Modelling and Simulation of Dynamic Systems
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Lecture – 24
System Transfer Functions

I welcome you all in this lecture on system transfer functions and this is a sub module for the course on modeling and simulation of dynamic systems, which you are going through, now in this chapter on system transfer functions, basically what we would like to do is that we would like to find out the ratio of say output by input, say for the second-order dynamic systems okay.

Then based on that ratio of what we define as the transfer function of the system okay and using that ratio we may try to predict or we may try to find out the output of the system for the given input and the transfer function, so that is the ultimate aim of it okay. So suppose if I am talking about a simple amplifier for that amplifier we know suppose there is certain input y to the amplifier.

There will be certain output from the amplifier for this given input depending on the gain of the amplifier and this gain is defined as output by input okay, the physical meaning for this equation that is gain is equal to output by input is that if the gain of an amplifier is 5, then for a given input of say 4 millivolt, the output is going to be 20 millivolt okay.

So this is what is the meaning of this relation, but for many system the relationship between output and input is not algebraic form as here otherwise we can conclude from this one, so these relationship rather than in the algebraic form, these relations are in the are in the form of differential equation okay and due to that the above formula.

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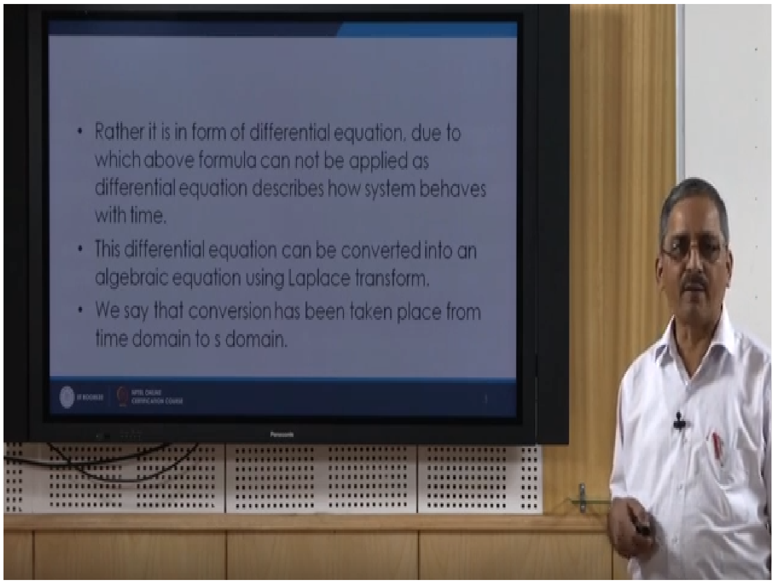

That is output gain is equal to output by input that cannot be applied because these differential equations describe how system behaves with time okay, these are not algebraic equation all right so there has to be some way to convert these differential equation into the algebraic equation and so that we can do the manipulation that is we can find out the ratio of the output and input okay.

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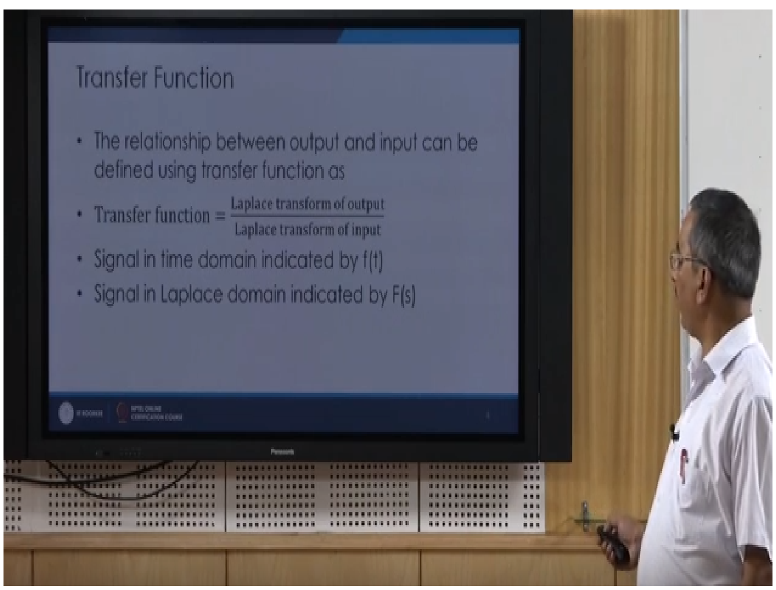

So these differential equations can be converted into an algebraic form using Laplace transform okay and what we say that the conversion has taken place from the time domain to S domain okay, time domain is usually differential equations are written in the time domain and of course the Laplace transform is carried out in the complex domain or what we call it as the s domain okay.

