Modelling and Simulation of Dynamic System
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Lecture – 22
Dynamic Response of the Second Order System

I welcome you all in this lecture on dynamic response of second order system, so last lecture which we saw was on the dynamic response of the first order system and we took the example of say hydraulic tank, discharge from a hydraulic tank okay. In this lecture, we will be talking about dynamic response of second order system and they are in fact many more examples of this system.

If you talk about mechanical system, then the spring mass damper system or one of the popular example of the second order system, then we can have example of torsional system or if you go for say electrical circuit then R-L-C circuit is again a good example of the second order system, of course which is an equivalent of the spring mass damper system as we have already seen.

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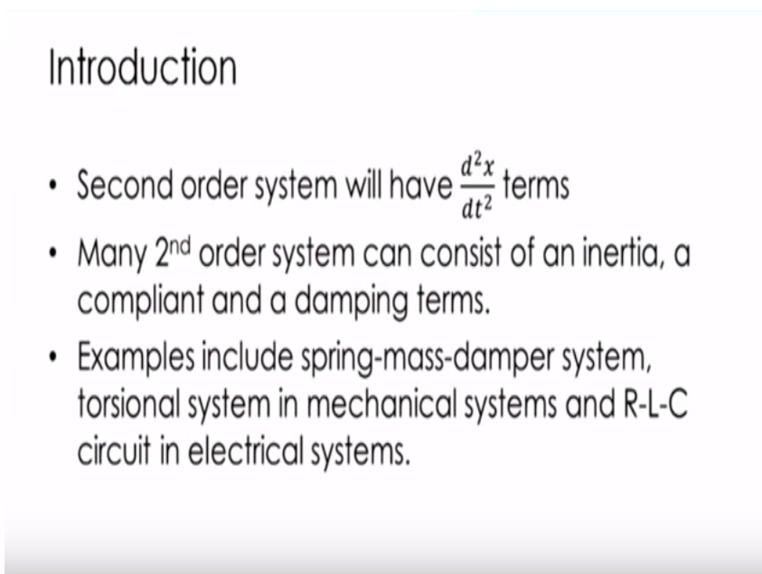

So the second order system will have the terms like d square x/dt square, so this is the way to identify the order of the system, okay so the second order system will have this term and of course many second order system can consist of an inertia, a complaint and a damping okay, so if you talk about mechanical system, mechanical translatory system then say we have a mass, a spring and we have a damper okay.

If I talk about say electrical system then we have inductance, a capacitance, and a resistance okay, so this way we can take the analogies among the different systems, so the examples as I explained to you is spring mass damper system, torsional system and mechanical system and R-L-C circuit in electrical system.

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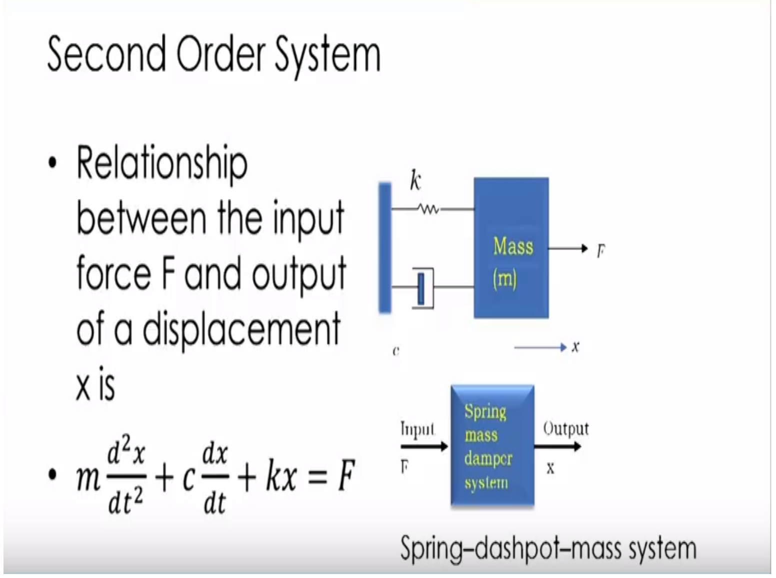

Let us take the example of the second order system okay, so also we say a spring mass damper system okay, this spring has stiffness k, the mass as mass of m and damping coefficient of the damped say c and say there is a force F being applied here and there is a displacement X of the system. So, here the input is F that is the force, which is being applied on the mass and the output is X for this spring mass damper system okay.

Now the relationship of the dynamic equation basically will have the relationship between the input force F and the output of a displacement X, so you can see this equation, the derivation of this equation we have already seen in mechanical system, in this lecture as I told you we are interested in looking at the response of the system, response of this second order system.

