Modelling and Simulation of Dynamic System
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Lecture – 29
Frequency Response

I welcome you all in this lecture on frequency response and this is a sub-module for the course, which you are going through that is modelling and simulation of dynamic systems. In our previous lectures, we have seen the dynamic model, the ways of deriving the dynamic model of systems and one of the method has been using what we say that we find out the equation, equilibrium equations okay, got the expressions in terms of the differential equations.

Now once you have these models, then the thing is that we are interested in finding out the response when it is rejected to certain input, okay, how that is the output is going to be, so previously we have seen when the system is subjected to, say the step input, ramp input, it is always interesting to know what happens, how the response of the system appears when the system is subjected to a sinusoidal input.

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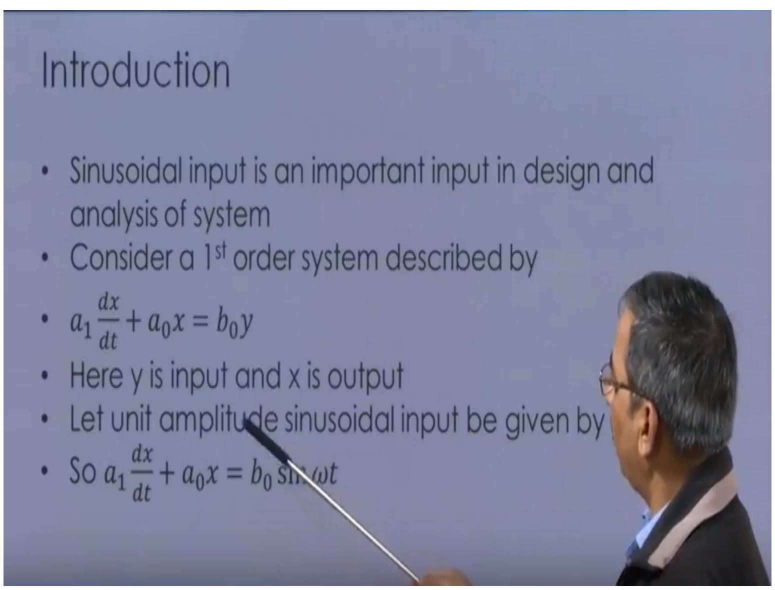

So this is what we are going to see today under the heading what we call it as frequency response and our aim of here is to see if my dynamic system is subjected to a certain sinusoidal input, what the output is coming okay or how the output, what is the output of it, okay, so as I said sinusoidal input is an important input in design and analysis of systems and say if you take a first order system.

Which is described by say this differential equation of first order differential equation say a1 dx/dt+a0x=b0y, where this Y is input and X is output ok now suppose this system is subjected to a certain sinusoidal input, let that sinusoidal input be of unit amplitude, a sinusoidal input, which is given by say Y=sine omega t.

Now when this system is subjected to this input, our interest is to see that what is happening to the X or how we can determine the X, so I can substitute this Y=sine omega t here in this expression and this is what we get b0 sine omega T and of course my left hand side remains as it is, so what I have is what I have is a1dx/dt+a0x=b0 sine omega t okay.

So this is what we are going to have. Now the thing is here we know that the sinusoidal function have the property that when differentiated the result it also sinusoidal with the same frequency ok, so if I differentiate sine omega t then what I will get basically is omega cos omega t ok.

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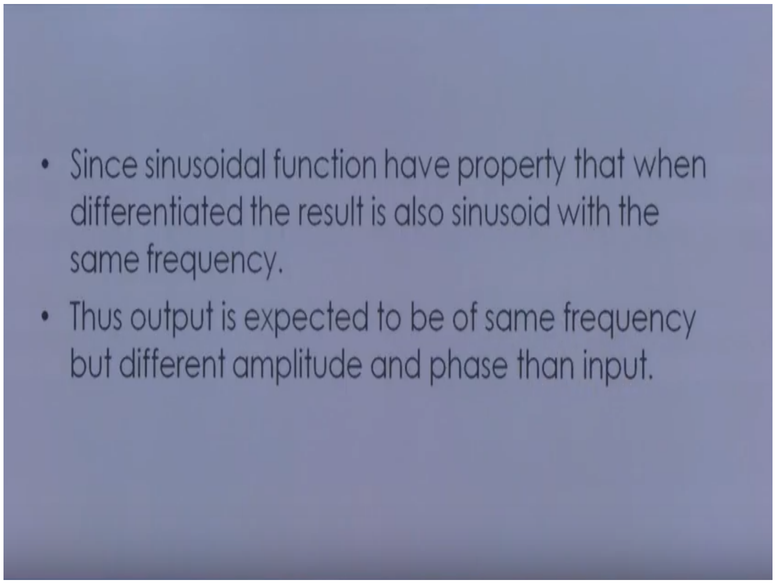

So you can see that this differentiation also has got the same frequency as that of the input, of course this amplitude has been changed, so thus output is expected to be of same frequency, but different amplitude and phase than input is there ok, so this is what is going to be and we can come to this conclusion that when system is subjected to a certain sinusoidal input, the output is also going to the sinusoidal.

