Introduction to Mechanical Vibration
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Lecture - 29
Damped Free Vibration

So welcome to the lecture on two degree of freedom systems. So we will consider now the damped free vibration of two degree of freedom systems.

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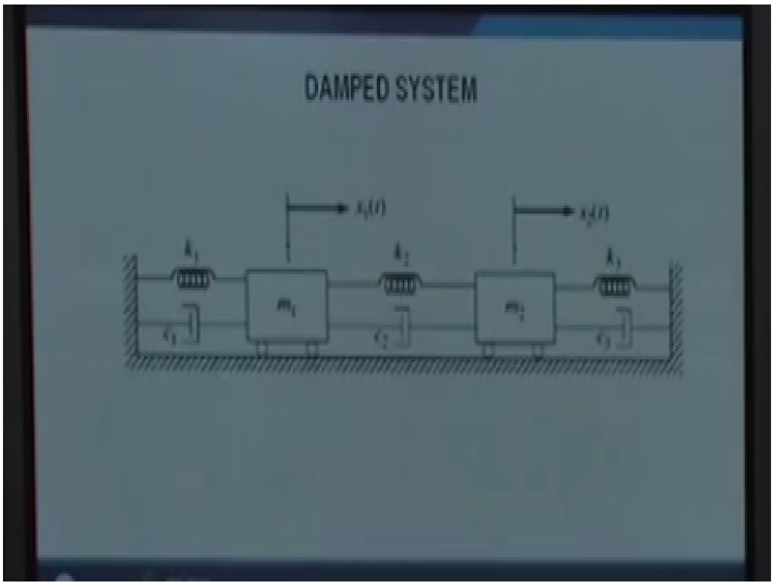

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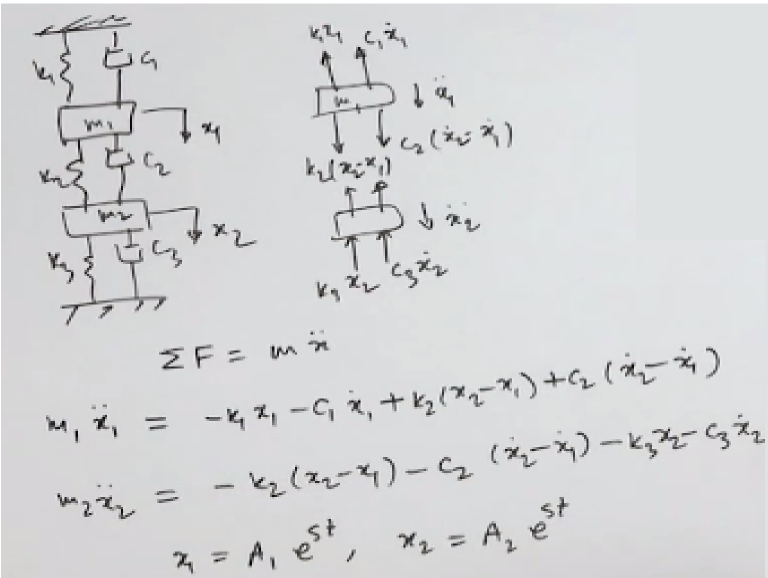

So we have such a system that is two degree of freedom, so this is the mass m1, m2; this is k1, k2, k3; c1, c2, c3 and this is the x1 and x2, the two degree of freedoms. Now to study these systems what we have to do, we have to first write the equations of motion so we make the free body diagram and here one more force is added in compare to the previous analysis that is the force due to the damping element.

So damping element c1, c2, c3; they will act certain forces in these equations and so we will write the equations and that we can write. So if this is m1, we have the spring force here, damping force here okay and then similarly here we have these forces k2 x2 - x1, c2 x2 dot - x1 dot and here k1 x1 and c1 x1 dot. Similarly, this mass will have same forces in the opposite directions and here these forces a3 x2 and c3 x2 dot.

