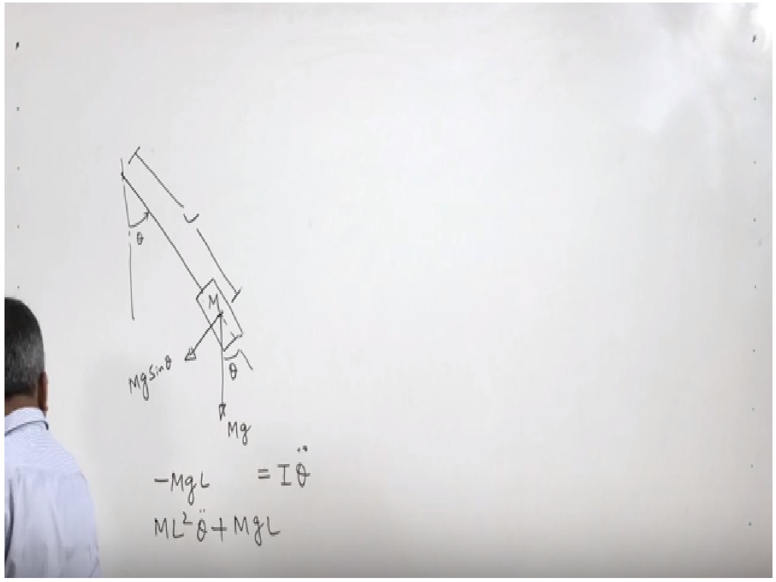

So it is this equation theta+double dot+g/L sine theta=0 okay, so this is the dynamic equation for the pendulum, now this equation you can see that this is a nonlinear equation, our aim is to find out the theta as a function of time, but you can see that this equation is non-linear because of the sine theta term okay.

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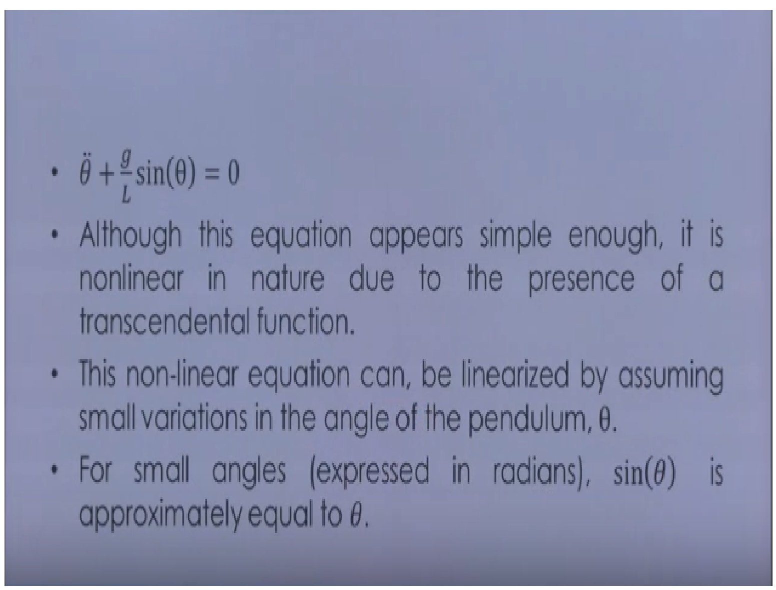

That is it has a transcendental function, so this nonlinear equation can be linearized or converted into a linear equation, if we assume that this theta value is small and we know that for a small value of theta, the sine theta is same as theta okay, so we can linearize this equation by assuming the small value of theta okay and of course as I said for small angle expression radiance, sine theta is approximately equal to theta.

So our equation turns out to be this one, theta double dot+g/Lx theta=0 okay and this equation describes the dynamic behaviour of the pendulum okay, now this equation you can see that this is a second order differential equation, so this equation can be solved by many ways, but here we would like to solve it using the Laplace transform.