So the transfer function can be defined as ratio of Laplace transform of output divided by Laplace transform of input okay, so this way we can define the transfer function, so here for algebraic equations, we describe the transfer function rather than describing the gain, there the gain has been output by input and here on a similar logic we define the transfer function as Laplace transform of output divided by Laplace transform of input.

Now signal in time domain are indicated by f that is ft that is function or time whereas signals in Laplace domain are indicated by capital F, for small f here, we are writing the capital F and for t here we are writing small s indicating that it is in the Laplace domain, so this is the convention followed globally for writing out of the Laplace or domain.

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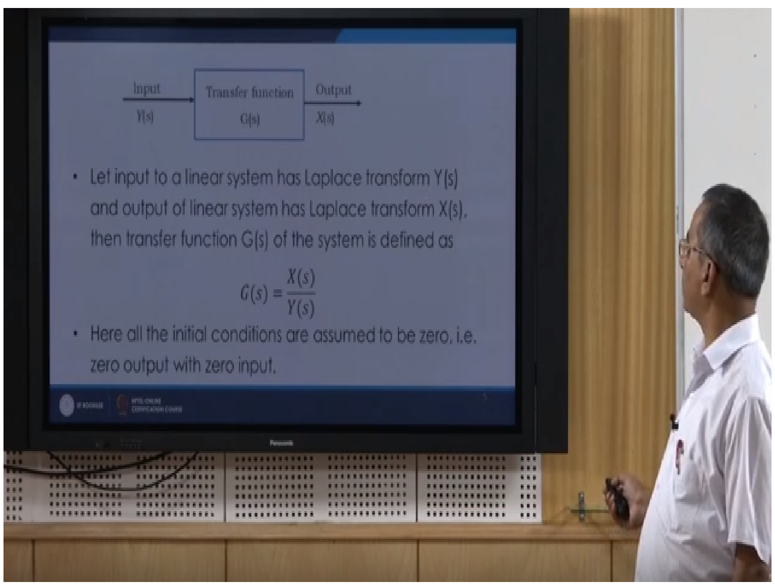

So here you can see that suppose we have a transfer function say Gs and input to the transfer function is Ys and output for the transfer function is Xs then we can define this transfer function Gs as ratio of Xs and Ys okay. So of course this is valid for a linear system and as I said if the linear system has Laplace transform Ys that is the input to the linear system as Laplace transform Ys and output of the linear system has Laplace transform Xs.

Then we can define the transfer function Gs as Xs upon Ys and of course here when we are defining, a transfer function one thing which we always need to keep in mind that we are assuming all the initial conditions to be 0 okay that is the 0 output with 0 input condition okay.

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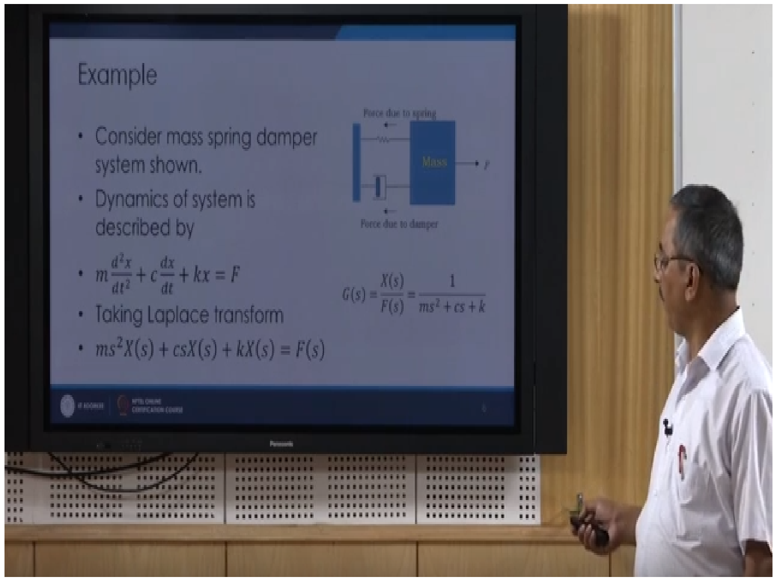

Now let us take an example of the transfer function okay so here I just take a simple spring mass damper system let this end of the spring and damper is anchored to the ground and other ends are anchored to the mass okay and here say stiffness of the spring is k and the damping constant of the damper is say c and let the mass be small m and a force F is being applied here.

