Introduction to Mechanical Vibration
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Lecture - 02
Addition of two harmonic motions and beat phenomenon

We will discuss the addition of two harmonic motions and beat phenomenon. As we discuss that, what is simple harmonic motions? When we are going to add the two motions, what are the resultants that we are getting out of these two motions? So, the contents of this lecture will follow the simple harmonic motion, we will recall it and then the phase difference and then we will discuss the addition of two synchronous harmonic motions, means they have the same frequencies.

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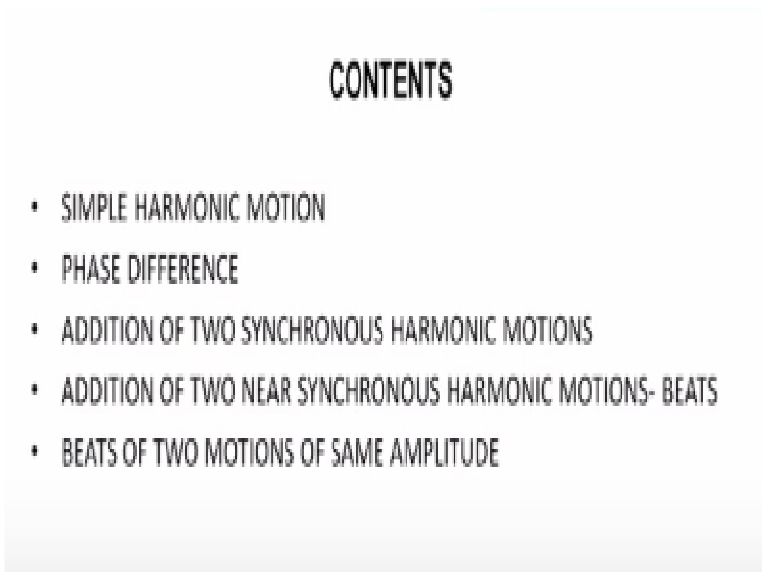

Of course there will be phase difference in the two motions. Then we will discuss addition of two near synchronous harmonic motions, means the difference between the two frequencies is slight and this will give rise to the phenomena like beats phenomenon. And then we will derive the expressions for these motions.

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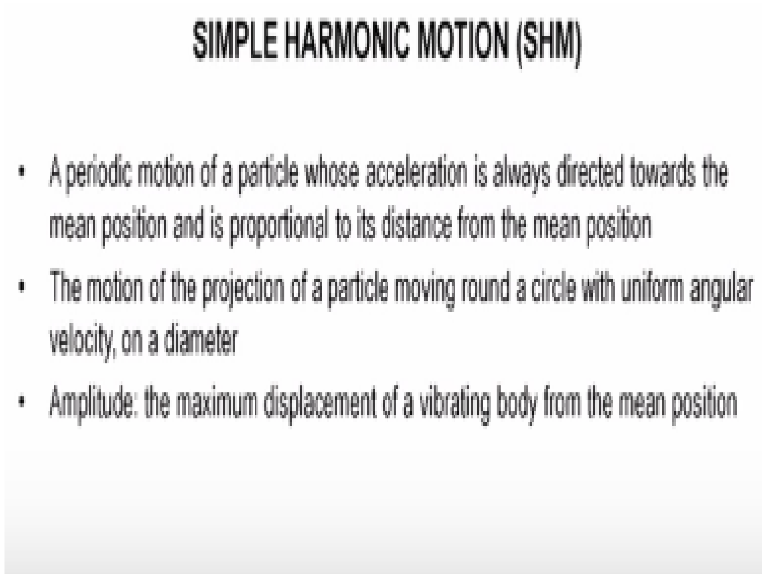

So here, we recall the simple harmonic motion that simple harmonic motion is the motion, a periodic motion that a particle makes, when the acceleration is proportional to the distance from the mean position and it is directed towards the mean position. And the simple harmonic motion is usually expressed in terms of sinusoidal or cosine motion.