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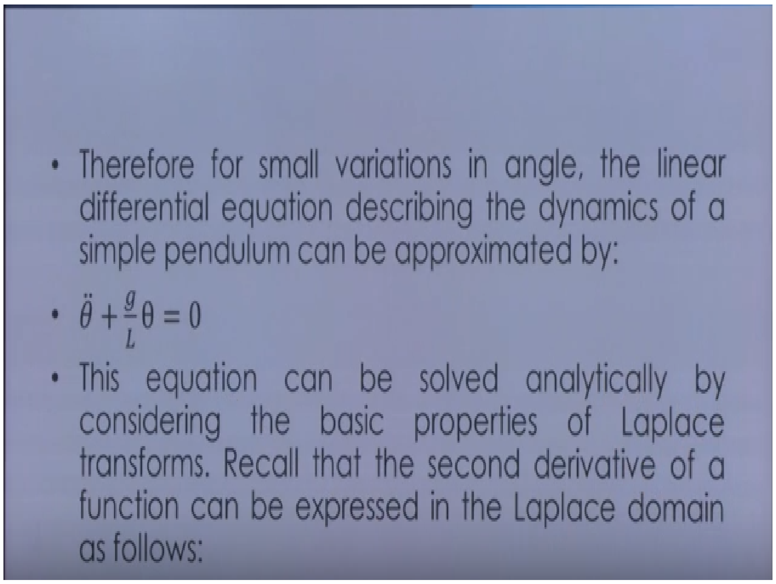

That is initially, we will convert this differential equation into an algebraic equation in S domain and then find out the value of theta in S domain and then, we will again take back the inverse Laplace transform and bring that solution in the time domain form okay, so this can be done, so the equation, which I am talking about is our basic equation that is theta double dot+g/L theta=0 okay.

And you know that in our lecture of the Laplace transform, which we were discussing, you see the Laplace transform for a second derivative is given as S square Laplace transform of the function–sx the value of function at say T=0–value of the derivative of the function at T=0 okay.

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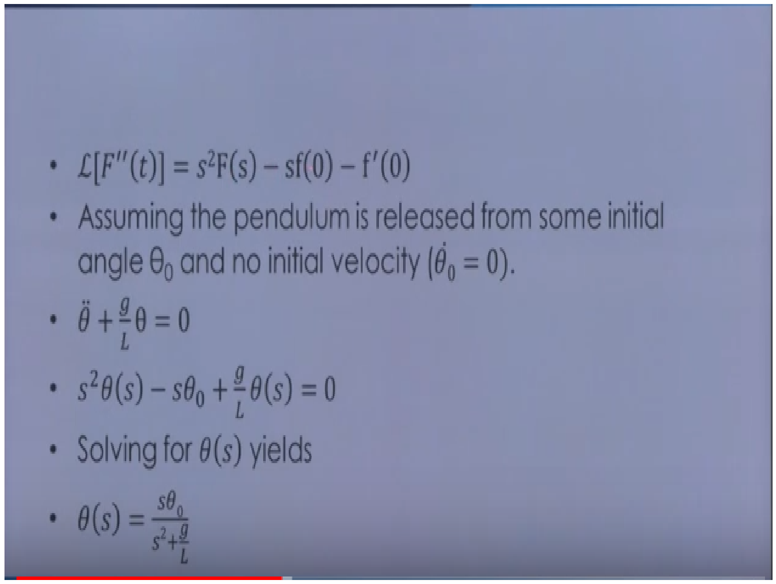

So in this case suppose if I assume that the pendulum is released from some initial angle, say theta 0 and there is no initial velocity alright, first let us write a generalized expression for this one, theta double dot, I can take help from above equation and so theta double dot I can write its Laplace, I can take the Laplace for this.

So this is s square theta s corresponding to this term minus we will have s here, and here we will be having the theta 0 value and this term I am taking this as 0 okay because I am assuming that there is no initial velocity, so this term will be 0 here okay, so this is there+we have g/Lx theta in Laplace domain and that is equal to 0 okay.

Now from this equation, I can solve for the value of theta s okay, so this theta s will give me s theta 0/s square+g/L okay, so this is what I get for the theta s value alright, now I can convert this solution from Laplace into the time domain by taking the inverse Laplace, now you see here what we are getting is S theta 0/s square +s/L.