Then, as we have seen that we can draw the free body diagram of the mass we can find out the unbalanced forces and then we can equate it to the inertial force that is mass into acceleration and we get the dynamic system equation okay that is mdx square x/dt square + csdx/dt+kx=F, now remember in this expression F is my input that is the force and the output are x.

Now in this case you can see that this governing equation is a second-order differential equation which depends on time alright. So in this case we cannot define the gain but rather we have to define the transfer function okay and to define that transfer function what we have to do we have to first take the Laplace transform and for this the Laplace transform will be ms square xs we will be discussing about the Laplace transform in coming slides.

Then I will explain you the detail about how we get this term okay plus this c into Laplace transform of a derivative is given by s times a Laplace transform of xxs+k this Laplace transform for this x is xs and the Laplace transform for force is F s okay so where these x are actually the function of time okay.

So this is the Laplace transform and from here I can define the transfer function Gs as Laplace transform of the output which is x here that is the displacement of the mass divided by the applause transform of the input which is Fs here. So if I take xs upon Fs so this is what I am going to get 1/ms square+cs+k okay. So this is how this is what we define the transfer function for a second-order system.

Likewise we can define the transfer function for a first-order system okay so that one example I will be taking at the end of this lecture will be taking an example. Now before we proceed let us study a little about the Laplace transform okay.

This method is very popularly used for solving linear differential equation the Laplace transform method is used for solving the linear differential equation using Laplace transform we can convert functions such as say sinusoidal, exponential into algebraic function of a complex variable s okay.

Differentiation and ending integration operation can be replaced by algebraic operations in the complex plane. So this is the advantage of Laplace transforms that we can do the differential and integral operation as a algebraic operation in the S domain and you know that we are much familiar with the algebraic operations rather than the differential and integral operations okay.

So this way is very convenient for us coming to the advantage of Laplace transform method there are certain basic advantages which we cannot draw from other method these are it allows use of graphical techniques for predicting the system performance without solving system differential equations that is one of the big advantage of this Laplace transform method that is you need not solve with our differential equation but you there are certain graphical techniques okay.

And you can use those graphical techniques to predict the system performance okay without solving the system differential equations. Secondly, when we solve differential equations using this method transient and steady-state components of the solution can be obtained simultaneously you remember when we were studying the behavior of the second-order system.

Then what we did initially we studied for the transient behavior separately and then we studied for the steady-state behavior separately and then our general solution was the addition of both the behavior that is the transient behavior and the steady-state behavior but using Laplace transform method okay, we can analyze both the behavior simultaneously that is also one of the biggest advantage of this Laplace transform method.

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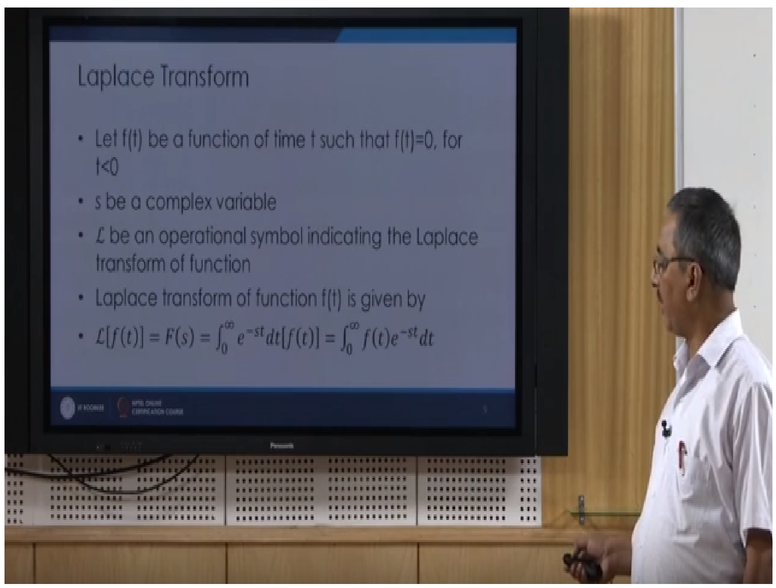

So let us see what the Laplace transform is okay. So let Ft be a function of time F be a function of time t such that Ft=0 for t less than 0 that is for the time value less than 0 the function value is 0 and sba complex variable then we can use this Laplace operator this symbol as operational symbol indicating the Laplace transform of the function okay and thus Laplace transform of a function Ft is given by this one.

