Modelling and Simulation of Dynamic System
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Lecture – 28
State Variable Formulation.

I welcome you all in this lecture on state variable formulation, which is a sub course for a sub module for course on modeling and simulation of dynamic system which you are going through state variable formulation is actually a way of representing the system equations and that is the equations of any dynamic systems.

So, till now in our some of the earlier classes we have seen the transfer function approach for finding out a relation between output and input, but the transfer function approach has got certain limitations okay. It is done in the complex domain transfer function approach workshops in the complex, complex domain and there are many more limitations but coming back to the modern approach.

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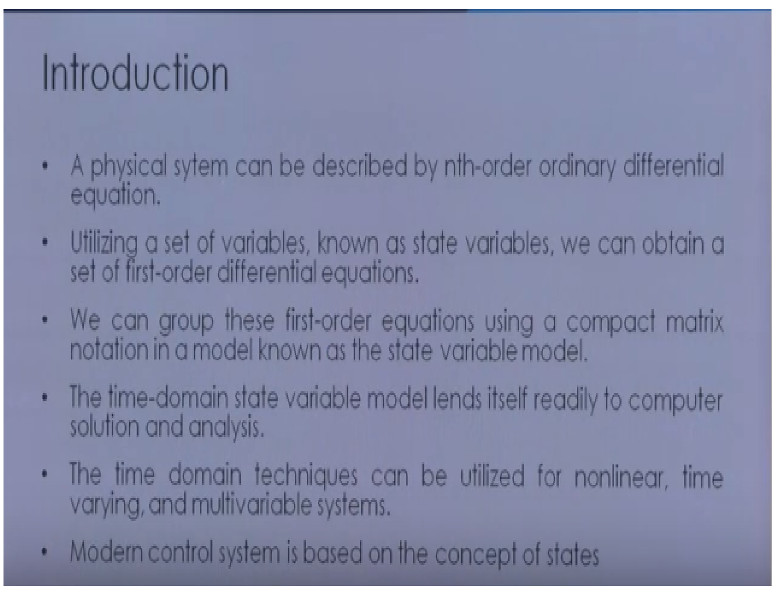

That is if you talk about say modern control systems where essentially we want to see we want to see the variation of the things with respect to time okay so there are this state space oppose is very much used so as we have seen a physical system can be described by nth order differential equations we have seen many examples of Mechanical system, Hydraulic system, Electrical system, Pneumatic system.

In our earlier classes and we have seen that we can describe the physical system with the help of nth order differential equation. Now the utilizing a set of variables which we call as state variable we can obtain a set of first order differential equations okay. We can group these first order differential equations using a compact matrix notation and this is what is called as the state variable model.

Now as I was telling you that time domain state variable model lends itself readily to computer solution and analysis it makes our analysis simpler and this time domain techniques can be utilized for a nonlinear system time varying system as well as multivariable system. So as I told you the modern control system is based on the concept of states and it utilizes this concept of the state variable approach.

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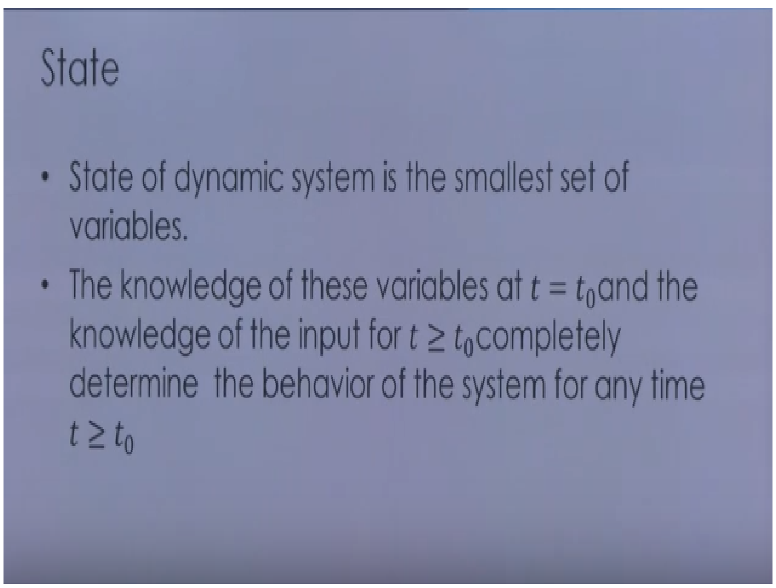

So before we proceed let us see some background what do you mean by state actually so the state of a dynamic system is the smallest set of variables and the thing about these variable is that that if we know these variables say at some time t=t0 and we know the about the input say for a time t=> 0 completely then we can determine the behavior of the system for any time t=>t0 okay.

So this is how we define the state and the state variables these are basically the variables making the small set of variables determines the state of the dynamic system so basically these are the set of variables that determine the state of the dynamic systems okay. If at least say very n variable say x1,x2,xn are needed to completely describe the behavior of a dynamic system.