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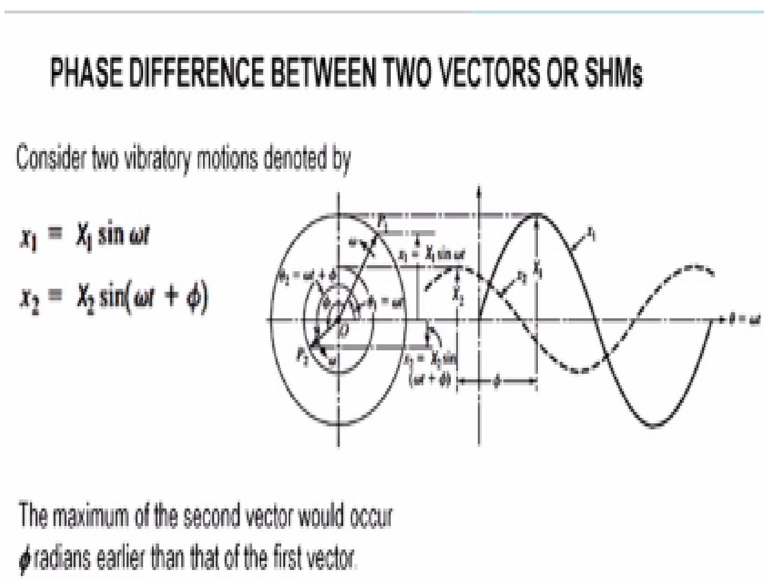

Like here, we can see that we have two simple harmonic motions. So x1 = X1 Sin omega t. Here the small letter is showing the overall function of this motion. And capital X is showing the amplitude of this simple harmonic motion. Then we have the second motion, x2, that is X2 Sin (omega t + phi). So these two motions have different amplitudes and there is phase difference. So, these motion x2 is leading with phi radiance to this x1.

So, we can see here. So, this is the vector diagram. So we can, on the vector diagram, we can see, so this is x1, so both have the same frequencies, so the both vector this OP1 and OP2, they are moving with the constant angular velocity omega. OP1 has this omega t, this theta and OP2 vector that is the angle omega t + phi. And this is the corresponding simple harmonic motion plotted on the theta axis. Now, we want to add these two motions. What is the resultant of these two motions? That, we have to find out.

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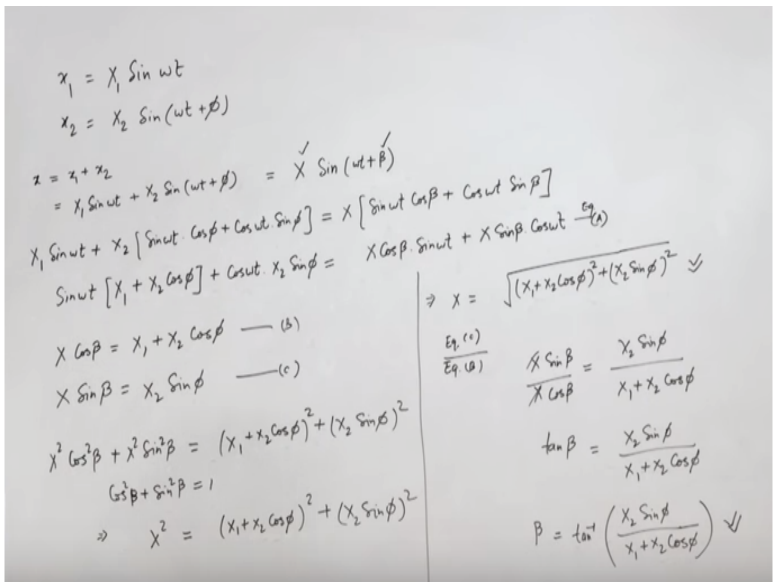

So, these are the two motions and we want to add these two motions. So, let us say that this is the resultant x that is x1 + x2. And so we can write x1 Sin omega t + x2 Sin (omega t + phi). Now, these two motions the same frequencies, therefore the resulting motions will also have the same frequency. And therefore let us assume that the resultant motion is also harmonic, so it is X sin (omega t + some different phase angle beta)