So this is how it is written Laplace operator of F T is equal to Fs okay, that is when I am taking the Laplace I am moving from time domain to the s domain and this is integral 0 to infinity e to the power -stdt and this is being operated upon function Ft okay or this can be written in a simplified way as 0 to infinity Fte to the power -stdt. So this is what we mean by the Laplace transform okay.

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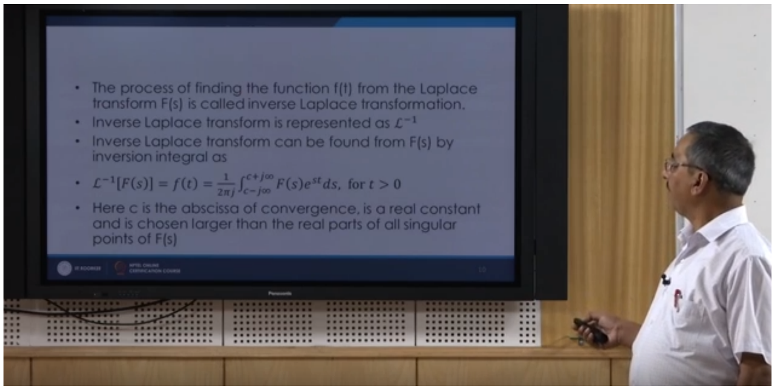

Now the process of finding the function Ft from the Laplace transform that is the inverse we are given the Laplace form transform and we want to find out the function a function which is a function of time F, which is a function of time t okay and this method is called inverse Laplace transform okay.

And this is represented by Laplace operator to the power minus 1 and this is what we call it as the inverse Laplace transform okay and the inverse Laplace transform can be found from say Fs by inversion integral as here. Laplace inverse of F S and of course this will be giving us a function in that time domain okay. So this ft is 1/2pij+c-j infinity to c+j infinity integration Fse to the power stds for t greater than 0.

Now here in this expression c is the abscissa of convergence and it is a real constant and is chosen larger than the real part of all singular points of Fs okay, there are other simpler way of finding out of the Laplace inverse of the Laplace transform that will be seen.

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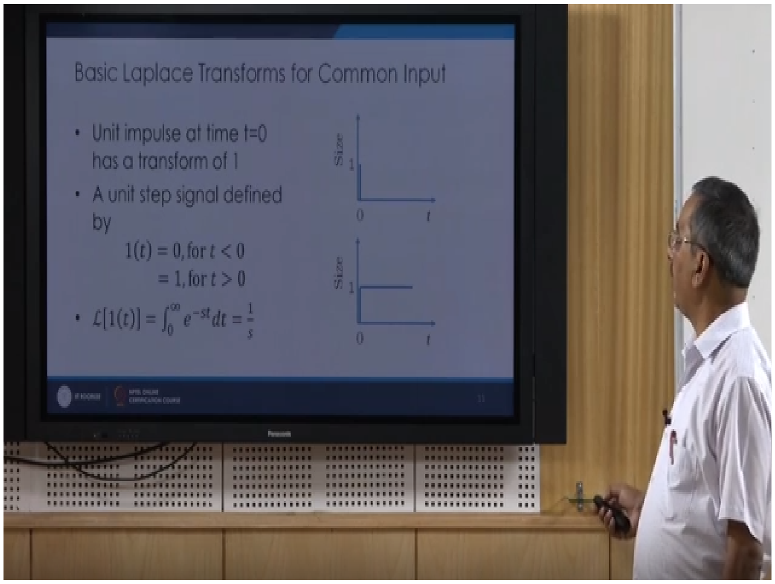

So let us see the basic Laplace transform for the common input okay, suppose first we have the unique impulse at time t=0, so here if I plot the size versus function of time the, so here at 0 time, I have got a unique impulse okay, so the unit impulse at t=0 has got Laplace transform of 1 okay, similarly I can define a unit step signal defined by say1t=0 for t less than 0 here, this is 0 and this value is 1 for t value greater than 0 okay.