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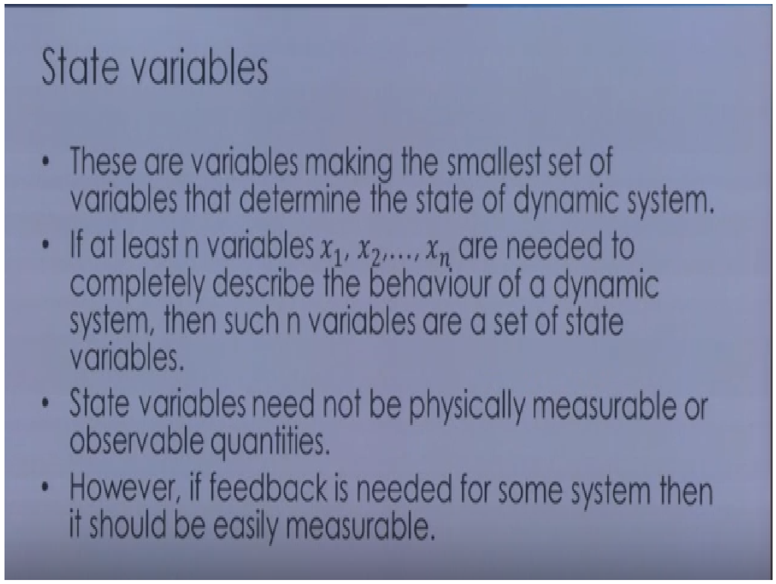

Then, search n variables are the set of state variables and remember one thing. These state variables need not be physically measurable or observable quantities but if you are trying to make certain feedback system then of course you need to know are these you need to have these state variables are measurable. Then only you can use it in the feedback system okay. So if feedback is needed for some system then it should be easily measurable the state variable.

So when we decide about the state variable these factors has to be kept in mind. Now state vectors if I is that these variables in the form of a vector or say as a component of the vector then or what we call it as the state vector okay. That is the end state variable can be considered as the end component of a vector X okay and this vector is called the state vector.

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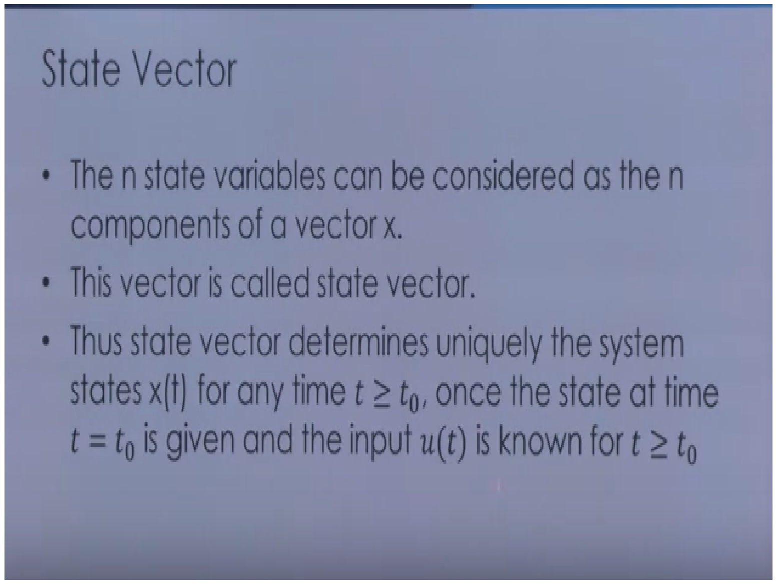

Thus, the state vector determines uniquely the system state xt for any time t > or = t0 once the state of at that time t is equal to t 0 is given and of course you know the input for time t > or = t0. Now state-space basically what we call it is a basically mathematical concept that is it's a n-dimensional space whose coordinate axis consists of say x1, x2, xn and this is called the state space.

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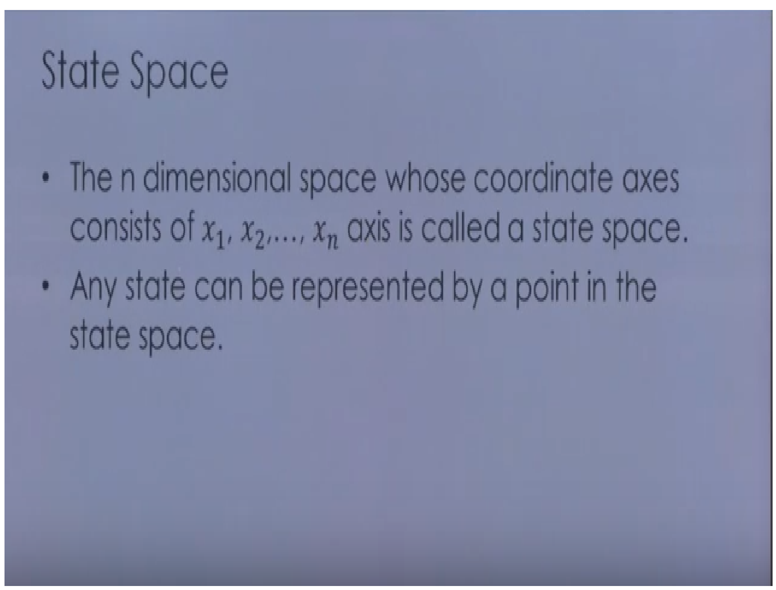

And any state can be represented by a point in the state space. Now let us look at the state space equation. Now in a state space modeling basically we come across 3 types of variables okay and these variables are the input variable, the output variable and the state variables and state space representation for a system is not unique okay.

So, you can have the different way of representation of the state space model for a system but the number of the state variable are going to be same the number of state variables are the same for any of the different space state space representation so you may have a different way of representing the system but number of state variables has to be same and the dynamic system must involve elements that remember the values of the input for t > or = t1.