So if I draw the free body diagram of this system that is represent of the forces here, say the spring force and damping force okay, and write the unbalanced forces and equate that mass into acceleration, this is what term we are going to get okay, so here this is the inertial force basically, this is the damping force and this is the spring force and this is the input force, which is being applied alright.

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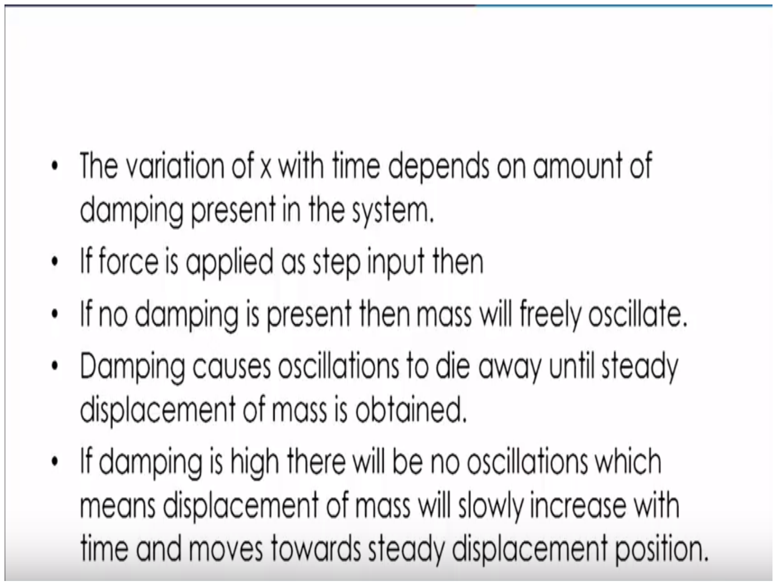

So this equation are here you can see that X are the outputs and F is the input okay, now here the variation of X with time depends on the amount of damping present in the system okay, as we have seen we have the damping term also coming here. If force is applied as step input then if no damping is present then mass will freely oscillate okay.

The damping causes oscillations to die away until steady displacement of the mass is obtained okay, and if damping is high, there will be no oscillation, which means that displacement of mass will slowly increase with time and move towards a steady displacement position. These type of behaviour we have seen earlier okay.

So here is the plot, so for this second order system when it is subjected to a step input say the force F, so if I plot the output that is X versus time, then if there is no damping, then you can see that they are going to be oscillations here and if there is some damping present, then initially there will be some transience and then after some time.

We will have the steady state value and if high damping is there, then these oscillations will be very small or almost it will not be there, the values will be gradually increasing and reaching to the steady state value, so this is how the system behaves with damping, so let us see the second order differential equation with no damping alright.

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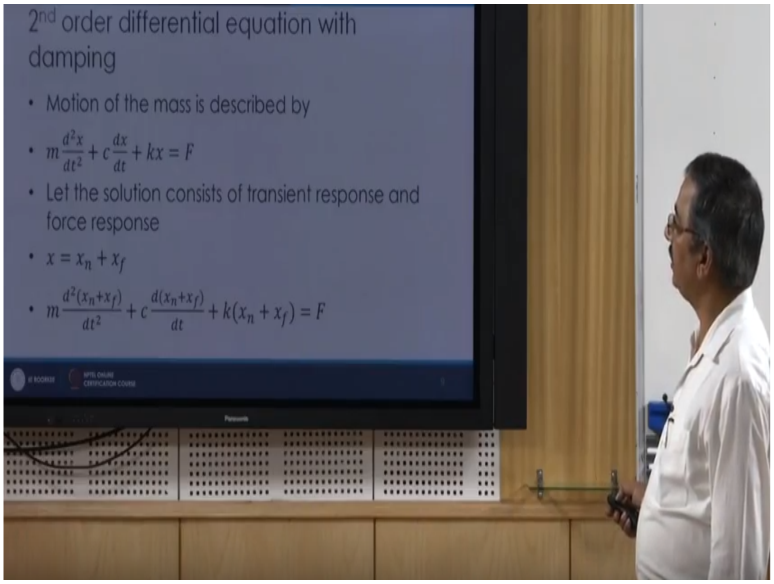

So again here I am going to consider the case of free vibration, so I am assuming this F to be =0 and since I am assuming there is no damping, so I can make the c term also =0 and whatever resulting equation will be md square/dt square+kx=0 okay, now let us assume that the solution for this is like this, x=A sine omega nt, where A is the amplitude of oscillation and omega n be the angular frequency of the free undamped oscillations okay.

Then this equation x=A sine omega nt, if I take its first derivative then this is dx/dt is A omega n, cos omega nt and the second derivative will be d square x/dt square is -A omega n square sine omega nt. Now here you see that this A sine omega nt is x itself, so A sine omega nt I can put that as x itself.

So here I have –omega n square x as this one okay, or I can write the equation d square x/dt square+omega n square x=0, so here what I have done basically is that I have derived the system equation from the assumed solution of the expression okay, so this is my derived equation from the assumed solution.