So, now we will solve these equations. Because we have, if you want to know the resultant, we must know X and we must know beta. So we can write this. So, X1 Sin omega t, we can write this like X2, so sin (a + b) = sin (a cos b + cos a sin b), so we can write (sin omega t cos phi + cos omega t sin phi). And here also we can open, X, so (sin omega t cos beta + cos omega t sin beta)

Now, we separate out the terms. Let us here we write, sin omega t and here we can write (X1 + X2 cos phi) + cos omega t into X2 sin phi. Now, let us say this is equation A. Now we compare from both sides, the co-efficient of sin omega t and cos omega t. So here we can write, here the co-efficient of sin omega t is X1 + X2 cos phi, here it is X cos beta. So we can write, X cos beta = X1 + X2 cos phi.

And similarly, we can write X sin beta = X2 sin phi. So, these are other two equations, let us say B and C. Now, we want to know X and we want to know beta. So, we will do square and under root. So, we do here X square cos square beta + X square sin square beta = (X1 + X2 cos phi) whole square + (X2 sin phi) whole square. Now X square, we know that cos square beta + sin square beta = 1.

So, cos square beta + sin square beta = 1. Therefore, we can write (X square = X1 + X2 cos phi) whole square + (X2 sin phi) whole square. So from here, we can find X = under root (X1 + X2 cos phi) whole square + (X2 sin phi) whole square. So here, we have got the amplitude X. And these, we know X1, X2 and we know phi, from these two motions. So, we can also find the X.

Now, we have to know the beta. So, beta we can know if we divide the equation c by equation b. So, we can find, like X Sin beta upon X Cos beta = X2 Sin phi upon X1 + X2 Cos phi. So, here X will be cancelled out and we will have Tan beta = X2 Sin phi upon X1 + X2 Cos phi. So, here we can find beta = Tan inverse X2 Sin phi upon X1 + X2 Cos phi. So, here we have got X and we have got beta. So, we have got the resultant that is the addition of the two motions.

So, it means that the addition of the two motions is X Sin omega t + beta, where X is under root X1 + X2 cos phi whole square + X2 Sin phi square under root. And beta = Tan inverse X2 Sin phi upon X1 + X2 Cos phi. So, the resultant of two synchronous harmonic motions is a harmonic motion with the same frequency. So, here we discussed the addition of the two synchronous harmonic motions.

Now, we will discuss the addition of the two near synchronous harmonic motions means there is slight difference in the frequencies of the two motions.

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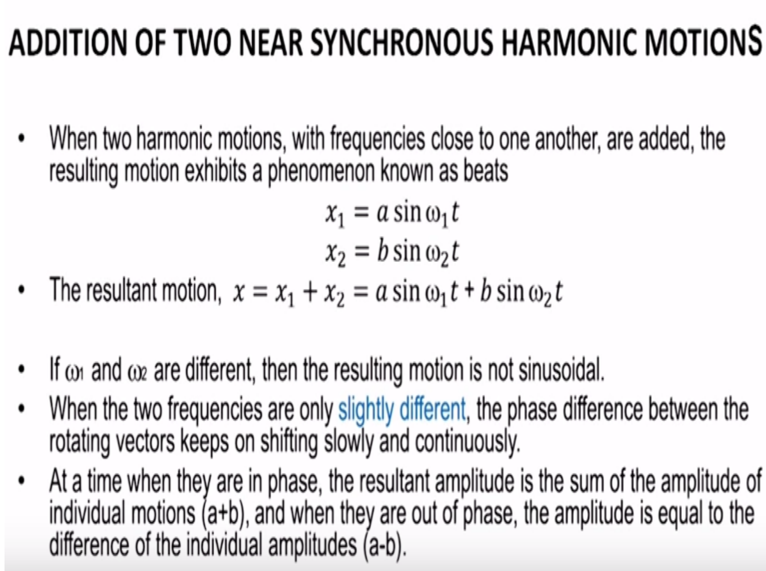

So, when two harmonic motions, with frequencies that are very close to each other, are added they exhibit some phenomenon that is known as beat phenomenon and if there are two motions, so, x1 = a Sin omega 1 t and x2 = b Sin omega 2 t, where a and b are amplitudes of the two motions and omega 1 and omega 2 are the frequencies and omega 1 and omega 2, when they are very close, but not the same, then the beats phenomenon occurs.