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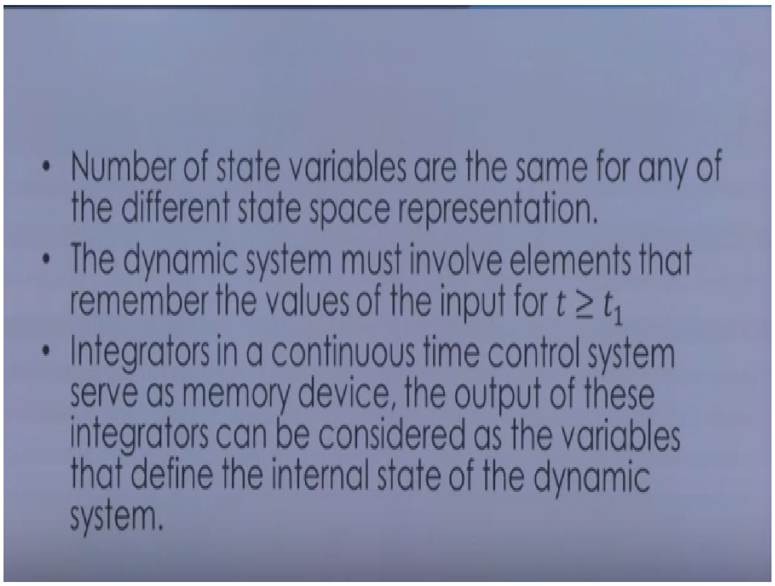

So, the question is that how to select by state variable or what should be the state variable? so if you remember or if you recall when we were discussing about the bond graph morning then I talked about how do we get the equations state equations from the bond graph model and that time.

If you remember, I said that the equations derived from bond graph model are in the state space form that is first order differential equations and the state where basically the observed causes and both absorb causes we're integration of the causes okay so its concept is basically the same here also that integrator integrators in a continuous time control system serves as a memory device.

The output of this integrator can be considered as the variable that defines the internal state of the dynamic system okay so these are basically the integrators okay which basically facilitate us in selection of the state variable okay. So the output of the integrator works as a state variable okay and the number of state variables to completely define a dynamic system is equal to the number of integrators involved in the system.

So, Let a MIMO that is multi-input, multi-output system involves say n integrators then suppose are there are R inputs we have say u1t,u2t,urt and we have say m outputs say y1t,y2t,ymt and suppose, there are any state variables say x1t,x2t,xnt okay These state variable as I said these are the output of integration okay.

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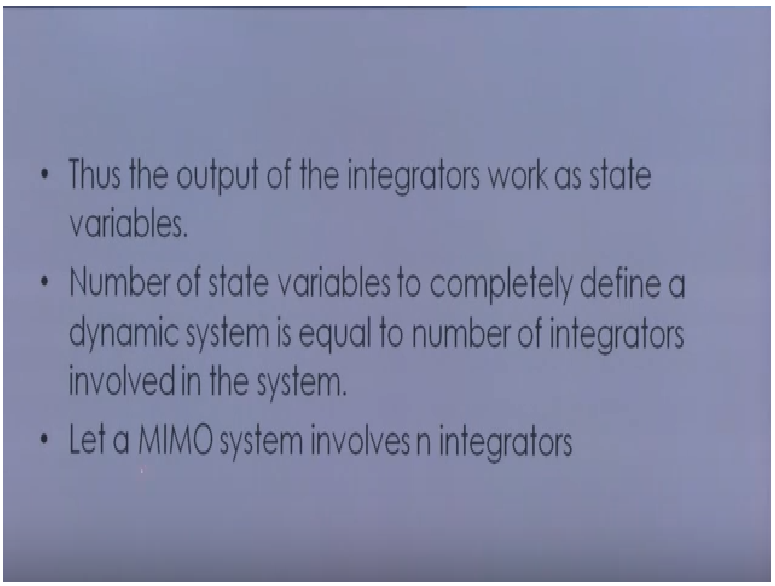

So the concept of selection of state variable is same whether it is bond graph modeling or whether this is the general way of modeling which we are talking about okay. So it is basically the integration of that is output of integration which is the state variable okay. So the we can consider the system can be described as say x1x1 derivative t=f1 and this is function of all the states say x1,x2,xn all the inputs u1,u2,ur and say time okay.

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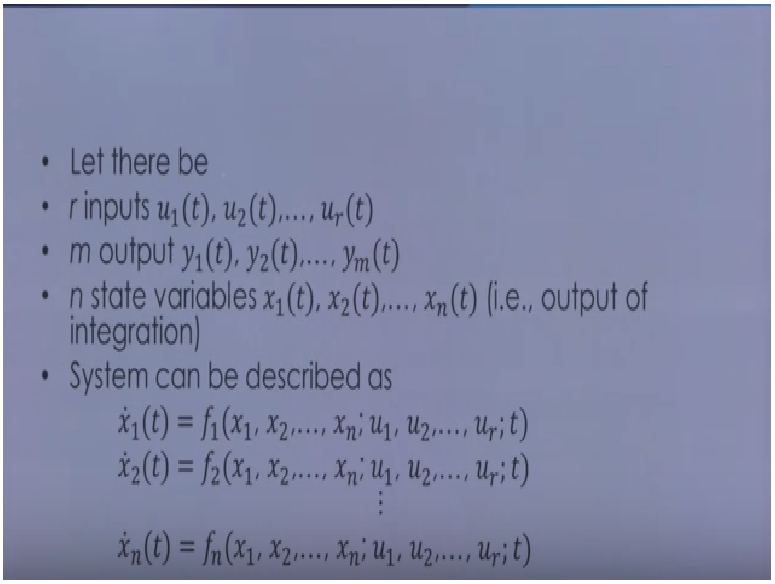

So, likewise I can describe the system are say by x2 dot here again f2 which is a function of all the info all the states all the inputs and time and likewise a xn derivative. This is function of all the inputs all by states all the states and all the inputs and time. We can describe output of the system okay say y1t,y2t,ymt there are m outputs as I told you then these outputs are function of say g1 that is g1 these are function of all the states all the inputs and time okay.