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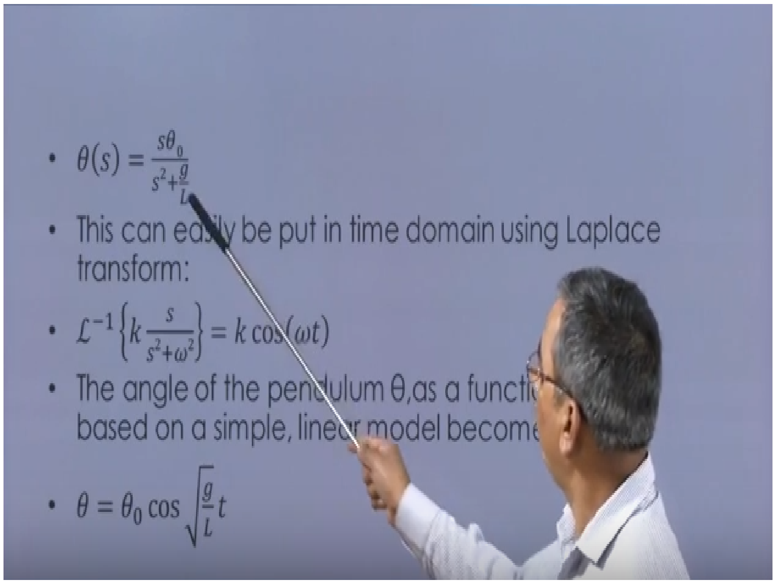

So this is basically s upon s square+omega square form alright, so the Laplace transform for some constant Kxs/square+omega square is given basically k cos omega t okay, so this is given so I compare this expression with this one and take the inverse Laplace for both the sides, so this becomes my theta t and of course I have theta 0 and this value will be that is from taking inverse Laplace will become cos of omega will be here basically if I compare this.

Then omega is my under root g/L, so this is under root g/Lxt so this way I can write the general expression for theta that is theta=theta 0 cos under root g/Lxt, so this is my general solution for this expression. Now to visualize this function okay for comparison purpose, we can make a simple Simulink model here.

So say I have a clock and clock signal is given to, this is actually the square root of g/L, which is gain basically alright and then I take the cosine of this term, so here basically I get this term and I take a cosine of this one and then I multiply this with the theta 0 value, so I have the theta 0 value from here, I multiply this and this is what I get the theta.

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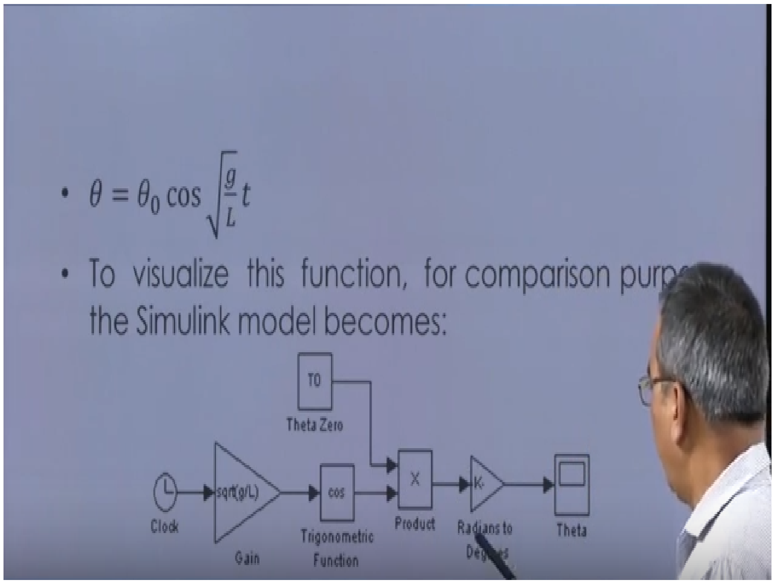

So here is my theta and then of course, I can use my radian to degree conversion using the simple formula that is theta/pi=D/180 so my D will be 180/pi x theta, so using this formula, I can convert radians to degrees and I can see that value of theta here, so this way I simulate the behavior or simulate the behavior of the pendulum in terms of the rotation theta.

Then let us try to make the model a little involved, let us assume that our model is non-linear okay and how do we assume this non-linearity that is the previous case, we remove the non-linearity basically assuming the theta value to be very small, okay so I removed that as assumptions from here okay and say that my theta value is not small okay.

