Introduction to Mechanical Vibration
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Lecture - 09
Viscous Damped Systems and Logarithmic Decrement

Welcome to the lecture on Free Vibration of Single Degree of Freedom System. Today we will discuss the different type of Viscous damped system and logarithmic decrement. So in the previous lecture we derived the equation of motion and we found the roots of the viscous damped system, we define the critical damping that is c upon c c - that is c c = 2 root k in to m and we define the damping factor zeta that is c upon c c.

So based on the different values of damping factor, we can classify viscous damped system in three categories, so they are critically damped systems, overdamped systems and underdamped systems. So today we will discuss the response of the systems in detail and we will discuss the logarithmic decrement.

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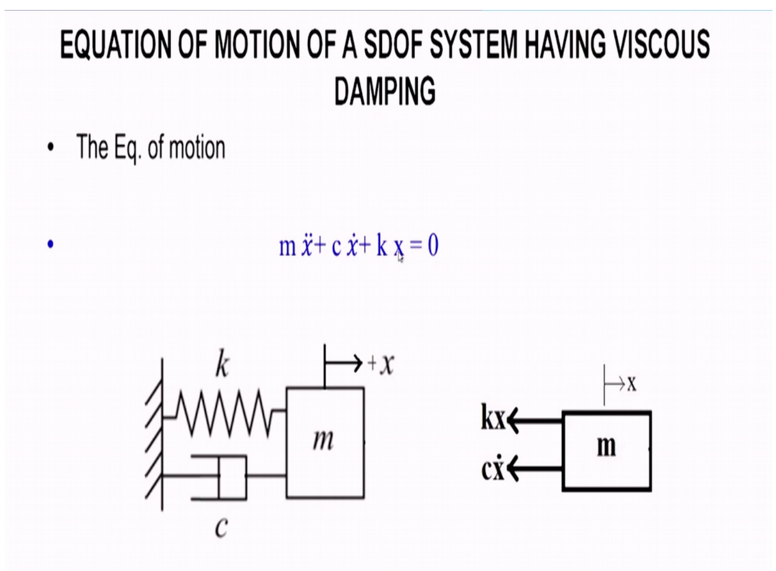

So - so this was our system, a single degree of freedom system consisting of m x double dot + c x dot + kx = 0, and we solve the system. So we found the roots of the equation.

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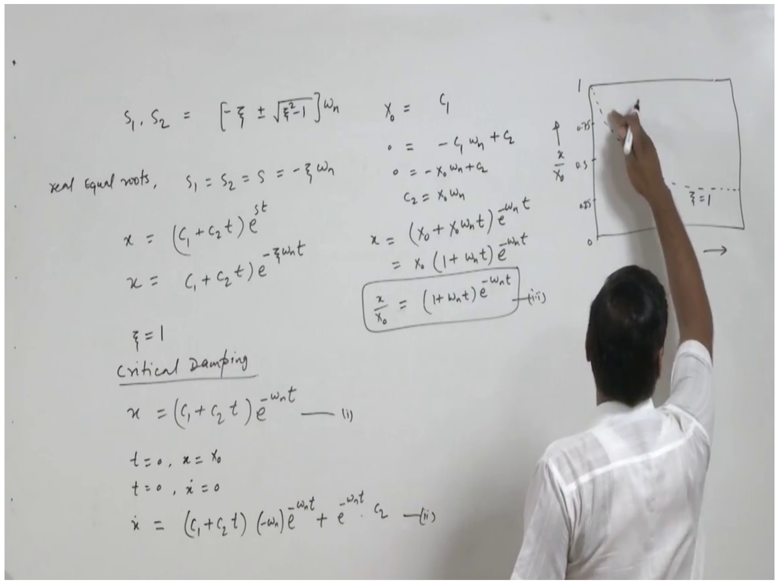

So s1 and s2 - so s1 s2 we found that was - zeta + - root zeta square -1 in to omega n, so t these are the roots of the solution of this system and we said that if we have the different like, we have the equal roots - we have equal roots, so we have equal and real and equal roots, so we have s1 = s2 let us say s and this is equal to - zeta omega n. So we say that in this case our solution is x = c1 + c2 t e power s t, and here c1 + c2 t e power - zeta omega n into t.

Now these conditions of real equal roots come when - comes when the under root part is 0 that is zeta = 1, so this is called, this case we called critical damping - critical damping when zeta is 0, so when we put here zeta = 1, we have x = c1 + c2 t e power - omega n t. So this is the response of single degree of freedom damped system with critical damping, here c1 and c2 they are the arbitrary constants and the values of these constants will be evaluated from the initial conditions.