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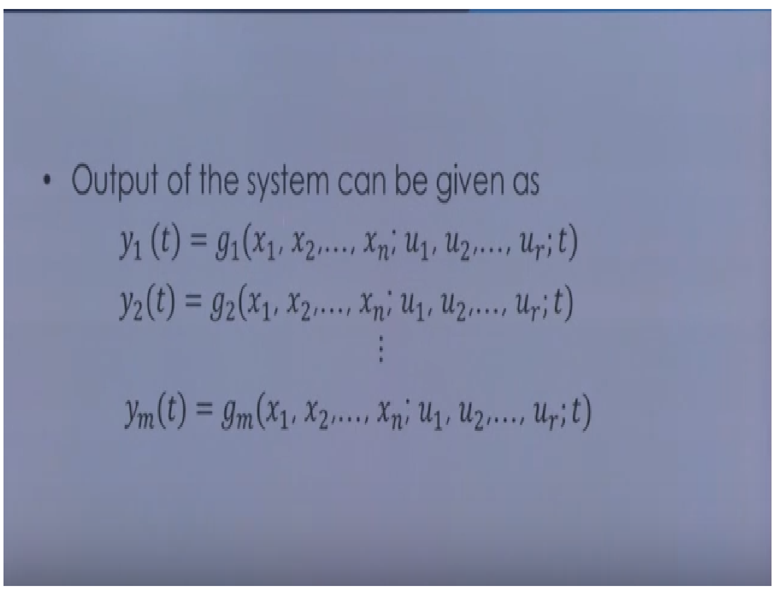

Similarly, y2 is function of all the states all the inputs and time. Similarly ym is function of all the states all the inputs and time. Now if we define say ut as this u1,u2,ur that is a input vector okay and xt say these are the state vector x1 to x2xnt and fxut if I define like this okay f1 which is a function of all states all inputs time Likewise f2 and fn.

I am ever to define then for say output vector I define as y1,y2,ym and say this g which is a function of states input and time if I define like this g1,g2,gm all the states all the inputs and time. So if I do that then I can write the state equation x dot t=fxut okay. I can write the output equation as yt=gxut. So this is basically the compact form representation of the state equation and the output equation okay.

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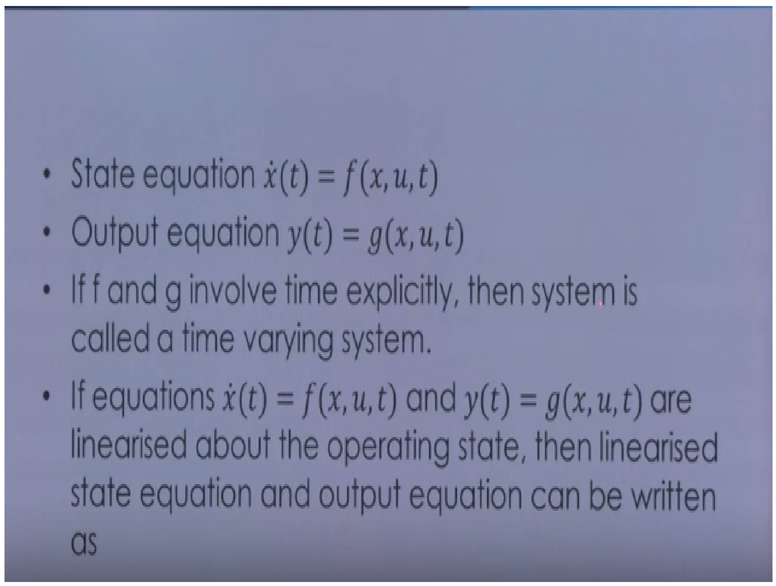

Now if these f and g involves time explicitly then the system is called a time varying system okay. And if we are able to linearize this equation about the operating states then the linearized state equation and output equation can be written as this way okay. So here the state equation x derivative t is written as say Atxxt+Btxutand yt is written as Ctxxt+Dtxut. Now here what are these ABCD these matrices are very important.

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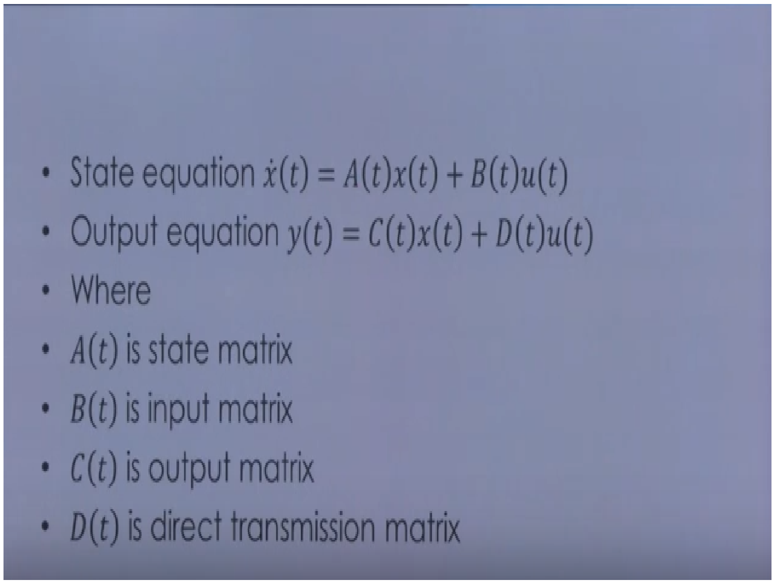

So, the A matrix we call it as the state matrix okay. This matrix is being multiplied by state vector we call it as a state matrix. The Bt matrix is being multiplied with input vector. So we call this as the input matrix. The Ct matrix is multiplied with xp vector so we call it as the output matrix and the Dt matrix is being multiplied with the input vector ut so that we call it as the direct transmission matrix here this Ct is multiplied with the a xt that is the state vector.