We can conclude that because this right hand side is a sinusoidal, so naturally the left hand side also has to be sinusoidal, but of course as I said you are going to have the different magnitude and the different phase angle ok that we will be seeing further, so these sinusoidal signals, which I have been talking about, they are better represented by a phasor ok, say for a sinusoidal signal.

If I talk about say v=V sine omega t+pi, here V is the amplitude, omega is the angular frequency and pi is the phase angle. Now this sinusoidal signal can be represented in a phasor by a line of length say absolute value or magnitude of V making at angle pi with the phase reference axis, so this is how we can do it okay.

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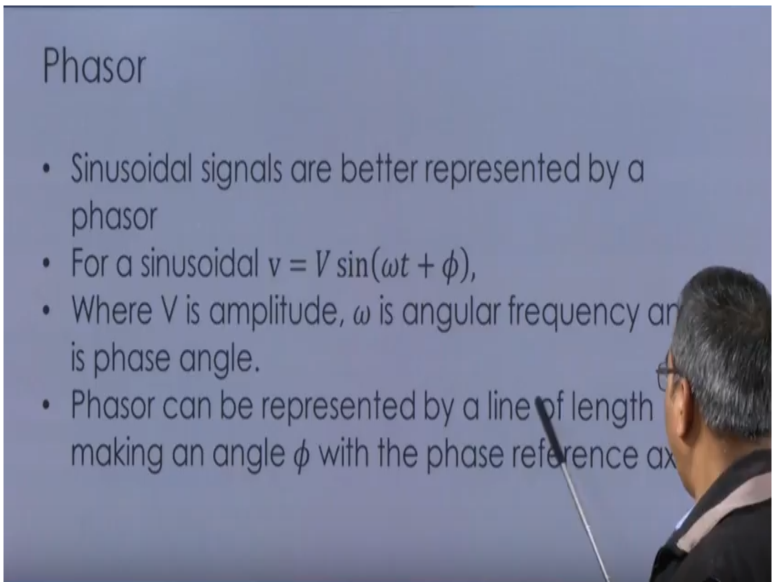

So we can see that the representation of the sinusoidal signal by a phasor, so suppose this is my sinusoidal signal okay, and here you have the angle relative to axis OA hence time, so here you see that this begins with say pi here, so there is a phase angle of pi and say this magnitude is represented over here ok and of course that has got say value of V alight.

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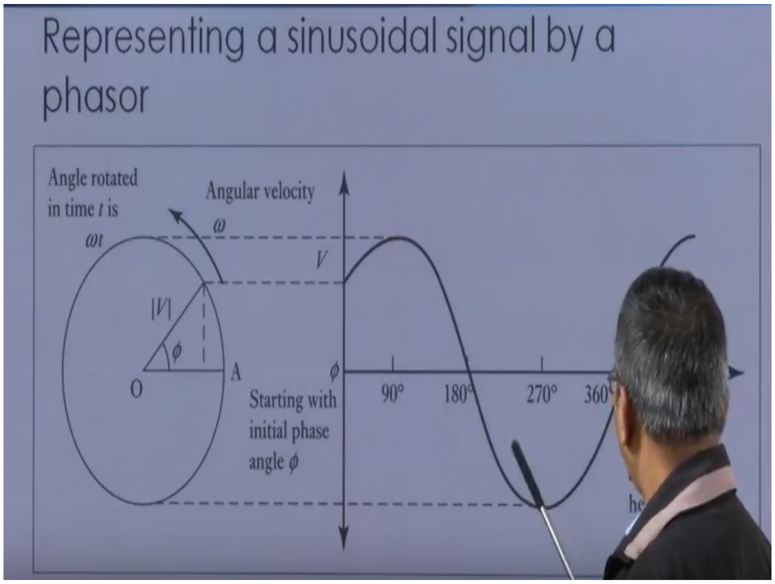

So this sinusoidal signal can be represented by a phasor here okay, and as I said this phasor has got an magnitude V and at a phase angle pi, so another interesting thing is that these phasor can be described by complex numbers okay, and we know that the complex number is represented by X+JY, where X is the real component of it and Y is the imaginary component of it.