So let us assume some initial conditions, so we take some general initial conditions that when t = 0, x = X0 some constant value and velocity is 0 at t = 0, so these initial conditions we will put to find the c1 values of c1 and c2. So we first we put x dot calculate x dot so x dot, so we differentiate the x the response of x that is let us say equation this is 1, so we differentiate this equation 1 with respect to t time and we here are the two functions of t, one c1 + c2 t.

And then e exponential function. so first the we do the differentiation, so the first function into the differentiation of second function + second function into differentiation of first function. So first function differentiation of exponential it is - omega n into e power - omega n t + differentiation of this is constant term so 0 and this is c2. now we put the values of initial conditions, so we have t we put in equation one the first initial conditions.

So X0 = c1, so here x = X0 and t = 0, so c1 and here t = 0 so this is 1. Now we put the second initial conditions in the equation 2, so here we have 0 = here c1 into - omega n so - c1 omega n this is 0 and here it is + c2 - + c2. now c1 = X0 so we put here so 0 = - X0 omega n + c2 so we find c2 = X0 omega n. now we have got c1 value of c1 and value of c2, we put in equation 1 and we find response, so x = c1 is X0 + c2 is X0 omega n t and here e power - omega n t.

So = X0 1 + omega n t e power - omega n t, so here we can write x upon X0 = 1 + omega n t e power - omega n t, so this equation three is the response of a critical system. We see that here the response contains a negative exponential with time, so e power - omega n t, so it means the response is decreasing with time, so we can if you want to put this in a graph. So here let us have this, so this is x, y, X0.

And this is omega n t, so it is 0 and this is1 this is 0.5, 0.75, 0.25 so here we will start this curve for get tending, so almost in this exponential so this is zeta = 1, so this is the response that is with the time tends to infinites tending towards 0 so this is this kind of motion is aperiodic motion. So what is the importance of critical damping?

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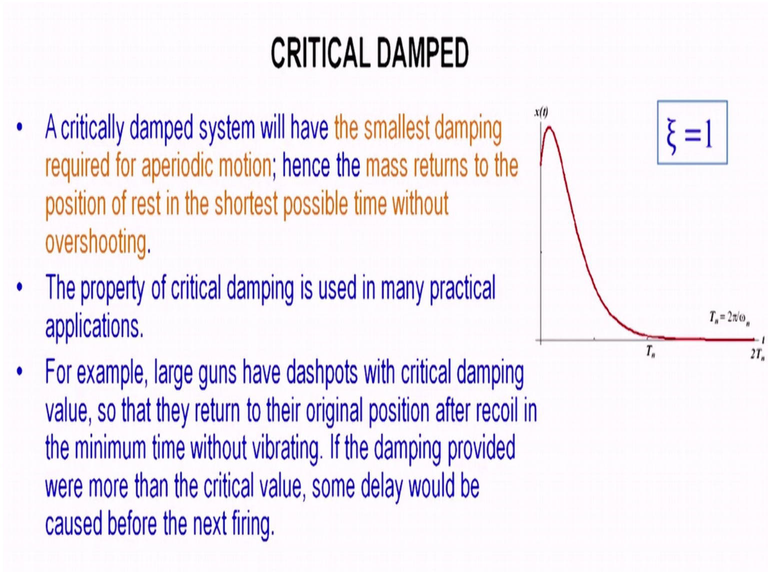

So this is the smallest damping okay that restrict the oscillations motion oscillation oscillatory motions, so it means that this is the minimum damping that we will provide and the system will go for aperiodic motion non-periodic or non-harmonic, and the mass will return in the shortest interval of time without any overshoot. So this critical damping finds its application in several mechanical systems like this guns when we fire a gun and there are some vibrations coming.

So before the next firing okay there should be recoiling before the next firing the - the - the system we should return into initial position in the minimum amount of time and therefore is critical damping is important in this application. We will discuss now the over damping that of course if you provide over damping system will return, but it will take more time than the critical damping and under damping will provide oscillatory motion.

So therefore the critical damping is the least damping that will keep the aperiodic motion. Now we come to the second type of system that is over damped systems, so here so again we start from in these roots s1 and s2.