Now if the function F and G do not involve time T okay explicitly then the system is called a time invariant system okay. And in that case the stratification and output equation can be written as that is this A B C and D they are not going to be the function of time okay. So I can write this as Axt+But+YtCxt+Dut and these are the state and output equation for a linear time invariant system okay.

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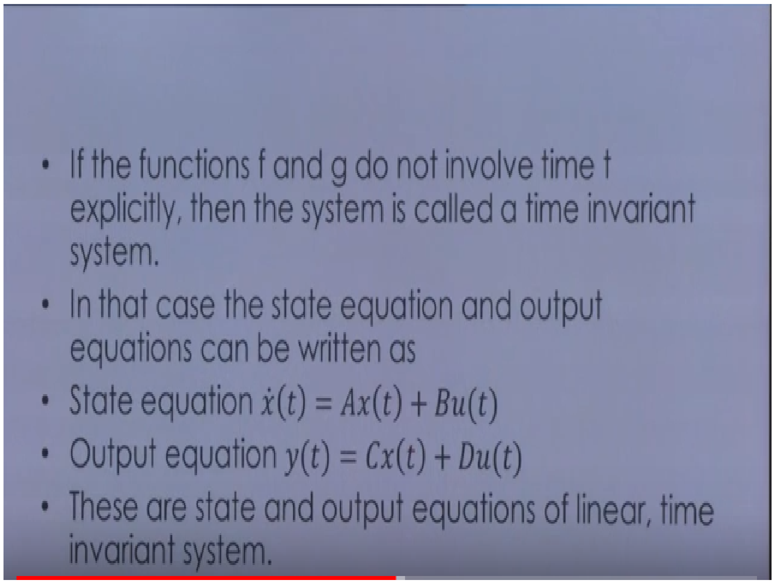

So remember these are 2 equations are very popular equations that is state equation is Axt+But and the output equation y is Cxt+Dut We can take an example to illustrate the concept of derivation of the state space equation okay I will take again the very well known case of the spring mass damper system you can take in fact any system.

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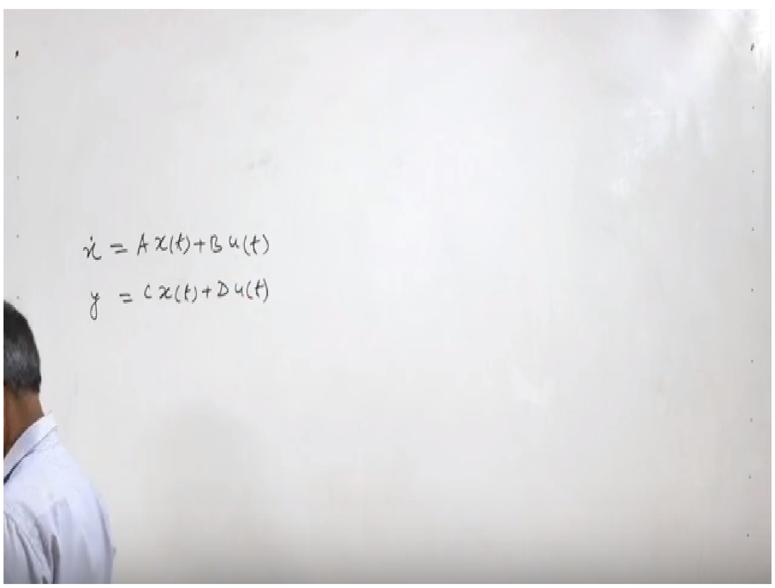

We can take an example to illustrate the concept of the derivation of the state space equation okay. I will take again a very well known case of spring mass damper system, you can in fact take any system. We have seen the several examples during our course of electrical hydraulic pneumatic thermal. Any system can be taken but here I am just illustrating this concept of derivation of state space equation of by taking this spring mass damper system okay.

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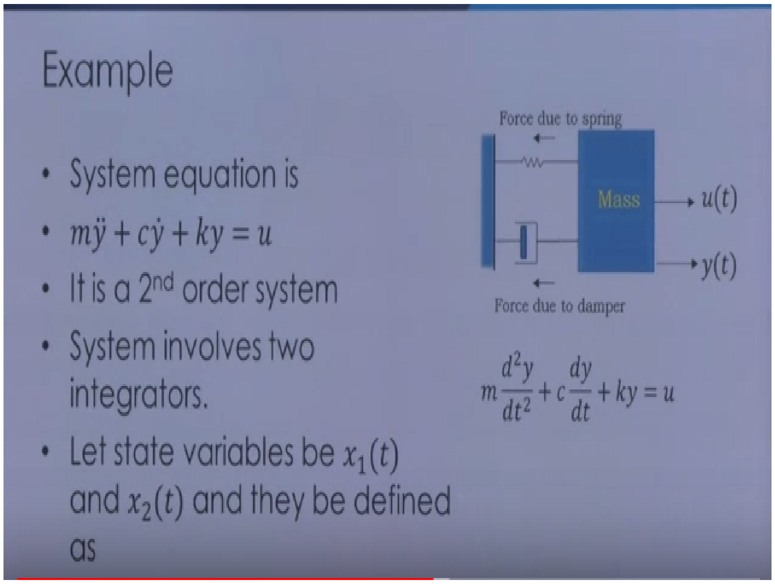

So in this system we have a mass m there is a spring and there is a damper say this is subjected to certain input Ut and say Yt is the output okay. So we have seen the conventional way the system equation is given as this one and of course all of us know how we did that okay. We drew the free body diagram for it and we found out all the unbalanced forces which are going to act on this body.