Now in absence of c and F, the system equation as I said it is going to be md square x/dy square+kx=0 alright, now if I make a say write this form, I take this m to the denominator here, so we have d square x/dt square+k/mxx=0 and if I compare these 2 equations then you can see omega n square=k/m, so this term can be compared, so we have omega n square=k/m.

So my solution which was xa sine omega t now that solution will become A sine under root k/mt and this is going to be the solution of differential equation for the free undamped vibration case okay, now let us consider a case of second order differential equation with damping, so the previous was there was no damping and we considered the free vibration.

Now I am going to consider the force vibration with damping okay, so the motion of the mass can be described by this equation, here you can see that this F is the input force and of course damping is present so we have this term, cdx/dt is there okay, now let the solution consists of transient response and the forced response okay.

Transient means I am calling it as natural response okay, so I am using here the subscript xm okay, so the total response of the system is composed of the transient response of natural response xn and the force response xf okay, so let us substitute this x=xn+xf in this equation, then this is what we get, md square/dt square of xn+xf cd/dtxn+xf+kxn+xf=F okay.

Now let us play with this equation a little okay, we segregate the terms related with xn okay that is the natural response or the transient response and segregate the term related with xf that is the force response okay, so we have from here, we can look at it this is what we get okay, md square xn/dt square+cdxn/dt+kxn this part here and md square xf/dt square+cxfdxf/dt from here +kxf=F okay.

So now you can see that we have basically 2 parts of this equation and this=F. Now if I assume that this =0 okay, because I know the natural response, the input force is 0, so let us assume that this=0, then we must have this part=F okay.

So now let us see the solution of transient equation that is we are going to look at the solution for this part and then we will look at the solution for this part and then we will sum up the solution and we will get the general solution, so let us take the first part, the solution of transient equation, so this is my transient equation md square xn/dt square+cdxn/dt+kxn=0 okay.

Let us assume the solution for this to be xn is Ae to the power st, where this a and s are constant. Now if this is solution of this equation, then we can substitute this in this equation okay, and of course as you know here there is a differentiation 2 times and here is differentiation 1 time, so when we do that this is what we get okay.

s square term I am getting here because there is twice differentiation and s term I am getting here because there is once differentiation and here we are substituting as it is. Now if I take this Ae to the power st term outside then this is what we have. Now you see this into this product is going to be 0, so either one of them has to be 0.

This term cannot be equal to zero because we have already assumed this =xn, if you take this term to be=0, then what will happen that our transient response itself will becomes 0, so I do not want to take this term =0, so this means that this term will be =0, so this equation ms square+cs+k=0 is very popularly known as the auxiliary equation okay.

And you can see that this is a quadratic equation and I can write the solution for this quadratic equation s=-c+(-under root c square–4mk upon 2m) okay, and we can take these 2 terms here with c as well as I can take this 2m term here inside, so I can write it like this–c/2m +(-under root this whole square–k/m).

Now here what I do basically I write this first term in terms of k as a coefficient of k/m, so this is c square as it is, this is my 4msquare, so m I am distributing, so 4m and m here and multiplying k and dividing by k, so that this value does not change. So this is how I have written it –k/m is this one.

Now as we have seen this k/m can be written as omega n square okay and if we define say zeta square=c square/4mk for this term, then zeta I can write as c/2 under root mk okay, so as I said solution for S in terms of omega n zeta is given by this equation that is -theta omega n +-omega n under root zeta square -1.

Now you can see here the solution for S is going to depend on the value of zeta, which is popularly called as damping ratio in the vibration terminology okay, so here if zeta is going to be more than one then naturally we are going to have the 2 different real roots because this number inside here is going to be a real number okay, more than one.

So this we are going to have the 2 different real roots and I can write those real roots as s1 and s2-zeta omega n, one value will be positive here and another value will be negative here okay, so once I have the values of s1 and s2, I can write the general solution as xn=Ae to the power s1t+Be to the power s2t okay.

Since here we are assuming zeta to be more than one or damping ratio to be more than one, this type of system is said to be over damped okay, now we can have another case when zeta=1, so if zeta=1 here in this equation then you see that this term is going to be equal to 0, so you are going to have the 2 equal roots of -omega and because I am substituting zeta=1 there okay.

So in case of 2 equal roots, I can write the solution xn as at+bx exponential e to the power -omega nt okay and this case that is of zeta=1 is said to be critical damped okay, now let us see the third case that is if zeta is less than 1, so here if this zeta is less than 1, then you see that what happens this term becomes a negative term and the square root of negative is going to be a complex number okay.

So we are going to write a bad thing, little slide different way that is I am going to write 1–zeta square and whatever negative term square root is coming that I am going to write as k okay, so S will be–zeta omega n+-j omega n under root 1–zeta square and here this zeta is always less than 1, okay.