So, the Laplace transform for this will be given by I can put this in the definition of the Laplace transform 0 to infinity, integral 0 to infinity, -stdt. If I carry out the integration, I get the value as 1/s. Next, we can find out the Laplace transform for a unit ramp function okay, now this is how we define the unit ramp function, Ft=0 for t less than 0 and Ft=t for t greater than or equal to 0 okay.

So the Laplace transform for this unit ramp t will be again here we put integral 0 to infinity, t for –stdt and if you carry out this integration. Of course here we have to do the integration by part and you will be getting 1/s square.

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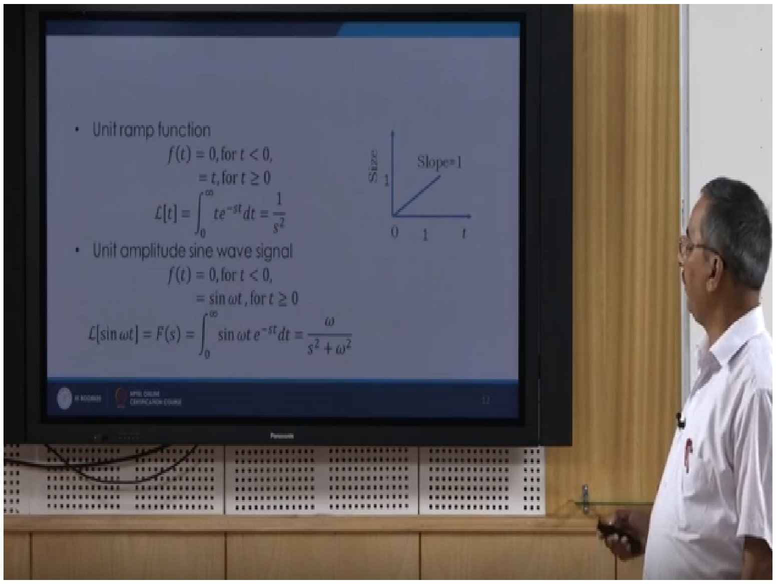

Similarly, we can carry out and find out the Laplace of unit amplitude sine wave signal, so if we have a unit amplitude sine wave signal that is Ft=0 for t less than 0 and for t greater than or=0, Ft is given by sine omega t, remember here I am taking the amplitude say a as 1 okay. So if I do this the Laplace transform of sine omega t given as Fs and this will be, we can again put in the definition of the Laplace transform and if we evaluate, we are going to get omega/s square+omega square.

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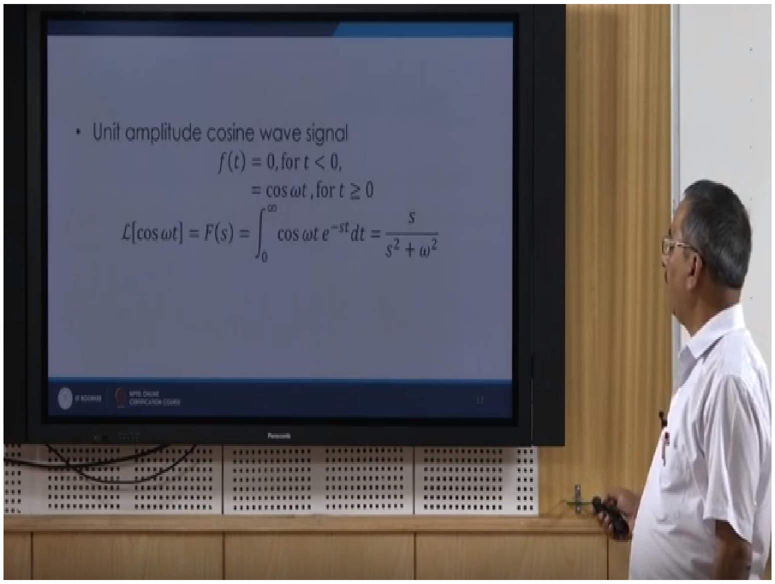

Likewise, we can find out unit amplitude cosine wave signal okay, so for cosine wave signal I can find out Laplace transform that is I define the cosine wave signal Ft=0 for t less than 0 and for t greater than or=0. We have this as cos omega t okay so the Laplace transform for this will be Fs, 0 to infinity, cos omega te to the power of –stdt or this is s/s square+omega square, so this is how we can evaluate the Laplace transform.