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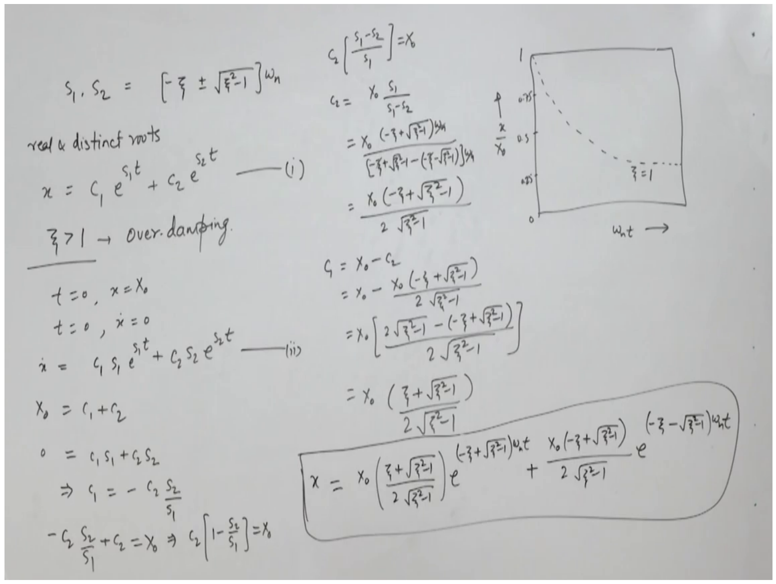

So we say that if our roots are real and - and distinct - distinct roots, so our x was c1 e power s1 t + c2 e power s2 t. so this was the solution of the differential equation when our the both roots they are real and distinct, and we see that in this case to be real means the under root quantity should be real, so zeta should be greater than 1, so here zeta > 1 and this is case of over damping - so this is the case of over damping system, okay.

So for over damping we have this equation let us say equation 1. now we similar process we will do, because we have to find the constant c1 and c2 so we take some initial conditions so let us take t = 0, x = X0 and t = 0 x dot = 0, so the same similar initial conditions as we took in case of critical damping. now we have to differentiate equation one to find x dot so x dot = dx by dt so here, c1 s1 e s1 t + c2 s2 e s2 t so here it is very simple.

Now we put the, so this is let us say equation number two, now we let us put the values of initial conditions, so the initial conditions1 the first we will put in equation one, so x = X0 and here c1 + c2, because exponential part will be unity so c1 + c2. Now we put second initial condition in equation two so x dot is 0 = here c1 s1 + c2 s2, okay. So from here we can find c1 = - c2 s2 by s1 - so c1 = - c2 s2 by s1.

And let us put this in this equation so c1, so we have c1 = - c2, so we put in this first this condition so - c2 s2 by s1 + c2 = X0, okay. So when we solve this we take c2 out and we get 1 -

s2 by s1 = X0. now here we can find c2 s1 - s2 by s1 = X0 and here c2 = X0 into s1 by s1 - s2 and X0, so here, s1 is - zeta + root zeta square -1, so it is - zeta + root zeta square -1 into omega n upon s1 – s2.

so when we say, s1 - s2 so - zeta + root zeta square -1 and - - zeta - root zeta square -1 and this here is omega n, so here omega n will be cancelled out and here - zeta + root zeta square -1 and this is - - + zeta so this zeta will be cancelled out, so here we will have X0 - zeta + root zeta square -1 upon 2 root zeta square -1, so this is c2. Now if you want to find c1, so c1 = X0 - c2 so we will have X0 - so this is c2 so X0 - zeta + root zeta square -1 upon 2 root zeta square -1 and so here we can take X0 out.

And we will have here 2 root square -1 - - zeta + root zeta square -1 upon 2 root zeta square -1. So here we have 2 root zeta square -1 and this is - zeta square -1 so we will have X0 zeta so -, - + so zeta and + root zeta square -1 upon 2 root zeta square -1. So we find c1 and we find c2 and then we put in equation one to find the solution. So here we have x = c1 e power s1 t, so c1 here is X0 zeta + root zeta square -1 upon 2 root zeta square -1.

So this is c1 e power s1 t, e power s1 is - zeta + root zeta square -1 omega n t, so this is s1 t and then + c2, so c2 is here X0 - zeta + root zeta square -1, and here it is2 root zeta square -1 and e power - so s2 so it is s2 t, so it is - zeta - root zeta square -1 omega n t, so this is the response of the over damped system. And we can see here that exponential here is because zeta is greater than 1.

