Introduction to Mechanical Vibration
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Lecture - 03
Fourier Series and Harmonic Analysis

Welcome to the lecture on Fourier series and harmonic analysis. We know that the vibration motions can be period, harmonic and random. Similarly, the vibration force, the excitation forces can be period, harmonic or random. So, we know that a period motion is a motion that repeats itself after an equal interval of time. Moreover, harmonic motion also repeats itself after an equal interval of time.

But additional condition is that the acceleration is proportional to displacement and the direction is always towards the mean position. So, today we will discuss whether there is any relation between period motion and harmonic motion. Means can be express a periodic function in terms of harmonic functions. So, therefore, the lecture contents are:

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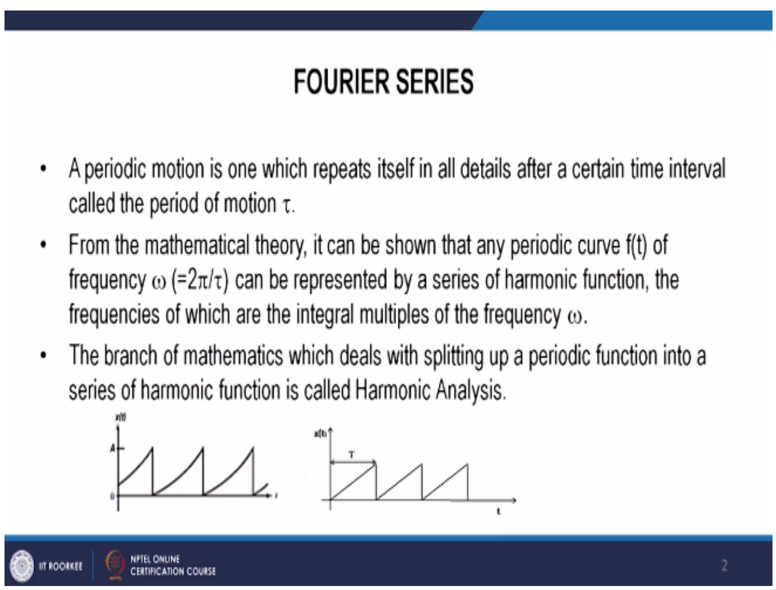

We will discuss. So, what is Fourier series? So, Fourier series we will discuss. Fourier series expresses a period function in terms of a harmonic function and the branch of mathematic that deals with splitting a period function into a series of harmonic function is called harmonic analysis. So, here we can see there are period functions they not harmonic functions but according to the Fourier series we can express these functions in terms of harmonic functions.

Harmonic functions, means sine functions and cosine function. So, here the Fourier series says that if we have a period function FT

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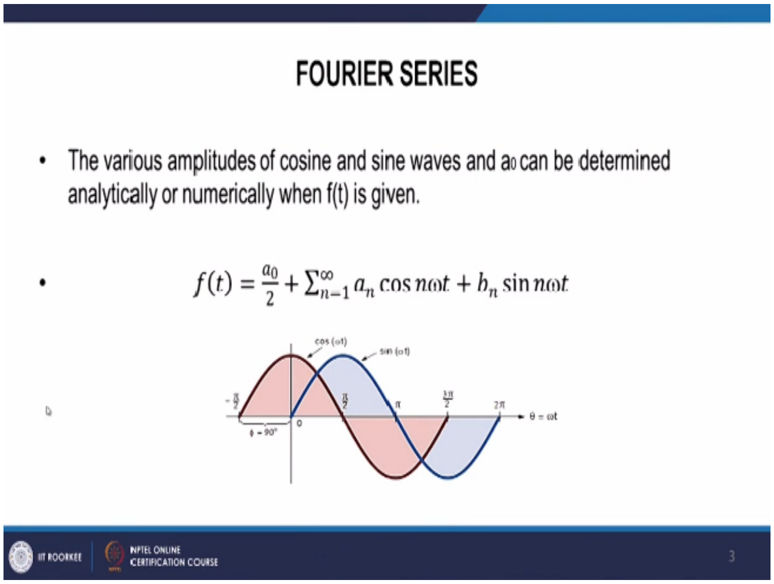

we can express this period function FT as equal to a zero by 2 + sigma n = 1 to infinite a n cos n omega t + b and sin n omega t where omega is the frequency of the period function and we can see the basic harmonic function they have sine. It starts with zero maximum at pi by two and then again zero at pi then negative minus at three pi by two minus and then again zero at 2 pi.