And then we equated those unbalanced forces to mass into acceleration and we got or this equation okay. So this D square/dt square+cdf/dyydt+ky=u. So this equation can be written the system equation can be written like this by my double dot+cy dot+ky is=U. Now see it is a second order system because we have the highest order.

Here is the 2 now it means back this system will involve the 2 integrators okay, because it is a second-order system. So if there are tool integrators it means that they are going to be two state variables in this case alright and let these state variables be say x1t and x2t okay. So I am taking these 2 state variables x1t as x2t and let me define x1t as yt and x2t as y derivative t okay.

Then, if you look at it then what happens we as I said we are interested in finding out the state space form for x1t I am writing as yt and x2t I am writing as y derivative t fine. So what is my x1 derivative so if I take x1 here this will be basically y derivative and y derivative we are already writing it as x2 so it is x2 okay, basically right hand side I want to write in terms of states okay and my states are x1 and x2.

So that is why I am doing this manipulation okay and my system equation as i have shown you is my double dot plus cy dot+ky=u okay. So far here I can write y double dot=-say c/my dot-k/my+u/m. So I have the ydouble dot here and y double dot is basically what it is going to be x2 dot. So I have x2 dot=this y double dot okay and y double dot is this one.

So this is -c/m and this y dot we already taken as x2 so I just write -c/mx2-k/m what is my y. I am writing as x1+ this is u/m okay. So it is this equation basically which we get x1 dot=x2 and x2 dot=-k/mx1-c/mx2+1/mu okay and what is my output equation output.

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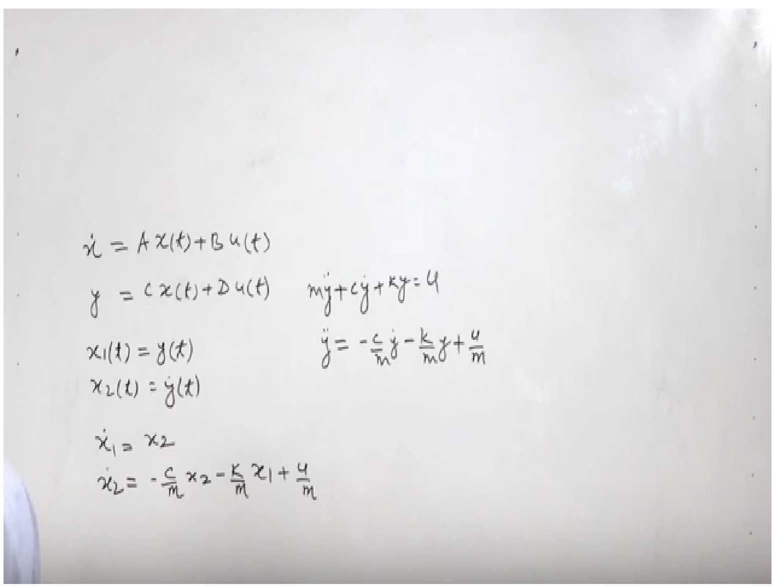

I am interested in knowing why okay my output is y and what is y here this y=x1. So this is my output equation again. So this is my output equation fine, now what I can write it I can write this equation in the vector matrix form and I remember I want to put my expression in this form, so that I can do using these 2 equations, these 2 equations that is I write the derivative of the state x1 dot x2 dot derivative=my A matrix.

Here +I want to write my state vector so it is x1x2+I, will have the B matrix I do not know, what it is I have to find out + my U okay. So far here you see that X 1 dot=X 2, so I can just write 0 and 1 here so when this gets multiplied I get x1 dot=x2n and I put 0 here. Similarly, x2 dot is say -k/mx1 so I right here -k/m this is going to be multiplied with x1 so -k/mx1-c/mx2 and then +1/mxu. So this is 1/mxu.

So, this is my equation that is equation in this form okay. Similarly, I can write the output equation that is y and remember this output equation has to be written in this form that is cxt So the there is a c and there has to be xt that is a state vector so x1x2 here and my y is x1 this has to be 1 and this has to be 0 and of course. I do not have anything else here. So, my d is going to be a 0 here.

Of course, I will have the U so this is how I can express my system equation in the state space form okay. So I hope you have understood it okay. So we have the vector matrix form state equation and this is my state output equation so from here from these equation I can identify what is my A matrix what is my B matrix and what is my C and D matrix okay. So here you see that these are basically this term which is of this matrix which is with multiplied with the state vector.