So if this zeta is less than 1, this number will be a positive number, but of course that negative sign we have already taken care by introducing j here okay, fine so this is my general solution. Now let us define a term omega d here, which is equal to omega n x under root 1–zeta square okay, so if you do that this is what my solution is s, value of s going to–zeta omega n+-j omega d.

So I can write the general solution if I substitute for the values of s1 and s2 here, so this is Ae to the power s1t+Be to the power s2, I have substituted values of s1 and s2 here in terms of omega n and omega d, alright now I can split these terms that is this term, first term of exponential of this one and first term of exponential of this was, so these 2 terms can be split up and I can take this exponential term outside.

So this is what I am left with, Ae to the power j omega dt+ Be to the power–j omega dt and we know that here this e to the power j omega dt, I can write in terms of the cosine and sine terms okay, so cos omega dt+j sine omega dt, likewise I can write for minus of this one, like this okay, now if I substitute this 2 exponential terms here okay.

This is what I am going to get okay, so we will have the same term over here and here I am going to get A+Bcos omega dt+jA-B sine omega dt okay, now this A+B, I can write as another constant P here and A–B, I can write as another constant Q and this way, I get an expression for xn okay.

So this xn is this one okay, and this represents damped oscillation with exponentially decaying amplitude okay, you can see here this is amplitude is exponentially decaying and this term omega d=omega n1–zeta square is called frequency of damped oscillations okay, so now let us see, so these 3 cases where for the transient cases okay.

Now let us see the solution for forcing equation okay, now for forcing equation, I can assume because in this case there is going to be an input okay, so I can assume say let there be a step input of size F at time t=0 okay, and I assume the solution as xf=A, where A is a constant okay, so if I take xf=A and I substitute it here, these 2 terms are going to be equal to 0.

So what I will have is kA= F and from here I can find out A as F/k and this is going to be my solution, so xf=F/k is my solution for forcing equation, so the complete solution can be written as summation of the transient part and the forcing part okay, so now let us write the complete solution for all the 3 cases that is the overdamped case, when zeta is greater than one.

So we have this part is the transient part and this part is the forcing part okay. Similarly for critically damped case, we have this solution as we have seen is for the transient part or this is for the forcing part and similarly for the underdamped case, this solution is for the transient one and this is for the forcing one okay.

Now by seeing these 3 equations,1,2,and 3 that is equation for overdamped case, critically damped case and underdamped case, we can see that when T tends to infinity here okay, all these terms tends to become zero okay, these terms will be decaying to 0 and this will be leading to a steady state value of x=F/k and of course, which is the steady state solution.

Thus we can write a general form of the second order differential equation and this could be like this A to d square x/dt square+a1dx/dt+a0x = b0y, okay, where y is the input and x is the output. Remember the case, which we have seen it was md square x/dt square+cdx/dt+kx=F okay, and for this case, we can write omega n square as a0/a2 and zeta square as a1 square/4a2 a0.

So this is the general expression, if we remember this, it will help us in treating the systems, second order system, which are there from the other energy domains as well. So if I plot this behaviour, then you can see that this is going to be the overdamped case okay, this is going to be the critically damped case and this is going to be the underdamped case and see this Fyk is our steady state value okay.

Which, corresponds to the force response okay, which corresponds to the force input fine here, Now as I said, we can use the same concept of seeing the response of second order system in many different system models okay, for example if I take this R-L-C circuit and suppose if I am giving some step input here.

If I want to know the behaviour of this system that is either in terms of what current is going to be there through the circuit or if I want to know that say how the voltage will be getting build up across the capacitor, all that analysis I can carry out similar to one which we have seen previously. Similarly, we can analyze a torsional system also okay.

So we can have say a disc with shaft here okay, I suppose if certain torque is being applied here okay, then we can have the opposing frictional torque and the opposing torsional torque here, which actually corresponds to the damping torsional damping and, which corresponds to the torsional stiffness of the shaft okay.

Then we can analyze, we can write the system equation and we can analyze the system in the similar way as we have seen. Here is another practical example for this, the strip chart recording, here you can see that there is a fixed rigid support and there is a pole here through which a spring is mounted, there is a mass here, which is being supported on say bearing and there is a pen paper contact okay.

And say there is a force being applied in this direction okay, so this example we can write the system equation for this case, also this expression for this case is going to be similar to what we have seen in the simple spring mass damper system and we can analyze the behaviour of the system in a similar manner.

So if you want to further read this, you can refer to as I have been telling you W. Bolton Mechatronics by Pearson Education, then we have our book Intelligent Mechatronic System and another introduction to Mechatronics and Measurement System by Alciatore, where you will get ample amount of study material to consolidate your fundamental further, thank you.