So, this completes one cycle of sin function and similarly cosine function is maximum at zero then minimum at pi by two and negative maximum at pi and then again zero at three pi by two. And then again it will be maximum at 2 pi. So, it will complete one cycle. So, we can express these f t into harmonic functions.

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$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos n\omega t + b_n \sin n\omega t \quad \text{—(1)}$$

$\omega = \frac{2\pi}{\tau} =$ frequency of the periodic function

$\tau =$ time period of the periodic function

$$\int_t^{t+\frac{2\pi}{\omega}} f(t)\,dt = \int_t^{t+\frac{2\pi}{\omega}} \frac{a_0}{2}\,dt + \int_t^{t+\frac{2\pi}{\omega}} \left[\sum_{n=1}^{\infty} a_n \cos n\omega t + b_n \sin n\omega t\right] dt$$

$$= \frac{a_0}{2}\int_t^{t+\frac{2\pi}{\omega}} dt + \int_t^{t+2\pi/\omega} \sum_{n=1}^{\infty} a_n \cos n\omega t\,dt + \int_t^{t+2\pi/\omega} \sum_{n=1}^{\infty} b_n \sin n\omega t\,dt$$

$$= \frac{a_0}{2}\left(t + \frac{2\pi}{\omega} - t\right) + 0 + 0$$

$$= \frac{a_0}{2} \times \frac{2\pi}{\omega}$$

$$\boxed{a_0 = \frac{\omega}{\pi}\int_t^{t+\frac{2\pi}{\omega}} f(t)\,dt} \quad \text{—(2)}$$

So, f t is a period function that we can write a zero by two + sigma n = 1 to infinite a n cos n omega t + b n sin n omega t. So, if we know because we know omega. So, omega is 2 pi by tau where this is equal to, this is natural frequency or we can call it frequency of the period function and tau is time period of the period function. So, we can evaluate a zero, a n and b n.

Because a zero n and b n they are the constants that we must know then we can express our period function f t into harmonic function. Now, here we can see that these harmonic functions vary n = 1 to infinite so we can have infinite terms and they will assemble to make this period function. In order to get a zero n and b n let us say this is equation number one we have to find. So, what we do we integrate both side of this equation with respect to t for one cycle.

So integrate f t d t for a cycle so any instant of time t. So, two t + 2 pi by omega so one cycle this is time period 2 pi by omega is time period and we integrate. So, a zero by two d t, d to t + 2 pi by omega and then + here we integrate t, t + 2 pi by omega sigma n = 1 to infinite a n cos n omega t + b n sin n omega t d t.

So here we can write a zero by two dt because a zero by two is constant, we can write t t + two by omega + sigma n = 1 to infinite a n cos n omega t dt + b n sin, n omega t dt and sigma n = 1 two infinite and then integrate t, t + 2 pi by omega. So, in order to solve these integrals, we will use some formula of integration. So, from here we can write this a zero by two and we can write it is t + 2 pi by omega minus t +.

Now, these terms so we have integral for one cycle t to t + 2 pi by omega and we have, n is constant and for n = 1 it is a 1 for n = 2 it is a 2 and cos omega t d t. Now, according to expression here we can see t, t +

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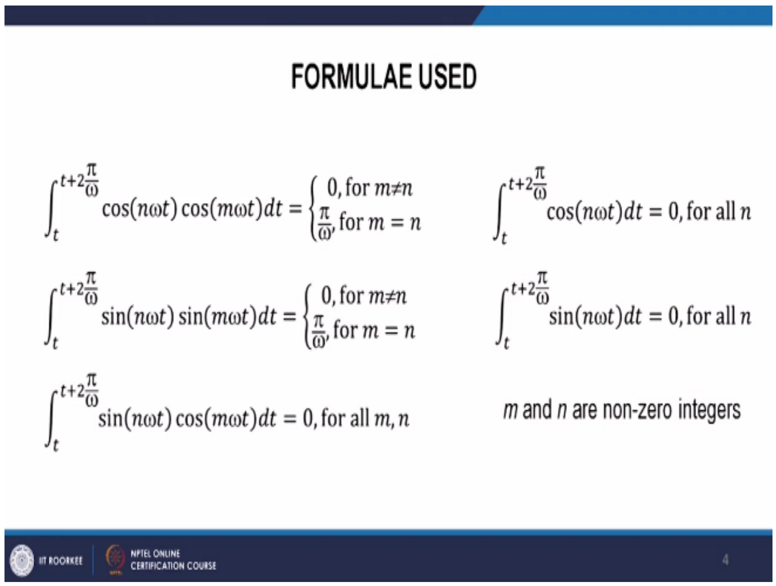

2 pi by omega cos n omega t dt = 0 for all n. So, it means the integration a n cos n omega t dt = 0 or any n value. So, this will be equal to 0 similarly we can see that sin n omega t dt = 0 integral t to t + 2 pi by omega is zero for all n values. So here this also the summation will be zero of integral so we will have only a zero by two into 2 pi by omega.