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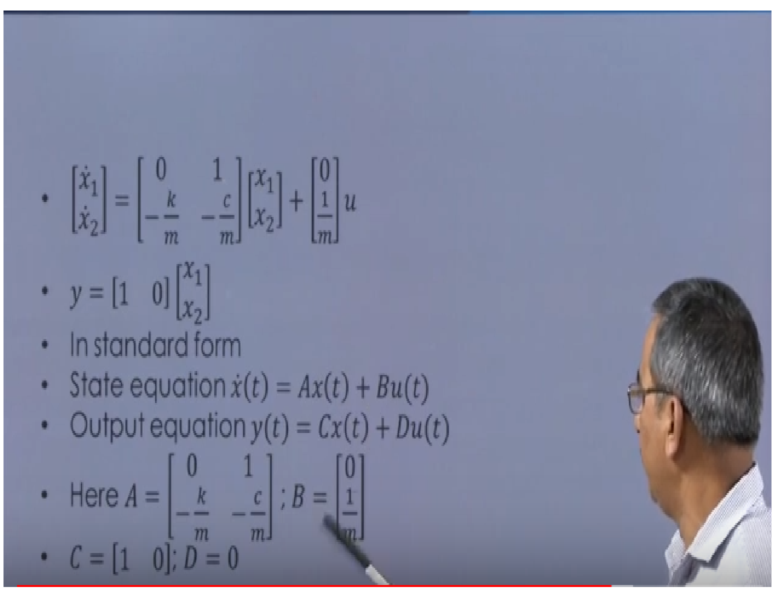

Here this is your A matrix this matrix is basically your B matrix and this matrix is your C matrix and this matrix is your D matrix okay. So this way we can identify the A B C and D matrix okay. Now another very important concept is that how can we derive the transfer function from the state equation, so the state equation which we have seen here from this state equation we can derive the transfer function okay.

As you know that once you have the transfer function or when you have you have moved into the complex domain again there are many number of control tools available to analyze and see the behavior of your system. So the transfer function of single input single output are SISO system can be obtained from the state space equation so we have these state equation and this is our output equation okay.

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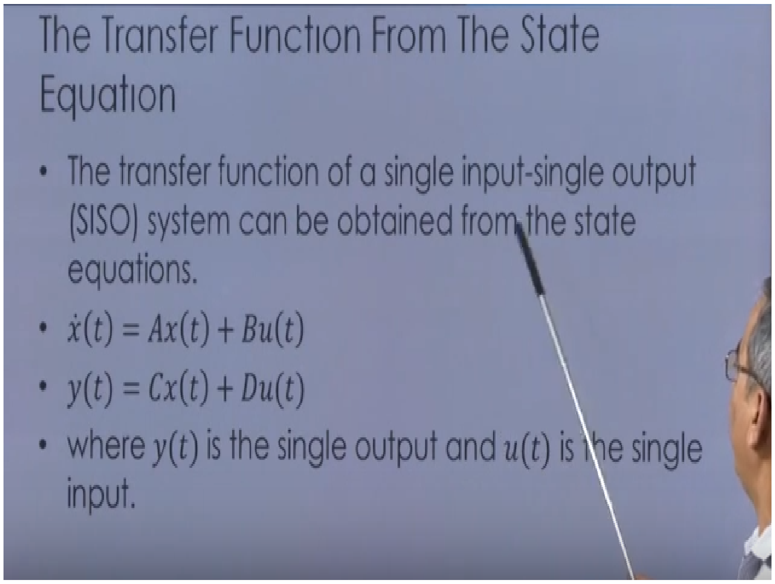

So, I have explained all that associated terms A B and C and D we are here Y T is the single output and ut is the single input. Now if I take the Laplace if I take Laplace transform for by this state space equation okay. So this first derivative I can write as sxs-x0 okay and this=Axt+Bus you know we use the capital later representation to represent, represent the parameters in the Laplace domain okay.

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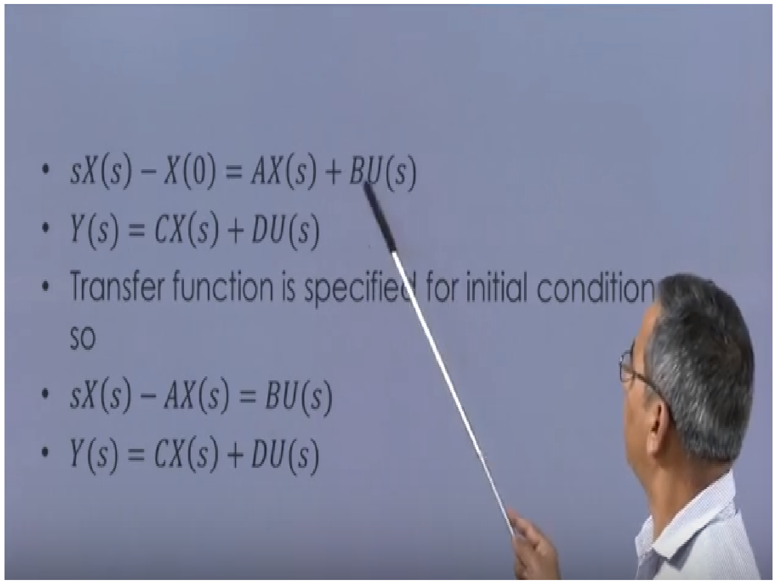

Here, S basically indicates that these are in the Laplace domain. Similarly, I can take the Laplace for this one that is Ys is CXs+DUs. So this way I can do that and when we are talking about transfer function, so as I told you at the beginning of the transfer function lecture that this transfer function is valid only if our transfer function can be derived only if we assume that all the initial conditions are 0 okay.