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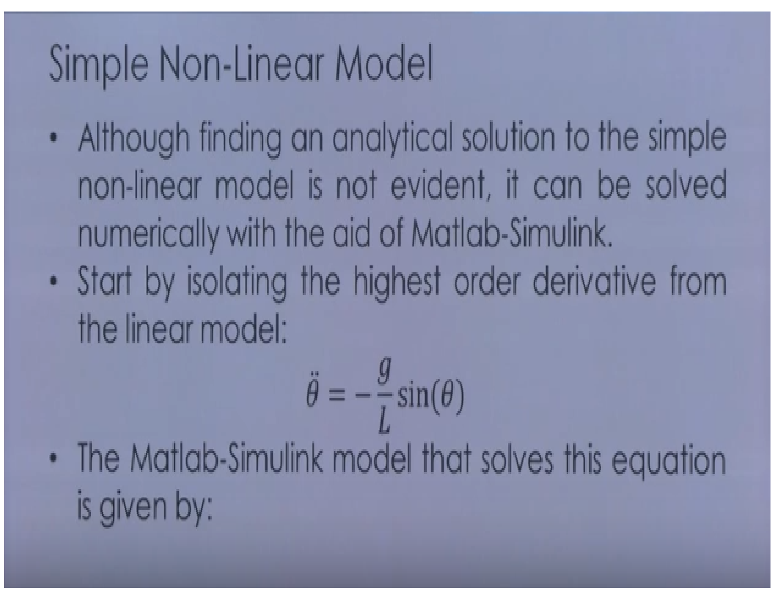

So we can create a MATLAB Simulink model to solve for this equation, okay so this is our equation of course as I said, I am not taking my theta to be smaller here okay, so this is my basically equation, which I want to simulate alright, so here you can see that say let us say that this is the point where I have the theta double dot value okay.

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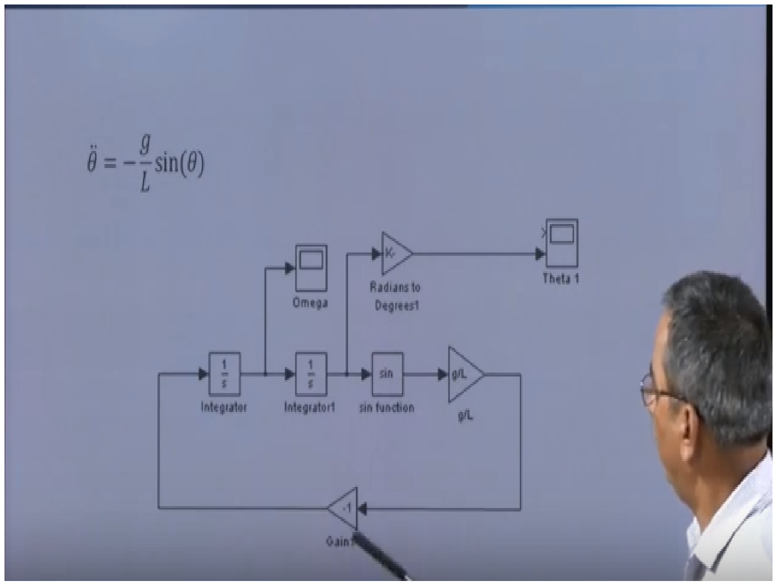

So I put an integrator here, 1/s, I get a theta dot value here and of course I put a display here to see that okay, so that I am calling say as omega, so this is my theta dot value. I further integrate that basically say using integrator 1 and I get a theta here, now this theta is in radian, I put a gain basically in order to convert the radians to degrees and of course I can view that theta value here, so here I have got the theta value.

Now I put a sine function here, so basically I will be getting this sine theta here, then this sine theta Ixs/L here okay, through this gain, so I get here g/L sine theta and again you see that we have got a minus sign, so I put again gain here with the gain as minus gain block with the gain as -1, so this is what I get minus g/L sine theta and this is nothing but the same theta double derivative okay and the rest of the say steps are repeated okay.

Now let us take further complicate our model basically taking the non-linear model including a constant aerodynamic drag force, so let us assume that there is a constant aerodynamic drag force fd is acting on the bob okay, now we can assume that the aerodynamic drag force acts only on the bob and the drag on the thin rod is negligible. The drag force imposes additional moment about this point O as we can see from here.

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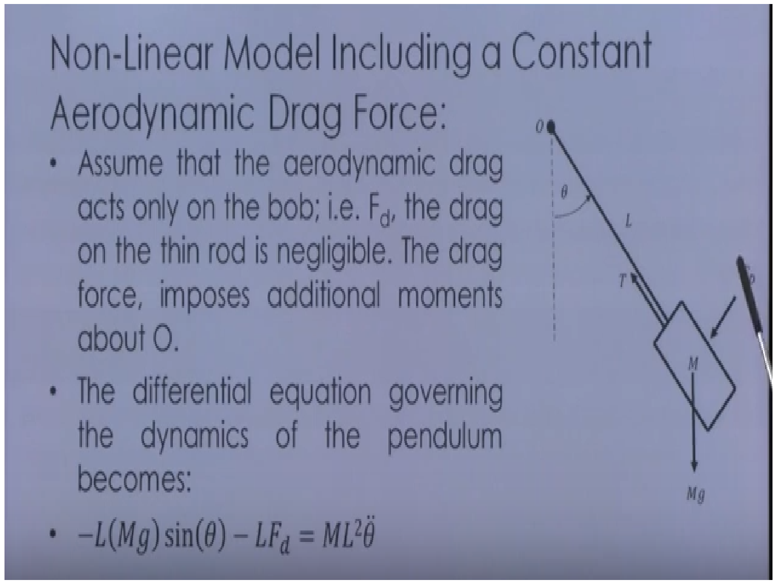