Now, we can cancel out two and two so we can write a zero = omega by pi integral t, t + 2 pi by omega f t d t. So our equation gives us the value of a zero that is a zero = omega by pi integral t to t + 2 pi by omega f t d t. So, it means that this is period f t is the periodic function and we integrate it from t to t + 2 pi by omega or any cycle. One cycle, we will find one cycle we will find the value of a zero.

Now, we have to find the value of a n. So, in order to get the value of a n let us first multiply this equation both side with cos and omega t and then we do the integration for one cycle.

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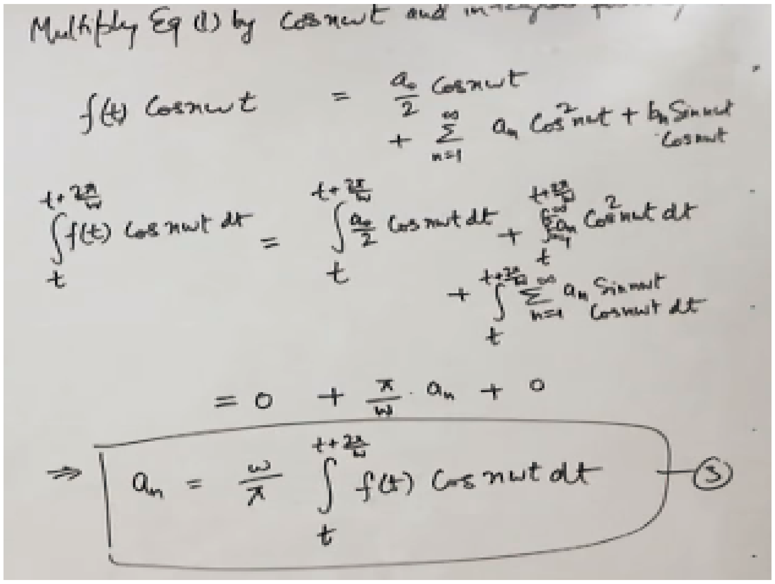

So, let us first multiply, so multiply equation one by cos and omega t and integrate for a cycle. So here f t cos n omega t and here it will be a zero by two cos n omega t then + sigma n = 1 to infinite a n cos n omega t square + b n sin n omega t into cos n omega t. So, this should be multiplied with cos and omega. Now we have to integrate for one cycle so we integrate so f t cos n omega t d t.

When we integrate t to t + 2 pi by omega = a zero by two cos n omega t dt and this is t two t + pi by omega + again we integrate t to t + 2 pi by omega a n cos square n omega t and here will come the summation n = 1 to infinite dt + integrate t to t + 2 pi by omega sigma n = 1 to infinite a n sin n omega t cos, n omega t d t. So, this is our resulting equation. Now again we will apply the formula that we can see in the slide.

So, a0 by 2 cos n omega t, a0 by 2 is constant so we have integral cos n omega t d t. We have integral cos n omega t dt and for one cycle we can see it is zero for all n. So, here we can have zero. Now we come to another integral that is cos square n omega t d t. So, we see that for cos square n omega t dt the first expression says that cos n omega t cos n into cos m omega t dt for one cycle that is t to t + 2 pi by omega = 0 for m not = n and pi by omega for, m = n.

So, when our n will go for a limit n = 1 to infinite for all the terms this integral will be zero except for n because n = n so we will have this integral pi by omega into a n. So, we have the integral is pi by omega and the constant a n remains. Again here the third term that is sin n omega t into cos n omega t and we can see the third formula it says that the sin n omega t and cos n omega t dt = 0 for one cycle for all m and n.