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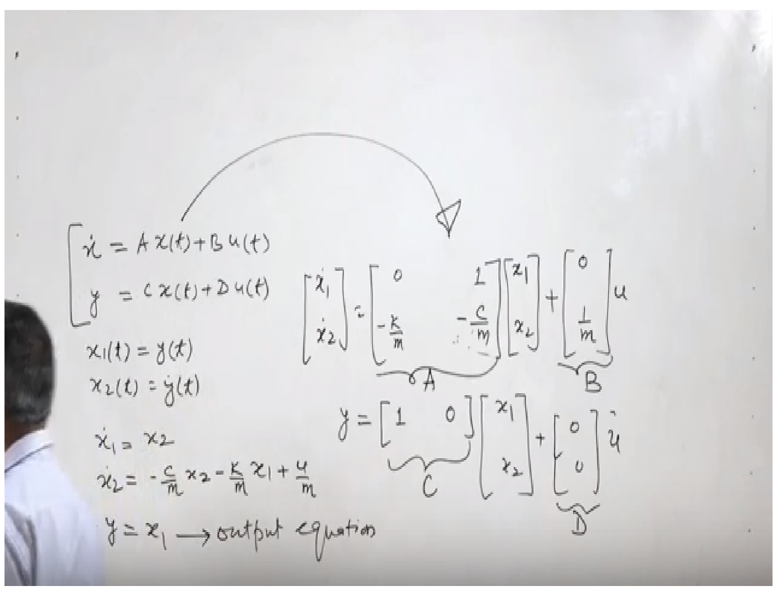

So, this x0 I have to take it as 0 so what I get SXs and I take this to the left hand side here. So-AXs=BUs okay and of course I have my Ys=CXs+DUs Now I can further simplify this equation this I can write I take Xs outside so this is SI-A where I is the identity matrix Xs=BUs.

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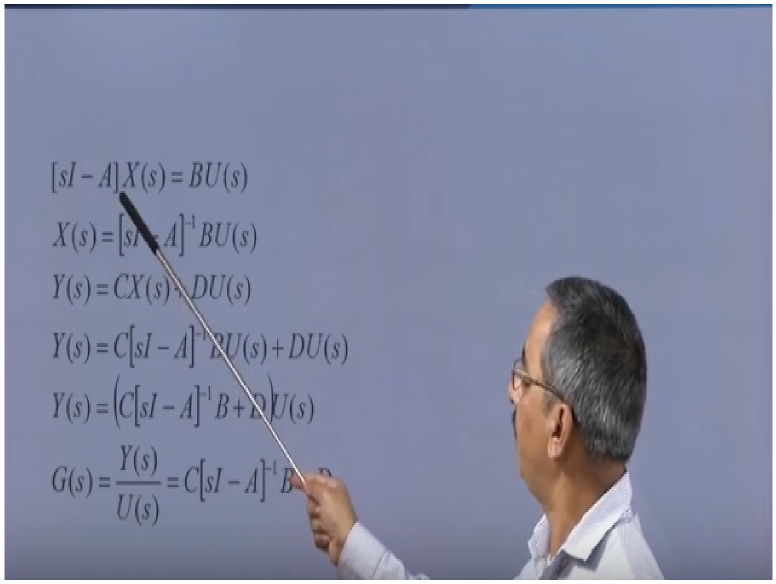

Now, I can do the pre multiplication both the sides by the inverse of sI-A here. So this inverse mine x this one will be giving me the identity matrix so I get Xs=SI-A inverse x BUs and you know our Ys relation is CXs+DUs or I can have Ys=C this X S I can substitute from here that is SI-A inverse x BUs+DUs are this Yf I can take Us outside from here.

So have this CsI-A inverse xB+D this Us is out outside here and now I can define the ratio between the output and the input that is Ys/Us of course with the initial condition of 0 value initial value so this will be giving me C and this complete term I will be getting and I can just explain this concept using the same problem which we have considered here.

So again, the same spring of damper system and this is by the dynamic equation one way of doing it is that I can take the Laplace transform directly here okay and assume the condition to be 0 and I can find out the transfer function the other way through state space approaches that these are my state equation and this is my output equation.

So, I can find out the transfer function Ys/Us as the expression which we have just derived and then i can substitute the value of C value of a value of B and of course B is 0 here. So if I put these values here and I work it out take the inverse and do the multiplication I get this equation and this is the very popular form of the transfer function for a second order system that is 1/Ms square+Cs=K okay.

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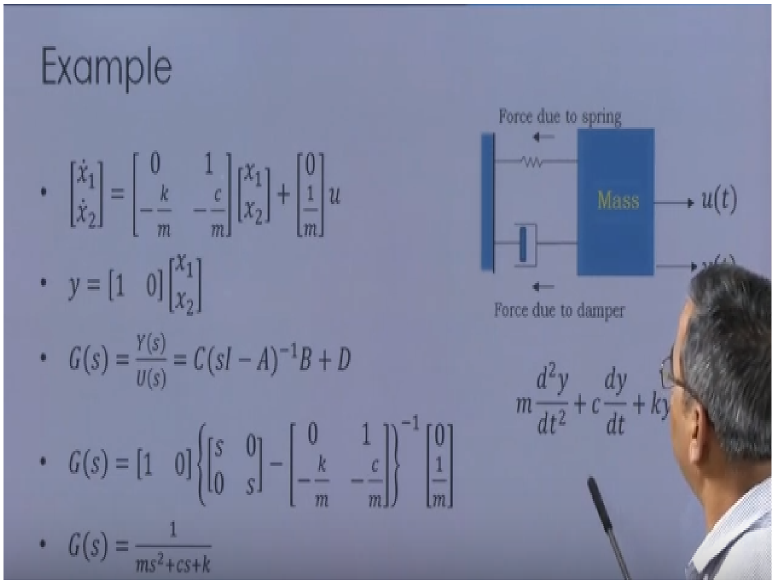

So with this, I will complete this lecture and if you want to have further reading if you want to see many more examples please refer Ogata Modern Control Engineering textbook. Thank you.