So here you can see that we have real components say X and imaginary component Y, then the complex number can be represented as X+JY okay, so on this graph you see that the imaginary component is represented along the Y axis and the real component is represented along the X axis, so corresponding X, Y actually these are the cartesian point and these cartesian point X, Y actually represent this complex number okay.

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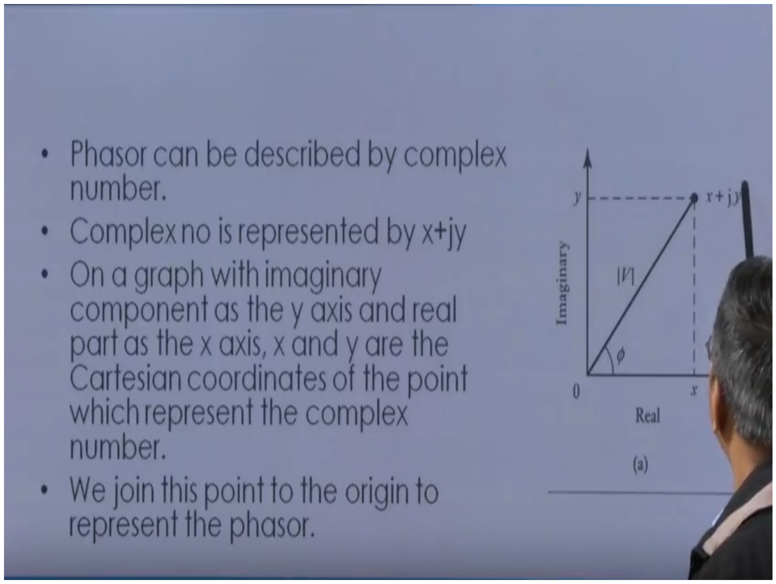

If this X,Y point I join with say origin of this coordinate system, then this represents the phasor, so here the phasor in this you can see that the phase angle that is we can find out if I just take the tan pi of it, so this tan pi will be Y/X and the length of the phasor will be basically under root X square+Y square, so the length of the phasor is under toot X square+Y square and the phase angle is given by tan pi=Y/X okay.

And again you see that this X is actually the horizontal component of this, so I can write X as say amplitude of Vx cos pi and Ys the vertical component of it, so I can write this Y as magnitude of Vx sin pi and then I can represent this sinusoidal signal as a phasor V=x+jy or V cos pi+j sine pi okay, so you can see that what basically we have done is that the sinusoidal signal we have converted into a phasor okay.

So sinusoidal signal has been converted into a phasor with the say phase angle pi and the magnitude V, so there are certain interesting thing, which we can look at for the phasor say for example here, say we have a phasor of unique magnitude along the say real axis okay, so I can represent this phasor as 1+j0 because here the imaginary component is 0, okay.

Now suppose I rotate this phasor by 90 degree like this okay, then what will happen, this will be coming to this position okay, and this phasor will be given by what only j component ok because real component is going to 0 here. Now you see that what essentially we are doing is that if I am going to multiply this 1+j0/j okay.

Then I am going to this one okay, so what does this actually mean is that the multiplication by j to this complex number is actually meaning that we are turning this phasor by 90 degree okay, again let us say that this phasor further turns by 90 degree that is it aligns with the real axis along the say negative direction here okay, now this can be represented by -1, okay.

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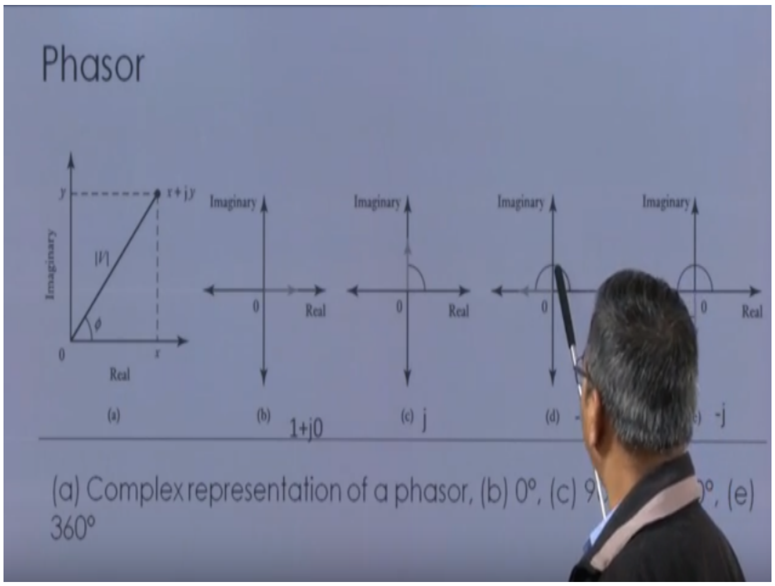