So, this term will be zero from here we can find a n = omega by pi integral t t + 2 pi by omega f t into cos and omega t d t. So, this is our equation number three that gives us the value of a n. So, we know the periodic function we multiply it by cos and omega t then we integrate for one cycle and multiply with omega by pi we will get a n. Now our objective is to get the last value b n so that is the coefficient of sin and omega t.

So, in order to get b n we will follow the similar process that we did for cos to find a n. So we will multiply this time with sin and omega t. The equation one will be multiplied by sin and omega t and it will be integrated for one cycle. So, let us do.

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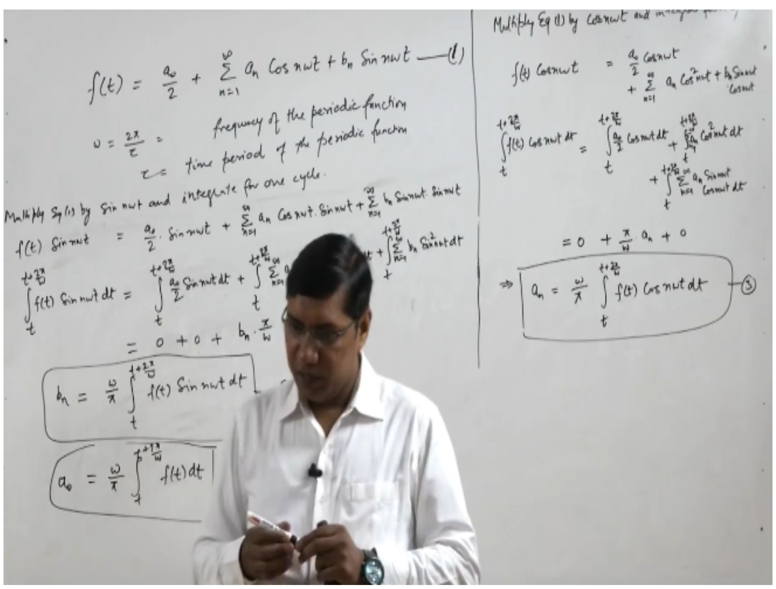

So, multiply equation one by sin n omega t and integrate for one cycle. So, if we multiply one we will get f t sin n omega t = a zero by two into sin n omega t + sigma n = 1 to infinite a n cos n omega t into sin n omega t + sigma n = 1 to infinite b n sin n omega t into sin n omega t or sin n omega t square whatever. Now, again we will integrate this equation from t to t + 2 pi by omega that is for one cycle.

So, we integrate so f t sin n omega t dt t two t + 2 pi by omega = again a zero by two sin n omega t d t, t two t + 2 pi by omega + t, t + 2 pi by omega limit and sigma n = 1 to infinite a n cos n omega t into sin n omega t dt + integral t to t + 2 pi by omega sigma n = 1 to infinite b n sin n omega t so here square d t. Now, we have written this expression with integrals so we will use the formula that we can see here.

So, here the first term in the right side, right hand side so that is a zero by two sin n omega t dt so we can see here that sin omega t dt is zero for any n. So, this term will be zero + the second term. So, second term is cos n omega t into sin n omega t dt and this for one cycle so we see here cos n omega t into sin n omega t is zero for all m and n so this second term will also be zero. Now, we come to the third term.

The third term we have sin square n omega t so again the second relation in the slide we can see integral sin n omega t into sin omega t dt = 0 for m not = n. So for all the terms there will be only one term where m = n rest all the terms in that infinite series infinite summation one to infinite will be m not equal one and that will be zero. So only one term that is m = 1 will be there and so it is value is pi by omega.

So, here we have been the constant into pi by omega, the integral. But now we can write here b n = omega by pi integral t into t + 2 pi by omega f t sin n omega t d t. So, this is equation number four. So, we have these, equation one tells about the Fourier series and then equation two, three and four gives the value of coefficient of these Fourier series. Now, we will take one example.