So now if I want to write the differential equation governing the dynamics of the pendulum, what are the unbalanced moments first I write and then I will equate it to the inertial moment okay, so the unbalanced moment here as I was telling MgL sine theta with the minus sign opposite to our assumed positive direction of rotation of the pendulum minus this Fd into this length as I said is L, so –LFd and this is our I alpha or this is ML square theta double dot okay.

So this way we can write that and now I can simplify this equation, so this is our basic equation alright, this is the moment because of the drag and of course this one because of the weight of the bob and this is the inertial moment fine, now we can isolate the highest order derivative here, so our theta double dot, I divide both sides basically say ML square, so this is what I get –g/L sine theta of course M and L, one M and this L gets canceled.

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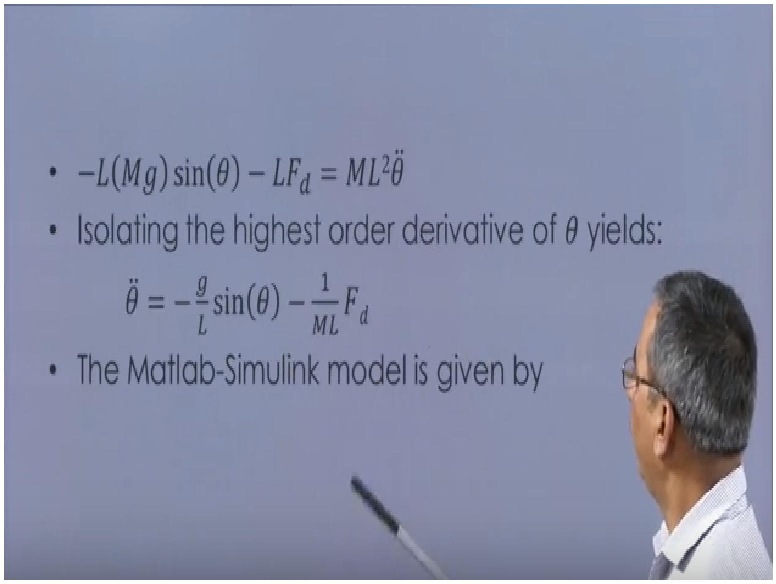

So I have –g/L sine theta–I am dividing this term basically ML square, so I have -1/MLxFD okay and corresponding to this equation, I can prepare Matlab Simulink model, so this Matlab Simulink model I can realize here, of course again we go by this expression okay. Now you see, say this value is my theta double dot, I take a first derivative here.

So I get omega 1 okay and then I take the first integration here the integrated 2, I get that is omega or theta derivative, theta dot and then I take another integration, I get the theta value okay, so here again the radian to degree conversion and of course we have got the display and here we have the theta value then I take the sine theta, this model is similar to what we have seen in the previous slides.

So I have sine theta and then I multiply with g/L and I take a minus sign here, so what I get, I get the minus g/L from here. Now this we have to model this term basically so to model this term what I do is that say I have got the drag force Fd okay, I will explain this a little later, so I have this Fd and this FdI/1 here and then I put a minus sign here okay, and both these did appear will basically give me the theta double dot.