Now there are certain basic rules in working with Laplace transform and these rules are the Laplace transform of say constant axft is going to be the a times Laplace transform of Ft that is Fs okay. Likewise, Laplace transform of say sum of 2 functions Ft and F2t=summation of the Laplace transform of F1t and the Laplace transform of F2 that is the Fus, F1s+F2s Laplace transform of first derivative of a function can be found out, Laplace transform of dft/dt=sfs–F0, but as I said for the transfer function.

We assume all the initial conditions to be 0 that is why we can take this F0 to be 0, so the Laplace transform of a derivative is given by sxfs okay, so this is what we have seen in that example of the spring mass damper system okay. Similarly I can define Laplace transform of d square F (t) upon d(t) square so this is given by s square Fs–sf0d/dt at dt/dt of F at t=0.okay.

Again I just say that initial condition R 0 then for evaluation of the transfer function the Laplace of Laplace transform form of d square Ft upon dt square be given as F square Fs. The Laplace transform of an integral of a function can be written as Fs/s okay. So this can be evaluated one can verify that. Then I was ever telling you inverse Laplace transform that is if you are given the Laplace transform okay.

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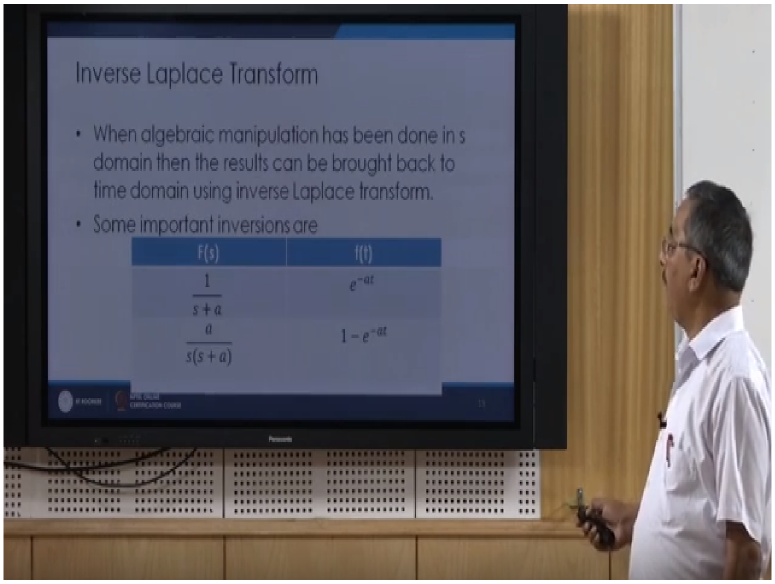

That is the value of the function in S domain and if you are interested in finding out the value of the function in the time domain then we can take the inverse Laplace transform okay and that we find out some of the extended values of the inverse Laplace transform are given here 1/s+a is the inverse Laplace transform is e to the power–at here, and if it a/s, s+a it is 1-e to the power–at.

Similarly, if it b–a/s+axs+b it is e to the power –at–e to the power –bts/s+a whole square it will be 1-atx e to the power –at similarly for a/s square xs+a it is t-1-e to power at upon a. okay. So this way we can find out the inverse Laplace or we can define the inverse Laplace transform and we can find the values for some of the extended functions.

Alright, Now many times here we may have the denominator either we will have the quadratic term or the cubic term then in that case what we do is that basically we resolve that equation or we rewrite that equation in the form of some partial fractions and then we find out the inverse Laplace transform okay.

Now as I promised you at the beginning we will take up an example of the first order system and we will try to find out the transfer function of the first order system .okay. So our first order system is defined by this expression as I was telling you that we have the a1dx/dt+a0xx=+b0y. Now here this y is my input and x is my output okay.

So as I was telling you that the Laplace transform we take in the Laplace transform all the initial conditions to be 0 when we are interested in finding out the transfer function. Okay. So to do that I can just take say the Laplace transform for this one, so this will be a1xs times xs+a0. This will be xs and this b0 and ys okay.