And I will tell you how to apply this formula to calculate the coefficients of the Fourier series. So, let us take one example

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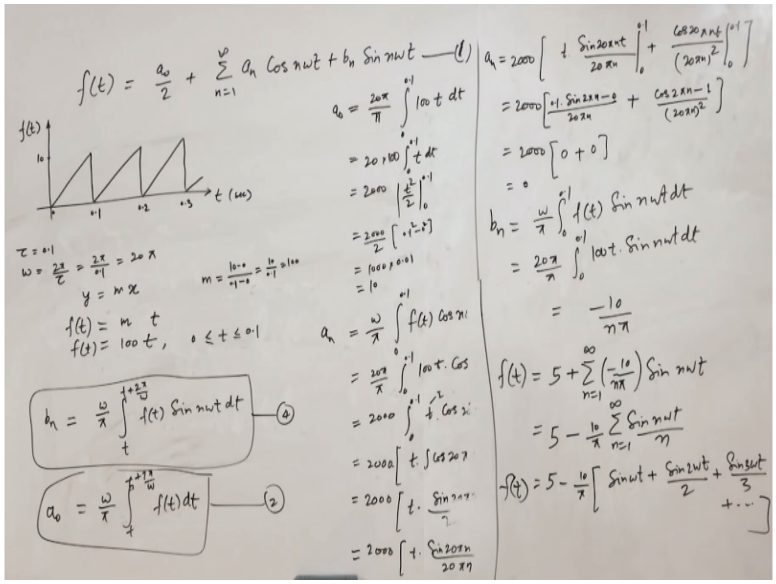

So, let us take one triangular shape periodic function. So in this triangular shape periodic function we see that this is zero and zero to zero point one, from zero point one there is repetition of the cycle then zero point two it repeats. It means we have the time period of zero point one. So we have tau, tau is zero point one. So let us say this is second so zero point one so we can find omega and omega is 2 pi by tau so that is 2 pi by zero point one.

So, it is twenty pi. So this is omega the frequency of the period function here omega is circular frequency that is in radian and now we have to write these f t the function f t we have to write the function because we need to know how these functions varies in one cycle. So let us write this function. So let us consider the cycle zero to zero point one in this cycle we have to express this function so f t.

So, we see that f t is growing from zero to point one as a lineately. So, it is an equation is line and equation of line that is passing through origin can be written as like y = m x. So, this is general equation of line when we have the x and y coordinates. Now, here our y coordinates is f t so we write here f t and x is t so we write here t now m what is m? So, m is the slope of the line so to find the slope m.

We write m = we take points these points and this point and we write this is y two minus y one upon x two minus x one. So, y two here means the value of f t, here. So, value of f t, here is ten and value of f t, here is zero upon. So, value of x two minus x one so that is time so t is point one minus zero so we have ten by zero point one = hundred. So, we can write here f t = 100 t. So, here f t = 100 t.

Now, we say it is zero less or = t less or = 0 point one. So the complete cycle it is defined as f t = hundred t. So, we got f t now our objective is to find a zero a n and b n so we will put these value of f t in these formula and we will compute. So, let us do it. So let us calculate a zero so a zero = omega by pi omega we calculate it is twenty pi by pi and cycle we start from zero to point one so the t we start t = 0.

So, it is zero to point one and then f t d t. So, f t is hundred t and dt now we solve this integral twenty into hundred we take out and t dt to zero point one t dt so we have two thousand into. So, t integral is t square by two and we have limit zero to zero point one. So, integral t power n = t power n + one by n + one so we have two thousand by two. We take two out and then zero point one square minus zero square.

So, we have one thousand into zero point zero one point one square so this is ten. So, we got a zero = ten. So, the first coefficient we got now we need to know a n. So, a n = omega by pi f t cos n omega t d t. So, it is zero to zero point one again we will take so it is 20 pi by pi zero to zero point one 100 t into cos. So omega is twenty pi so it is twenty pi n t two d t. So, here we can write two thousand because pi is canceled out.

And this come two thousand zero to zero point one t into cos twenty pi and n t d t. Now, here there are two functions of t so we have to first do the integral by parts and then we will apply the limits. So, we have two thousand. So integrals by parts so let us take this is first function and this is second function. So, we have the formula that we apply so the formula is that we write like first function into integral of second function.

So it is cos twenty pi n t dt minus integral of so differential of first function. So differential of t is one because dt d by one and then integral of the second function. So, this is cos twenty pi and t dt and then we will put the limit zero to zero point one on this. So, we can solve it two thousand we have t into so this we have cos will go to sin twenty pi and t by twenty pi n minus so again this integral so cos this is sine twenty pi and t upon twenty pi and d t.

And this is zero to zero point one. So, we will solve two thousand t into sin twenty pi n t upon twenty pi n minus. So, sin integral is minus cosine. So, minus, minus + and this is cos twenty pi n t by so twenty pi n again will be multiplied here. So, pi n square and this limit will go zero to zero point one. So, we can write here a n = so we give the limits to these so it is two thousand t into sin twenty by n t by twenty pi n limit zero to zero point one + cos twenty pi n t by twenty pi n square this again limit zero to zero point one.