So and so this quantity is less than zeta, so therefore this quantity is negative and here this quantity is negative, so again here there is the negative exponential and so the response will be decreasing, so here if you want to show on this curve so we have zeta = 1, let us say for zeta greater than 1 so let us say zeta = 2, so the response will be something like here like this, so here zeta = 2, similarly here zeta = 5, zeta = 10, and this zeta = infinite.

So what we see that for overdamped system to like critical damping system, the response is decreasing with time and it is aperiodic motion and when t tends to infinite the amplitude is tending to 0. we see that one more point we must not that, when we increase the damping okay like we are increasing the damping zeta = 1, then zeta = 2, then zeta = 5, zeta = 10 and so on.

So when we are increasing the damping there is more resistance of the damping force and so the - the motion becomes slower, so the system having more damping will be more sluggish than the lower damping, so higher the damping slower is the moment and so we see that critical damping and over damping system they do not have any oscillations they do not have any vibrations they just they are a periodic motion and with time tends to infinite they are tending to equilibrium position.

Now we will discuss the third and the most useful system that is the under damped system, why the under damped system are more important, because the value of damping in structures or systems are usually very less that is between 0 and 1 not more than 1, so this kind of systems are more in practice and therefore we will discuss in detail these systems so let us again start with this equation so we found that the when we have the imaginary roots that is we have s1.

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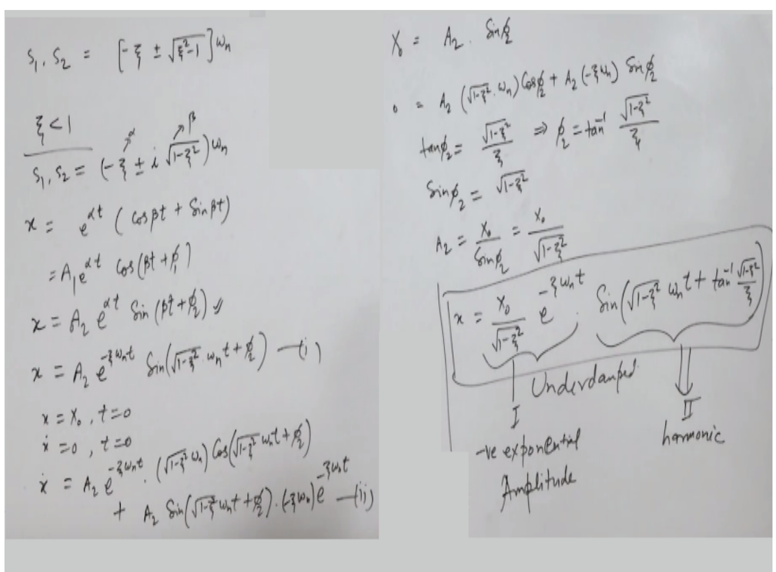

So this imaginary roots is only possible when the under root quantity is negative it means that zeta is less than 1, so when zeta is less than 1 we have s1 s2 = - zeta + - i root 1 - zeta square omega n. so we found that for such a system we have the - we have the solution x = so it was like e power alpha t cos beta t + sin beta t, where this is alpha and this is beta.

We can also write this as e power alpha t some constant A 1 e power alpha t and cos beta t + pi 1 or we can also write A2 e power alpha t sin beta t + pi 2, because these are same beta same frequency here and therefore we can write these two harmonics in terms of either cosine with some phase and either sine with some phase and these things we have discussed already in the first week lecture. So let us assume that we are taking this expression.

So we have x = A2 e alpha t sin beta t + pi 2, so here we write A2 here we have e power alpha is - zeta omega n t into sin, beta is root 1 - zeta square omega n t + pi 2. now again we will take the initial conditions, so we take the initial conditions and we have x = X0 at t = 0 and x dot = 0 at = 0. So let us differentiate x dot =, again here are two functions, so A2 e power - zeta omega n t and we differentiate this.

So here we have root 1 - zeta square omega n into sin is cos the differentiation 1 - zeta square omega n t + pi 2 + A2 sin root 1 - zeta square omega n t + pi 2 into so we differentiate this - zeta omega n into e power - zeta omega n t, so this is a question number one and this is equation number two. Now we put the values of initial conditions and if we put the initial conditions in one so we get X0 = A2.

So this is1 and this is sin pi 2, A2 in to sin pi 2, so here sin and we put in equation two so 0 x dot is 0 =, here we will have cos pi 2 so A2 into root 1 - zeta square omega n into cos pi 2 + A2 that is here - zeta omega n sin pi 2 - sin pi 2, so here we can write tan pi 2 = - so we can write here tan pi 2 - tan pi 2 = root 1 - zeta square upon zeta, okay. So from here we can find sin pi 2 = root 1 - zeta square, okay.