So from here I can get the output by input. xs/ys as b0/a1s+a0. So this is what is here. We have b0 /a1s+a0 okay and this xs/ys is my transfer function. And further what I can do is that I can divide the numerator and denominator by a0. So what I get b0/a0 /a1/a0xs+1 okay and say what is my b0/a0 this is basically the ratio of output by input in a steady state condition because in steady state condition, this term will be 0.

So my v 0/a 0 will be basically x upon y of course in the case of a steady state all right, so this is my say gain G in the steady state and this given by a0 is the time constant of the system. So this can be written as tau s+1 okay, so this way we can find out the transfer function for the first order system. Now when this first order system is subjected to unit step input okay that is I m given the value of ys as 1/s then I can find out what is my output that is what is my xs okay.

So from here I can find out xs=Gsxys or xs is gsi know, G/tau s+1 and the Laplace transform of a unit step input is 1/s as we have seen so into 1/s so this xs I can write as G into say I divide the numerator and denominator by tau, so Gx1/tau/sxs+1/tau, so this is what I can get or this xs is G1/tau/sxs+1 /tau.

If I take the inverse Laplace transform for this, I can write x which is a function of time as Gx1-e to the power–t/tau okay, so this way what I have done is basically that I have solved this first order differential equation using the transfer function and using the step input okay, so using the transfer function and step input, I have found out the time response of the function of say solution of the equation.

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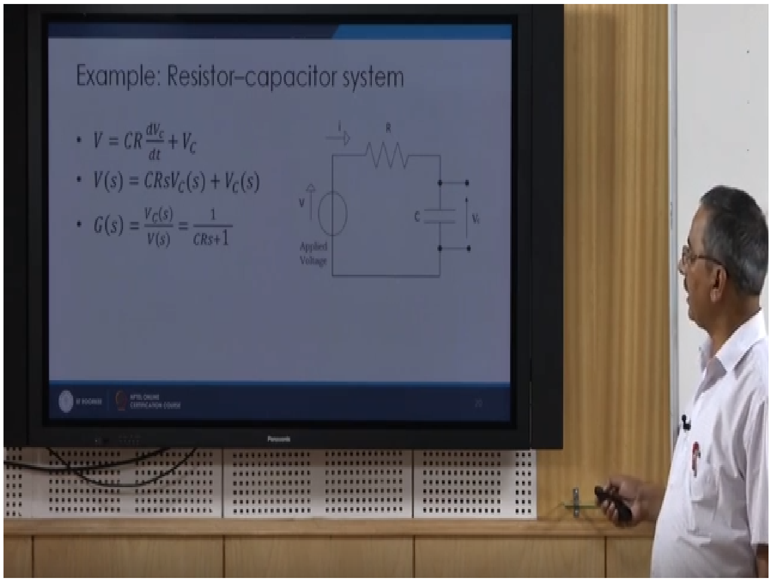

We can take the example of a physical system say resistor capacitor system here okay. Now in this case say resistor is represented by R and capacitance is represented by C and say the applied voltage is V and my interest is to find out the voltage V C that is V C is the voltage across the capacitor so that is my output okay. So using the Kirchhoff's voltage law, I can write the system equation as this one okay that is the loop equation for this loop.

V=this voltage will be CRdvc/dt=Vc okay. So this is the dynamic equation for this first order system and then I take the Laplace transform of both the sides, so Vs=CR sVcs+ Vcs and then from here I can find out the transfer function that is Gs=output that is Vcs/input Vs and this is 1/CRxS+1 okay.

So this way we can find out the transfer function for the first order system and what is the use of this transfer function is basically that we can see say by seeing, measuring the output okay as a function of time for the given transfer function and the input okay. So that we can do either we can take example of any first order system say hydraulic system or say thermocouple and we can workout the time response as here we have seen in the previous case.

So the application of this transfer function is that for a given differential equation, we can find out the output response without solving the differential equation okay, so using the Laplace transform we convert that differential equation in the form of an algebraic equation, manipulate that algebraic equation and then we convert it back those algebraic equation in the S domain into the equation in the time domain using inverse Laplace transform.

So this way we can use to find out the response of the system, so if you want to further read it, these are the 2 references, Mechatronics by Bolton and Modern Control Engineering by Ogata, you can refer it, thank you.