So and we take this like this. So these are the forces on the free body diagram and when we apply the sigma f = mx double dot for both the cases, we can write the equations of motion. So here for example m1 x1 double dot = - k1 x1 - c1 x1 then + k2 x2 - x1 and + c2 x2 dot - x1 dot. Similarly, for mass m2, so m2 x2 double dot = so all the forces are opposite to so they will - k2 x2 - x1 - c2 x2 dot - x1 dot - k3 x2 - c3 x2 dot.

So this is the equation of motion and to solve these equations of motion, we will assume some solution like we will assume x1 = A1 e power st and x2 = A2 e power st. So when we put here where A1 and A2, they are the amplitude so their ratio will define the mode shape and A1, A2 they are constant, s is also a constant and when we put in these equations, we will find s.

And we solve for the s that is the root of that characteristic equation and from there we will find the final solutions with the help of the initial conditions. So rather than solving on this system, we take some simple damped system to demonstrate.

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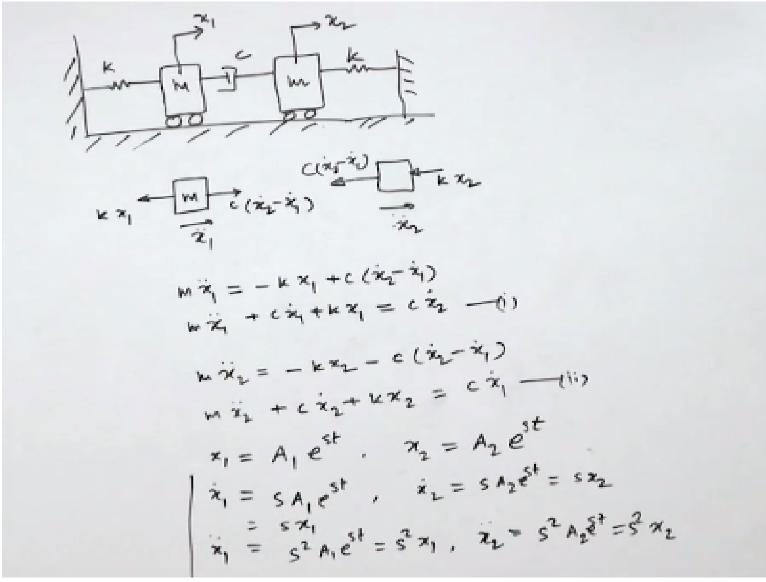

So we have this system. So here is this mass, here is a damper and there is a second mass and this is the rigid. So there is no friction between the mass and the surface and this is k and here is again one spring, so this is also k and this is c, this is m and so here it is x1 and this is x2. So this is the system and now we have to write the equations of motion for this simple system. So this is the first free body diagram of this mass m first mass.

So we will have here this force kx1 or if we pull here, it will be kx1 and then this is the damper and if we assume x2 > x1 so this damper force will pull it c x2 dot - x1 dot and this is x1. Now here the second mass, we will have the same force here c x2 dot - x1 dot opposite direction and this is spring because when it will compress, this will apply a force k times x2 and here it is x2.

So we can write this equation of motion for this system so here mx1 double dot = - kx1 and + c x2 dot - x1 dot. So here mx1 double dot + cx1 dot + kx1 = cx2 dot so this is equation one. Now for the second mass, we will have this so we have to write the equation of motion mx2 double dot = so - kx2 - c x2 dot - x1 dot so - kx2 and -.

So because this is in this direction and they are opposite forces so they will be having - sign. So here mx2 double dot + cx2 dot + kx2 = c times x1 dot. So this is equation number two. Now as I said that we have to assume the motion x1 = A1 e power st and x2 = A2 e power st. So here why we are taking such a form because s is a complex, it could be complex number and due to the introduction of damping the motion, the roots could be complex.