And we put this value here so we find A2 = X0 upon sim pi 2 = X0 upon root 1 - zeta square, so we find A2, we find pi 2. so here pi 2 = tan inverse root 1 - zeta square upon zeta, so we can write the equation like x = X0 upon root 1 - zeta square e power - zeta omega n t into sin root 1 - zeta square omega n t + here tan inverse root 1 - zeta square upon zeta, okay. So this is the equation for under damped system.

Now here if we note that here are two parts, so the first part this first part and this is second part. This first part is exponential - exponential - exponential and this is harmonic - harmonic and this is negative exponential so it is negative exponential. So here we see that if we have this harmonic motion and this is showing as amplitude - amplitude, the amplitude is decreasing with time, because here is negative exponential and so amplitude is decreasing with time.

So what will be the type of this curve? So we can see here response we can plot the response.

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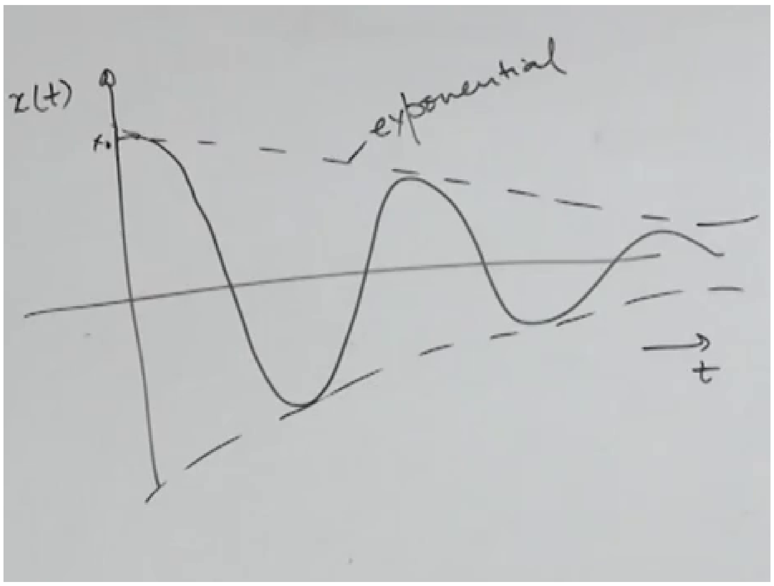

So we have here - so we have here the response like this is exponential, so amplitude is decreasing exponentially and this is the harmonic, so here this is X0 so here let us say X0, x t and this is t, so we can see here the under damped system also here the response of under damped system.

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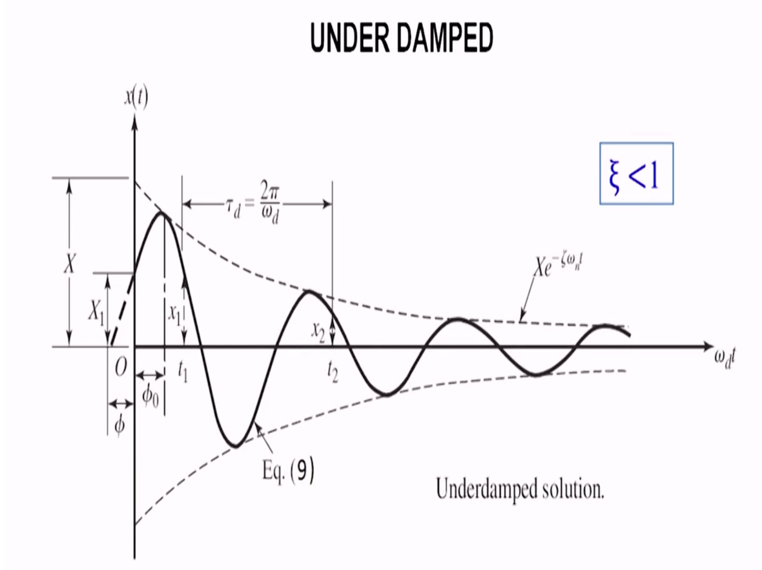

And we can see that it is decreasing - the amplitude is decreasing as x e power - zeta omega n t, where we can define here this is our x, X0 pi root 1 - zeta square and the motion is harmonic here. So thank you for your attention in this lecture and see you in the next lecture.