So and of course this operation we can achieve say it is only 90 degree rotation with respect to the previous one, so if I multiply this by j then what I will be getting is J square and the j square is nothing but -1, okay, so I get the -1 or what I can say that with respect to this one, there has been 180 degree rotation okay, or 180 degree rotation means that what I have done is that I have multiplied this by j square okay.

So if I do this then this is what I am going to get -1 okay, that is for a 90 degree rotation, you have to multiply with J, for 180 degree rotation you have to multiply by j square and similarly for 270 degree rotation, you have to multiply this by J cube okay, and you will be getting it –j because when you multiply with j cube, this is what you get and this you can write as jxj square and this is j square is -1.

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So this is what you get as the –j okay, so this is 1+j0 is 0, 90 degree is represented by j, rotation represented by j, then we have 180 degree and 270 degrees okay, they are some corrections here. We can take an example say this is not there, say your V is given by say we have V=20 sine omega t+45 degree okay, say this is my sinusoidal signal and we can take this example to represent it by a phasor, okay.

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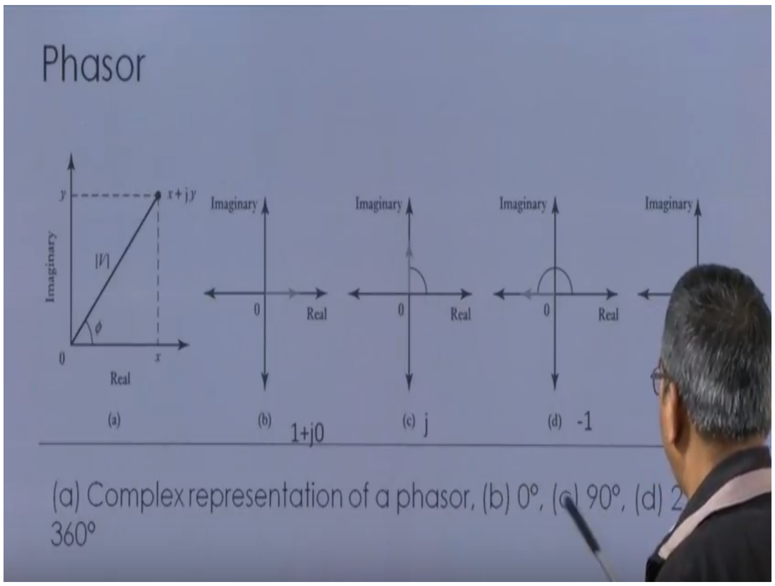

So here we can see the magnitude of this is 20 and so what we will be having basically is this, this is our real axis, here is our imaginary axis okay, and this is at an phase of say 45 degree okay, and this magnitude is 20, so we can represent this phasor say as w=20 cos pi+j 20 sine pi okay.

So this is how we represent and we can substitute the value that is 20 cos 45+jx 20 sine 45, so this is the phasor representation of this sinusoidal signal. Now let us see the phasor equations okay. Suppose, we have the expression X=sine omega T okay, then if I take the first derivative of it dx/dt=omega cos omega t or what I can write this as this is omega.

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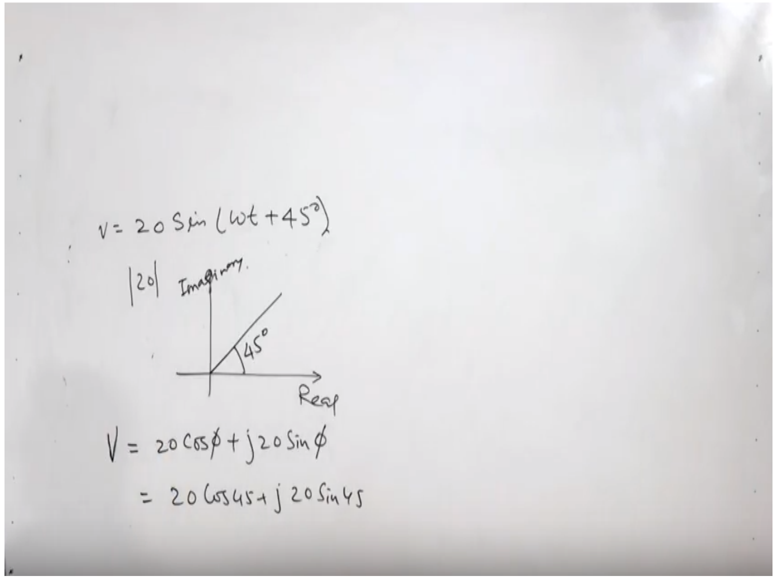