So that is why we take this kind of solution and so now we put this here. So what is x1 dot is s into A1 e power st and equal to sx1 and here x2 dot = sA2 e power st so that is equal to sx2. Similarly, x1 double dot = s square a1 e power st so that is s square x1 and x2 double dot = s square A2 e power st, so that is s square x1 and x2 double dot = s square A2 e power st, so that is s square x2.

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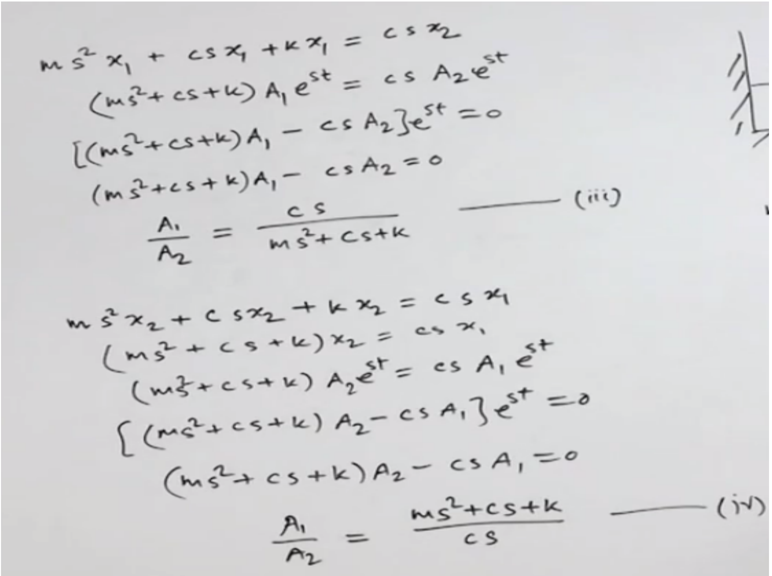

So now we put these values in equation one and two so what do we get, so mx1 double dot so x1 double dot is s square and ms square into x1 + csx1 + kx1 = csx2. Now ms square + cs + kx1 and x1 is = A1 e power st and equal to cs and x2 is A2 e power st. So of course we can have because e power st if we do like ms square + cs + kA1 - cs A2 e power st = 0.

Of course e power st exponential it cannot be 0, so we have to write ms square + cs + kA1 - csA2 = 0 and so we can write here A1 by A2 = cs by ms square + cs + k and this is equation number three. So here we have obtained the amplitude ratio as a function of s. Now we put these values in equation number two, so we get m so x2 double dot is s square x2 + c so x2 dot is sx2 + kx2 = c, x1 dot is sx1.

So here ms square + cs + k and x2 = csx1 so here ms square + cs + k. Now x2 is A2 e power st, so A2 e power st = cs, x1 is A1 e power st. So again here ms square + cs + kA2 - csA1 e power st = 0. Because e power st cannot be 0, we have to write ms square + cs + kA2 - csA1 = 0. So we can get A1 by A2 = ms square + cs + k by cs. So we get this amplitude ratio A1 by A2.

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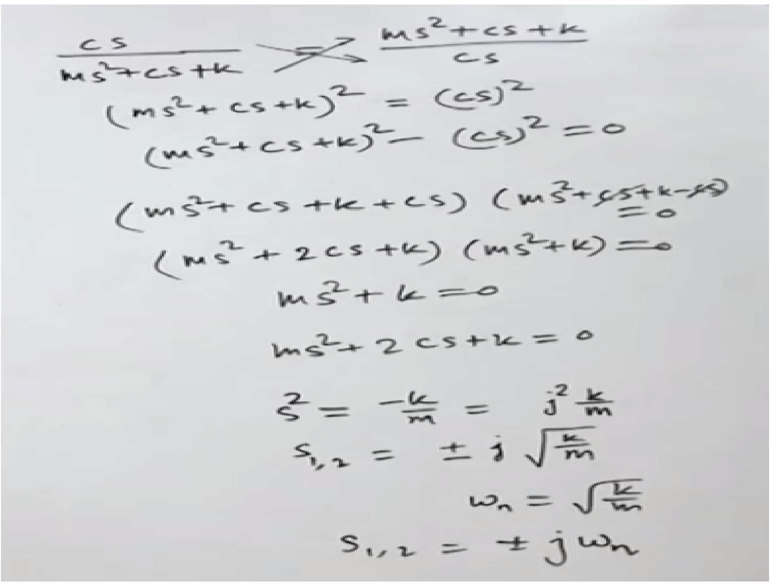