So, here point one we put zero point one into this is sin 2 pi n because t = point one. So, it is 2 pi n of course and minus because this is zero so t = 0 upon twenty pi n + this is point one so it is cos 2 pi n minus again it is zero so cos zero is one. So, one twenty pi n square = 2 thousand. Now, sin 2 pi n is zero because for any n = integer value n = 1 to infinite we will always 2 pi by four pi, six pi so these are always zero.

So, this is zero so it zero + here cos 2 pi n is always one and so one minus one for any n. For example, n = 1 it is cos 2 pi so cos 2 pi is one. Cos four pi is one so one minus one is again zero. So we get that a n is zero. So, we find that this the value of the a n is zero now we calculate b n so b n we will calculate similarly so it is omega by pi zero to zero point one f t sin n omega t d t.

So, can put here twenty pi by pi zero to zero point one hundred eighty into sin n omega t dt and we can solve this integral in the similar way and we find the value of this integral is minus ten by n pi. So, we go for solution similarly to n and we will find this as (()) (37:22). So, what is our Fourier series so for this period function so f t = a zero by two +. So, we can see this equation one and we can put these values.

So f t = a zero is ten so ten by two = five + sigma n = 1 to infinite. Now a n is zero so there will be no cosine term only b n and sin omega t so b n is minus ten upon n pi. So, here we can write f t = five + b n, b n is minus ten by n pi and here is sigma n = 1 to infinite and he had signed n omega t. Now, we can write like five minus ten upon pi sin n omega t by n and sigma n = 1 to infinite because n is one to infinite.

We keep all the n terms here so we can express this five minus 10 by pi and sin omega t + sin two omega t by two + sin three omega t by three + sin one. So, this is my Fourier series that contains n terms. Another thing is that there are even and odd functions so an even function is that if we put x minus t = a. If we put minus t in spite of t it is x t. So, this is even function and for even function we have only cosine, no sin terms.

But if we have odd functions means if we put x minus t = minus x t. It is minus x t so Fourier series contains only sin terms. So, for example this function the rectangular wave that we discussed this function f t = hundred t we put f minus t so if we put f minus t = hundred minus t so = minus hundred t = minus f t. So, it is an odd function and that is why we see that the cosine coefficient was zero. So, therefore we have only sin.

Now, for this triangular series we can see that the series is infinite because it goes from n = 1 to infinite but first few terms can

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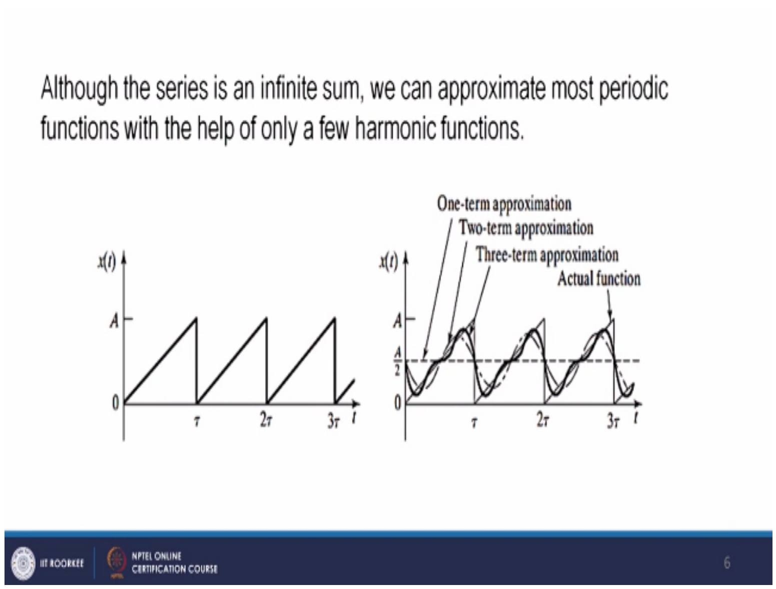

Give us a feeling that the harmonics are going to produce the periodic function. So, we can see here one term approximation that is purely the constant term a by two that is here ten by two so five and then the second term two term approximation it is some sin term that is in negative because there is negative sin and so the sin for sin term then two terms, three terms and they are going to result the actual periodic signal. Thank you for your attention and see you in the next lecture.