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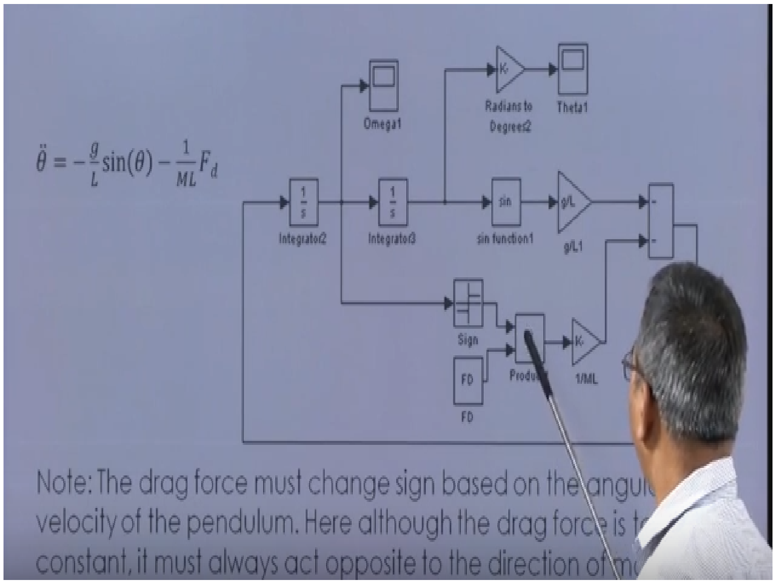
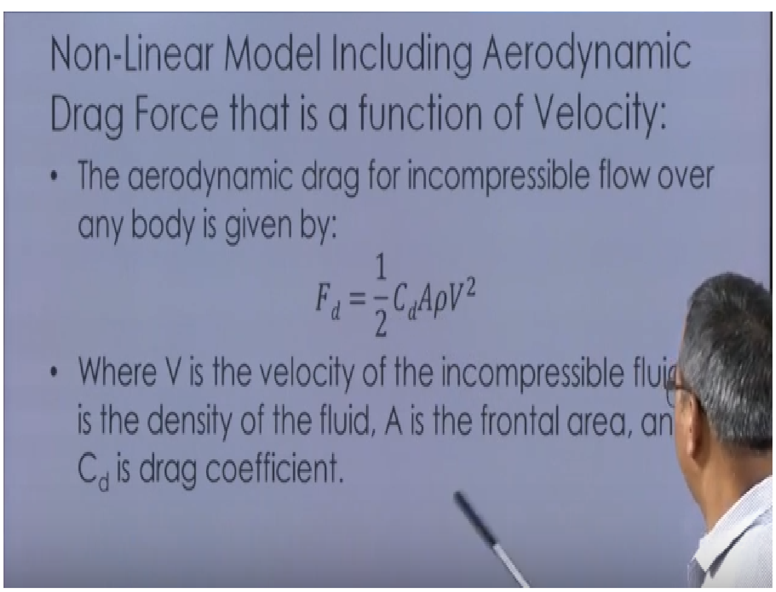


So here I get this theta double dot term. Now you see that this drag force actually changes sign based on the angular velocity of the pendulum okay, so here although the drag force is taken as constant, it must always act opposite to the direction of motion, so to take care of that, here we take the omega here and of course, we put a sign here and then this is given to be multiplied with Fd value, okay.

So this is how it is done then let us further add further features to our model, so here the non-linear model including aerodynamic drag force that is a function of velocity. The previous one we had the constant drag force was assumed, now we can take if my drag force is a function of velocity. **(Refer Slide Time: 22:30)**



So I can write the expression for that aerodynamic drag force for incompressible flow over any body and of course this is given by Cd rho V square, where Cd is the drag coefficient, A is your frontal area, V is velocity and rho is the density of the fluid, now the drag coefficient for the cylindrical body of course is a function of Reynold number RE, which is given by rho VD/meu, of course where meu is the viscosity okay.

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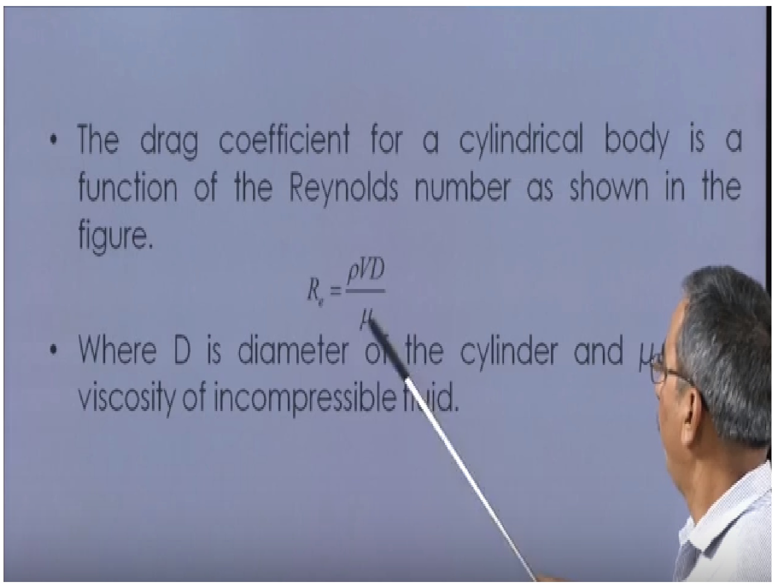