I can explain this cos omega t in terms of the sine itself as omega sine omega t+90 degree, so what does this mean, this means that the differentiation operation has what it has done is that, it has multiplied the previous magnitude by an factor omega and it has changed the phase angle by 90 degree, okay.

So the differentiation has resulted in a phasor with length increased by omega times and it has been rotated by 90 degree with respect to the original phasor, so this is what does the differentiation operation, so actually the complex notation, differentiation means as I said multiplication of the original phasor by J omega.

So if in phasor operation if I have a phasor written in terms of complex notation, then the differentiation essentially means that I multiply that phasor by factor j omega, now this multiplication by J will be rotating that phasor by 90 degree with respect to the previous one and the multiplication by omega will be giving or will be multiplying the magnitude by omega okay, so this is what is done.

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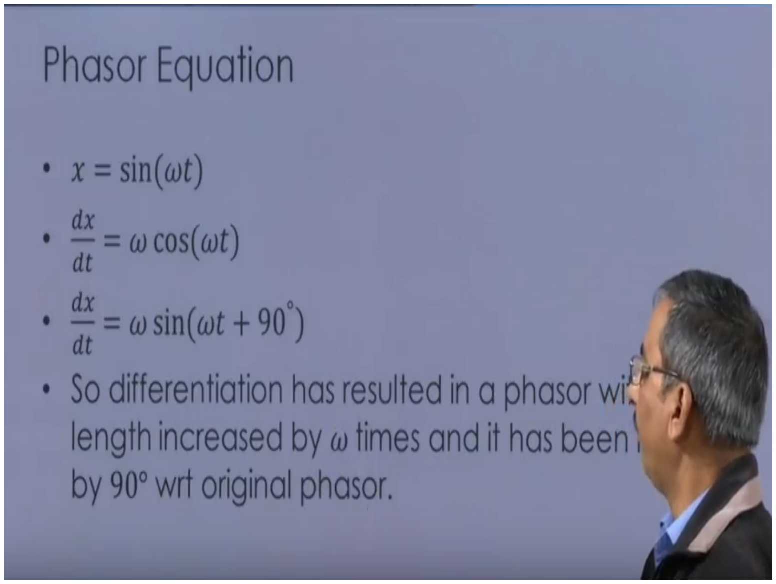

So we can take the differential equation like this say a1dx/dt+a0x=b0y the same expression and this can be written as a phasor equation as what see here differentiation involved, so what I do is say the phasor is x, then I multiply this x/j omega, so this j omega is being multiplied and x.

I have and this a1 value is my original coefficient here plus we have the a0x=b0y okay and from here I can get the ratio of output that is x divided by input and this is b0/j omega a1+a0. So this is what I get now this is my ratio x/y=b0yj omega a1+a0 and now, I hope you remember that for the first order system when we found out the transfer function okay that is we had the first order system okay.

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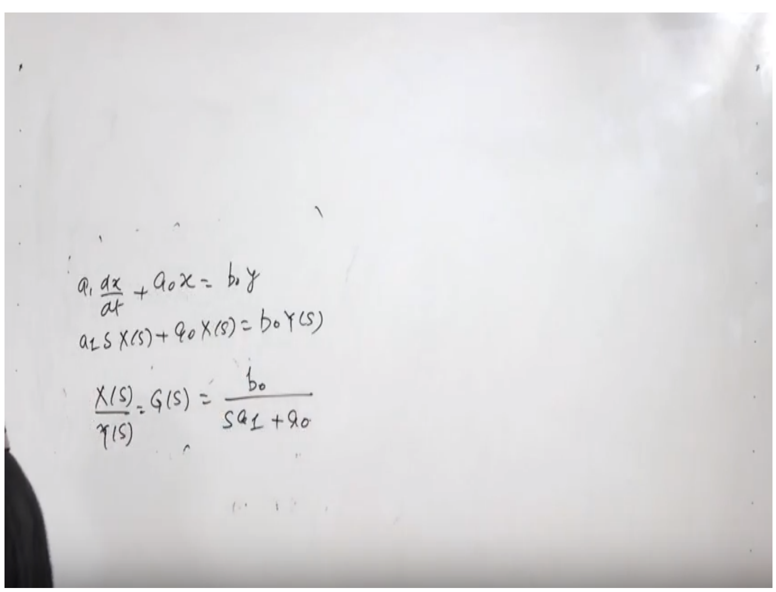