Now what we can do, we can equate these two equations three and four because both are the ratios of A1 by A2 and so therefore we find we do like cs by ms square + cs + k = ms square + cs + k upon cs. Now we cross multiply and so we get ms square + cs + k square because these are the same terms so they will be square equal to cs square. So we can write ms square + cs + k square - cs square = 0.

Now we can because a square + b square = a + b into a - b. So we can use this formula so ms square + cs + k + cs and second factor is ms square + cs + k - cs and this is = 0. So what we get ms square + 2 cs + k and here cs will be cancel out so we will have ms square + k = 0. So it means that ms square + k = 0 and ms square + 2cs + k = 0.

So now from these two equations, we can get the values of s. So when ms square + k = 0 so s square = - k by m and that is - complex number j square because - is = j square. So s = j root k by m + - and root k by m is if we say that omega n = root k by m, so we can write s1, 2 = + - j omega n.

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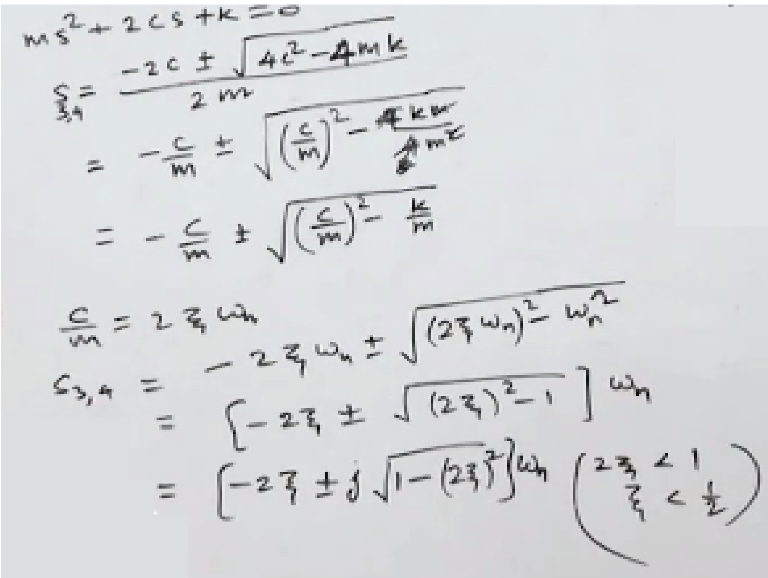

Now what about this equation? So if we write ms square + 2 cs + k = 0 so we have s = - b so - b is - 2c + - under root b square so b square is 2c whole square. So it is 4c square - 2ac so 2 into A is m and c is k by 2A so 2 into A is m. And so we will have this is third and fourth root of s where we got here two roots of course this is negative - j root k by m and this is + j root k by m and here 3 by 4.

So this we can write - c by m + - under root so we take m inside so we will get c by m whole square -. So here 2mk by 4m square so 2km by 4m square. So here it is 2 and m is cancel out, - 4ac so it is b square - 4ac so it is 4 into a into c. so it is 4 here it is 4, so 4 4 cancel out, we will get k by m. So - c by m + - under root, c by m whole square - k by m.

Now c by m = 2 zeta omega n so we have s3, 4 = - 2 zeta omega n + - under root 2 zeta omega n square - and k by m is omega n square. So we will get - 2 zeta + - under root 2 zeta whole square - 1 and omega n. Now we assume that this part is such that 1 - 2 zeta whole square. So 2 zeta, it is less than 1 so it means zeta is less than half so damping factor is less than half so we are taking basically the under damp system.