Viscosity of the incompressible fluid and D is the diameter of the cylinder okay, other terms we have defined, so we can use the standard literature available in order to find out the CD value for a given Reynold number okay, this is basically Reynold number is expressed in terms of the kinematic viscosity, which is basically meu / rho, anyhow for a given Reynold number, we can find out what is the value of CD is going to be and that CD value can be substituted back.

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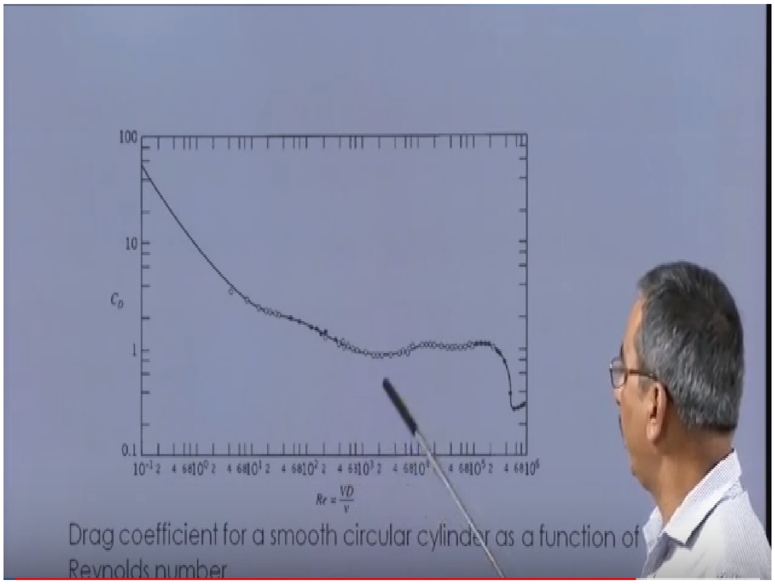

So to create the Matlab Simulink model, we have these 3 expressions, this is expression for Fd and here there is Cd term and of course this Cd I get from the plot for the Reynold number verus Cd okay, so I get the Cd value, I get the Fd and then I can substitute this F D in this expression in order to create the Simulink model okay.

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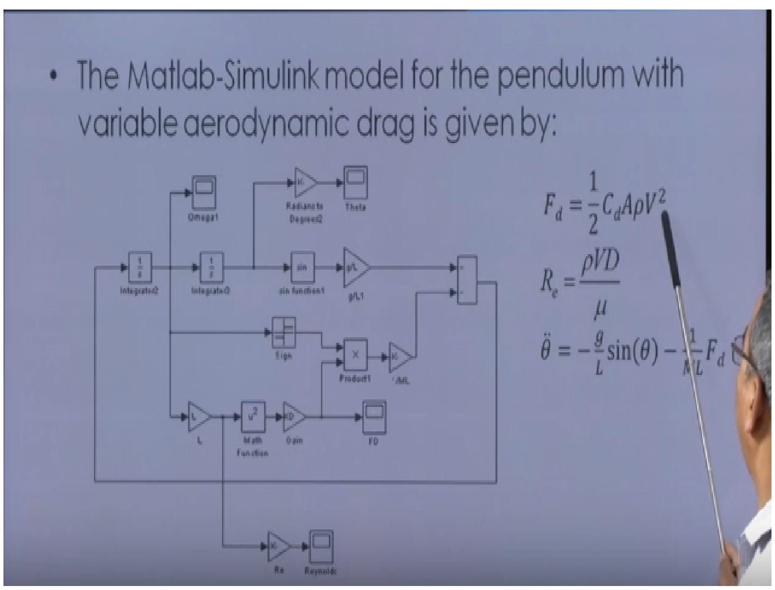

So rest of the things are same because we are building on the basic model only, so here we have theta double dot, first integration here, so I get the theta dot; second integration, I get the theta and I view the theta here in the display, I have to get the sign here, then g/L being multiplied by g/L, so this is the minus sign here, the same thing okay and this portion also we have discussed the same as the previous case, the only thing is that this Fd is being determined.