That is a1dx/dt+a0x=b0y okay, so when we found out the transfer function for this, what we did, we took the Laplace transform of both sides and we took the initial conditions as 0, so what we had is a1sxsaA0xs=b0ys, so from here we got Xs upon Ys, which we called as the transfer function and that was b0/sa1+a0, so this is what we have, b0/sa1+a0, so remember this is our transfer function, which is defined okay.

And this is what we have worked out for the frequency response fine, so if by seeing these 2 expressions, we can see that if you want to convert from transfer function to the frequency response function okay, in place of s what we have to do, we have to just replace this s/j omega.

So if you replace this s/j omega, you will get the frequency response function or what I have said is the frequency response function or frequency transfer function for steady state condition can be defined as Gj omega, we are placing this s/j omega and this the output phasor divided by the input phasor okay, so this is the output phasor divided by the input phasor okay.

Now let us see the frequency response for a first order system okay, so suppose I am interested in finding out the frequency response of a first order system, we can take an example, a first order system has a transfer function say Gs=1/1+tau s and where tau is time constant. Now what we are interested in, we are interested in finding out the frequency response function again okay.

So the frequency response function for this case will be what we are doing, we are replacing this S by is j omega, so it is Gj omega=1/1+s is being replaced by J omega and tau, so j tau omega, so this is there, now we can find out the magnitude as well as the first angle once we have the this way so basically ok.

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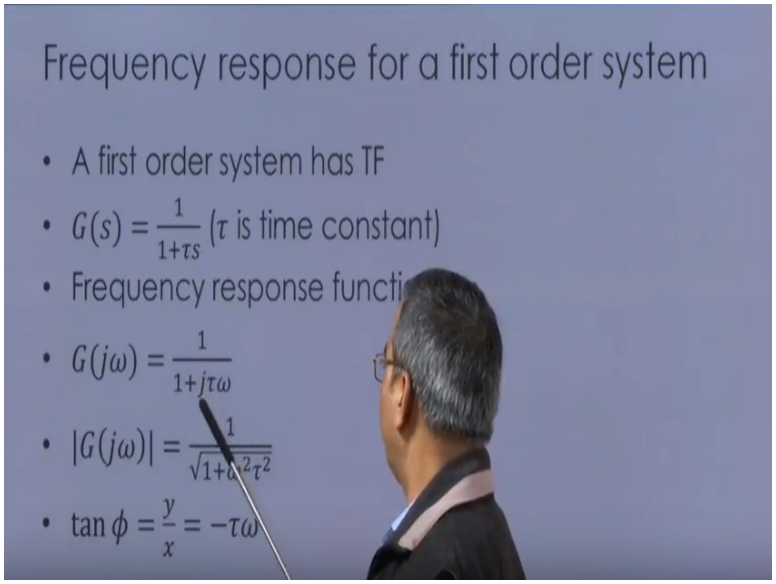

We can find out the magnitude, as well as the phase angle once we have the this phasor basically okay, we can find out the magnitude and phase angle and you remember this phasor is basically the ratio of output phasor divided by the input phasor okay, this is ratio output by input, so basically this tells that how many times the input has been multiplied ok.

In order to get the output okay, and how do we get this thing, it can be explained simply, these are just a small mathematical operation and through those mathematical operation, we can get that, so we have J omega=1/11+j tau omega, so what we do is basically we try to represent this in the form of a complex number, ok.

So we after representing in the form of complex number, we try to find out the x component, we try to find out the y component and once we have the x and y component, we can find out the magnitude, as well as we can find out the phase angle, so this is what we are going to do, so to convert this in the form of separate x and y component of the complex number.

What is done is that, the numerator and denominator are multiplied by this number with a opposite sign here, so this is what I am going to do, 1–j tau omega/1–j tau omega and so this what it makes is 1–j tau omega/this will be 1–j square tau square omega square okay. a+bxa-b that will be a square –b square, so this is what I get 1–j tau omega/1, remember j square is -1.