So here we will have - 2 zeta + - so we take it - outside so it is root - is j under root 1 - 2 zeta whole square and this is omega n. Okay so here we have obtained the roots, now what will be the solution of this. So here we see that we have the natural frequencies here so they are because they are the complex roots they are in pair.

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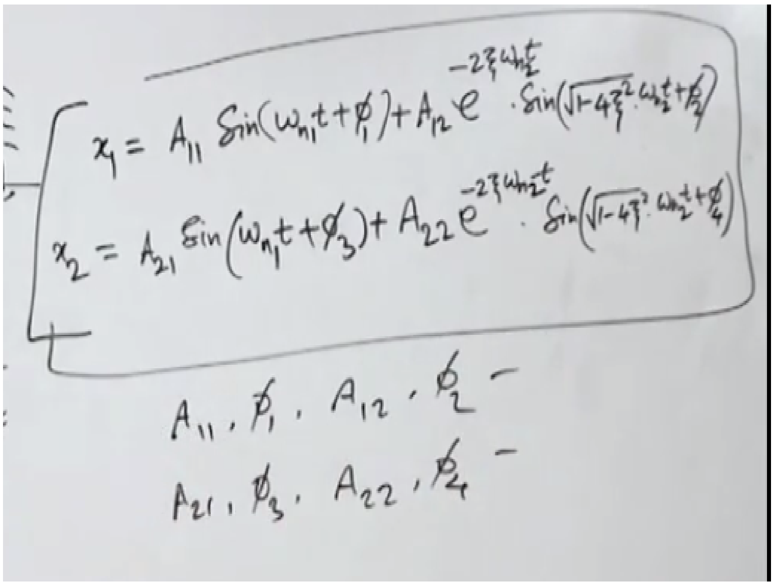

And so we will have a solution like x1 = A11 sin omega n1 t + phi 1 because this root will give us a solution like this, harmonic one and the other one due to the second frequency we will get the solution like A12 exponential - 2 zeta omega n t into sin root 1 - 2 zeta square so it is 4 zeta square into omega n t + phi 2. Okay so this you can understand from because this is complex root.

So we will have some exponential free decay vibration as we discussed to see the damped response of the single degree of freedom system. So this is similar, but only here the damping is not zeta, but it is 2 zeta so it is 1 - 2 zeta square is 4 zeta square and here again it is 2 zeta. Similarly, x2 will be A21 sin omega n1 t + phi 3 + A22 exponential.

So here it is omega n1 - 2 zeta omega n2 t into sin root 1 - 4 zeta square into omega n 2t + phi 4. So here we can see these are the free vibration response of the system and the values of this constants that is A11 phi 1; A12 phi 2 or A21 phi 3; A22 phi 4. They will all depends on the initial conditions.

Now if the system is given an initial condition equal disturbance so if initially we displace what the system with equal amplitude like 5 mm both then the system will vibrate purely with the first natural frequency because of course from here we get the mode shape corresponding to the first natural frequency and mode shape corresponding to second frequency. So when if the equal initial disturbance to both the masses, the system will vibrate with the first natural frequency.

And so only the first term that is sin omega n 1t + phi 1 will be there for both the equations. However, if we are giving completely opposite initial conditions, then the system will vibrate with the damped free vibration. So the second part will be there. Because if we put here this s12, we will get A1 by A2 = + 1 and when we put these values, we will get A1 by A2 = - 1.

So the system will be in case of omega n2 it will be vibrating as a damped free vibration. Okay so that is how we can study a damped vibration of two degree of freedom system. So in case of first natural frequency, damper is ineffective so there is no role off damper because both the system they are vibrating as pure sin wave okay, but in other case damper is effective with a damping factor of 2 zeta okay. So I thank you for this lecture and see you in the next lecture.