Now from here okay before being multiplied, so here we get the theta dot value and of course there is one characteristic length is there, then we have a Math function and from here basically I by putting a gain, I get the Fd value okay, so here this is basically V square term corresponding to here and of course I can always get this theta in order to evaluate the Reynold number and see the value of it okay.

So this way we can do that then we can also include apart from the aerodynamic drag, the friction at the pivot also okay, so as I said there I could potentially be a significant amount of friction at the pivot point and the Matlab Simulink model including the aerodynamic drag and coulomb friction at the pivot point can be created again by taking from that basic model.

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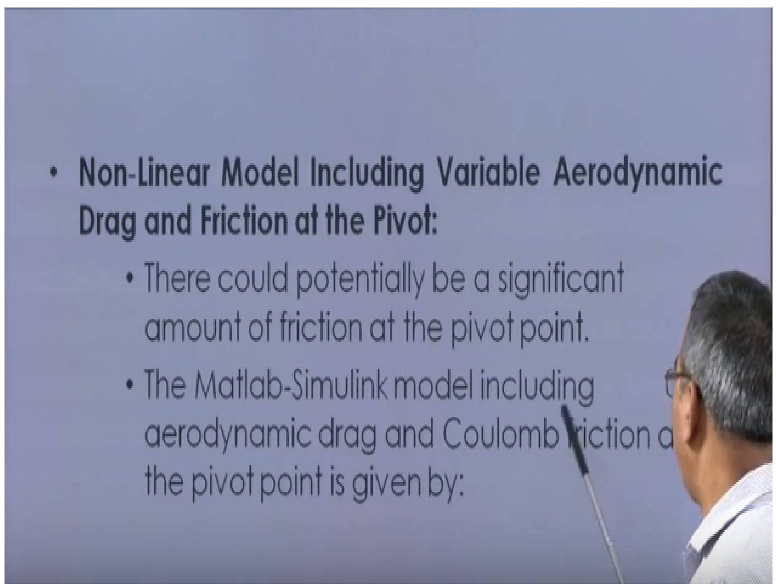

So these are the basic model, which we have already seen theta double dot, here we have theta dot and here we have theta, so sine theta here and multiplied by g/L–is taken here okay, so this part is actually I did in the previous part, these things are same as the previous one, so here is basically we take omega or theta dot and here is the frictional torque and we get product of it and then we divide by 1/ML square because you see our basic equation.

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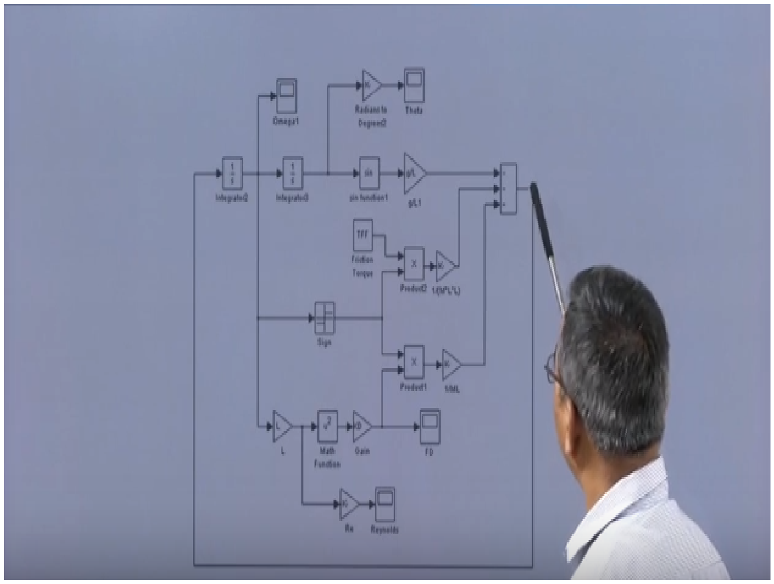

We have been dividing by ML square in order to get the theta double dot term okay in the left hand side and then this can be subtracted here and we have the theta double dot value coming from here and this way, we can create a Simulink model for the pendulum okay.

So this is what I wanted to tell you, we can see the response of the system based on an initial some pendulum angle, we can see the response, this response decline okay, the magnitudes are declined and we can take some value here I am giving some value for which this simulation have been done okay, so you can try yourself by giving the different values and observing the behaviour okay, thank you.