So this is 1+tau square omega square or what I get is 1/1+tau square omega square –jx tau omega/1+tau square omega square, so this is essentially my x component, which is the real part and this is my y component, which is the complex part okay, so I can find out the magnitude for this one, so if this is x, this is y, this is my real part and here I have got the imaginary part okay.

So this will be under root x square+y square and this pi, we can find out tan pi as y/x okay, so let us find out the magnitude that is the mod of Gj omega this will be under root x square, so it is 1/1+ tau square omega square, whole square+y square that is tau omega/1+ tau square omega square, whole square okay.

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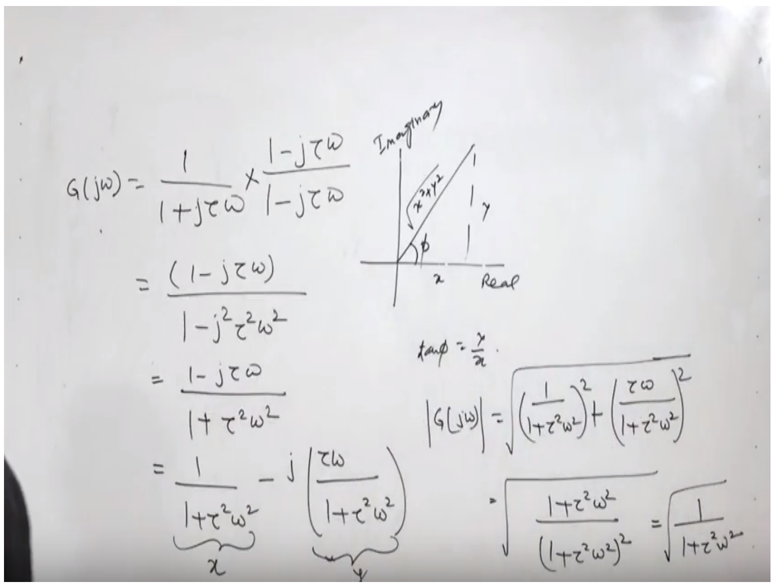

So this will be, what we will be getting is 1+tau square omega square/1+ tau square omega square, whole square basically okay, so this will be nothing but under root 1/1+ tau square omega square, so this is what we are getting 1/under root 1+tau square omega square okay, and I can find out the value of pi that is tan pi=y/x.

So this I can get what is my y here, so this is my y component basically that is tau omega okay, that is - tau omega /1+ tau square omega square and the x component is 1/1+ tau square omega square, so this is nothing but -tau omega, so this is my tan pi, so this way I can find out the value of tan pi as-tau omega okay.

We can take an example basically to illustrate say a system transfer function is given by say pi/S+1 and what we have to do is actually find out the magnitude and phase of the output when subjected to sinusoidal input of this one, so we are given the sinusoidal input and we have this, the system as the transfer function.

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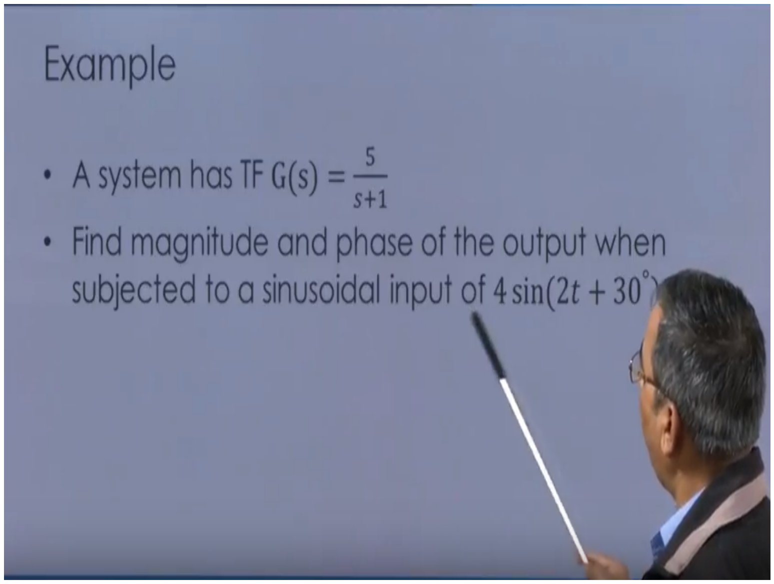

So we have to find out the magnitude and phase of the output okay, so to do this what I can just give you a brief idea how we can do that and basically this tells us how can we find out the system response in the frequency domain ok, and so to do this here you see that we are given GS=say pi S+1.

So we can convert this by replacing S by J omega, so this is pi/j omega+1 okay, now we have to convert this into the complex number, so basically what we are trying to do is that we will find out the magnitude of this Gj omega and the phase angle okay, and once we have the magnitude and phase angle, then we can find out the output phasor okay, by multiplying the input phasor with this G fine.

So that we can do the same operation here that is I multiply with –j omega +1 at the numerator and –jomega+1 at the denominator, so this is what I get pi 1 –j omega/here this will be what a square–b square, so we will have –j square omega square+1 okay, so this is b square and this a square, so b square-b square, so this is pi 1-j omega/this will be-1.

So -,-+1, so omega square+1, so this is basically pi/omega square +1 is the real part okay,-j x pi omega/omega square+1, this is our complex part okay, so from here I can find out the magnitude of it here that is Gj omega and this Gj omega will be given by under root this one okay.

So pi/omega square+1, whole square+pi omega/omega square+1, whole square that is under root x square+y square, okay, so this we can do or what we can write this further is this one, this pi square, pi square I think I can take out, so this is there and so here I will have 1+ omega square and here I have 1+ omega square, whole square.

So this will be basically pi/under root 1+omega square, so this is my value of Gj omega, its amplitude and I can find out the phase angle also by writing tan pi=this will be what –omega is it not, yes it will be –omega okay, so from here we can find out the value of pi okay, so I get the value of pi,IGe the value of magnitude, then you see that we have this input, so the input has got a magnitude of 4.

So whatever value I am getting from here this omega of course we know because we have given the input, it is 2 okay, so I can substitute for 2 here, so this will be 5/under root 1+4 or this is 5/under root pi or this is under root 5 okay, likewise this will be minus of that is2, so -2, so I can find out 5 how many degrees it is going to be there, okay, so the output basically will be we have the input as 4 sine 2T+the angle is 30 degree okay.

So the output will be 4 times this magnitude root 5 and then sine 2t and here we have to actually find out the angle with respect to the reference axis okay, remember this value is for our input okay, so this angle we know and the output will be this one, so basically it will be this angle, so we have in the output expression we have to actually write this angle value that is the value, which is coming with respect to this one okay.

So that we can evaluate and put it over there Similarly, we can take the second order system okay, so that is the frequency response for a second order system can be taken and this is the transfer function for the second order system omega n square/s square+2 zeta omega ns+omega N square.

Now frequency domain response function we replace S by J omega, so if I replace s/j omega, this is what I am going to get okay, and I can represent this J omega in the complex form like this okay, using the same procedure, which we have seen earlier okay, and once we have this Gj omega, I can find out the magnitude of it okay, I can find out the amplitude of it, I can find out the phase angle of it okay.

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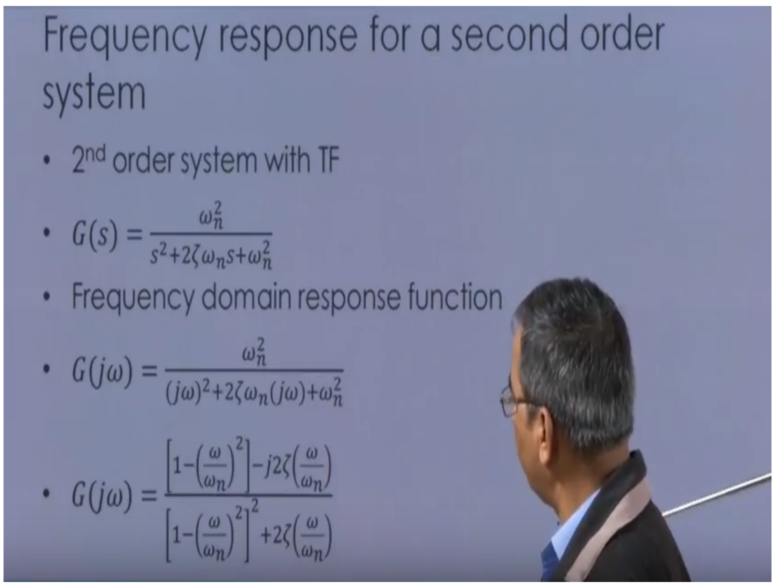

Again for a given type of input, we can find out the output response okay, so the sole exercise of this frequency response has been to find out the output for a given input okay, and the given frequency domain response okay, so I hope this is another way of finding out of the output for the given input and for the given system equations. You can refer Bolton, if you want to read more, thank you.