So, if omega 1 and omega 2 are different, then the resulting motion is not sinusoidal. And when they are slightly different, the phase difference vector rotates continuously and keeps shifting. So, at certain instant of time, when they are in phase, the resultant motion is maximum that is amplitude is maximum that is (a+b), sum of (a and b) and when they are out of phase, the amplitude is minimum and that is the difference of the two amplitudes of the two motions.

So, we will now write the equations for the beats phenomenon.

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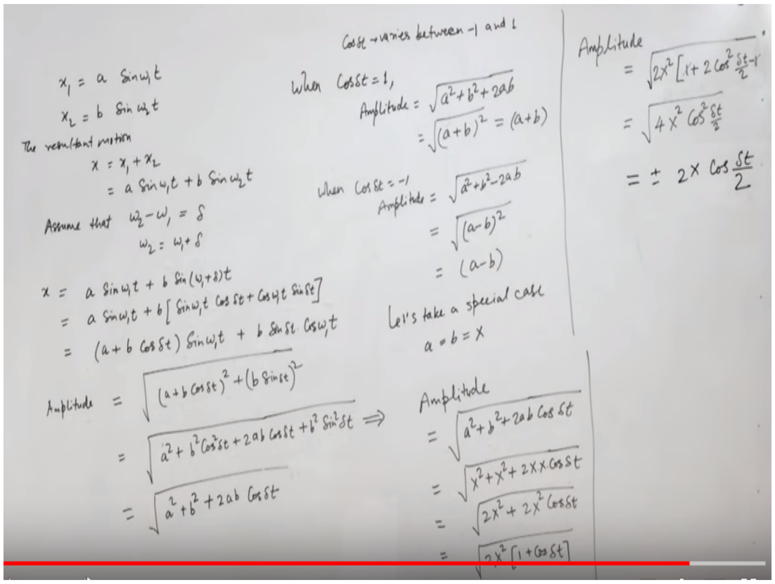

So, we have two motions x1 = a Sin omega 1 t and x2 = b Sin omega 2 t. The resultant motion you want to get x = x1 + x2. That is equal to a Sin omega 1 t + b Sin omega 2 t. Now here, we have assumed that omega 2 - omega 1 = delta, a very small quantity, so here we can write omega 2 = omega 1 + delta. Then we write x = a sin omega 1 t + b sin (omega 1 + delta t). = a sin omega 1 t + b, here sin a + b, so we will open it, so (sin a, so omega 1 t cos delta t + cos omega 1 t sin delta t). So we can separate out the terms of sin omega 1 t.

So, we can write (a + b cos delta t) to sin omega 1 t + b sin delta t into cos omega 1 t. So now we want to know the amplitude. And amplitude will be = this square plus this square under root. So here, b square cos square delta t and here is b square sin square delta t. So, we can write b square (cos square delta t + sin square delta t) and cos square delta t + sin square delta t = 1. So it will be b square. So it is like a square + b square + 2ab cos delta t.

Now when cos delta t = 1. Because this is cos delta t, where is cos delta t, where is between - 1 and 1, it where is between minus one and one. So when cos delta t = 1, the amplitude = root a square + b square + 2ab. And so it is (a + b) whole square under root. So, it is equal to (a+b). When cos delta t = -1, amplitude = root a square + b square - 2ab, because cos delta t = - 1, so it is - 2ab.

So we can write under root (a - b) whole square, because (a - b) whole square is a square + b square – 2ab. So it is (a - b). So therefore, when we have the two motions, x1 = a sin omega 1 t and x2 = b sin omega 2 t, the resultant motion will have the maximum amplitude (a + b). So resultant amplitude continuously keeps on changing from maximum that is (a + b) to minimum that is (a – b) with a frequency that is equal to the difference between the individual component frequencies.

So, this phenomenon is known as beats. Now, let us take a special case, so let us take a special case, when a = b = X. So, we assume that the amplitude of the two motions is same. So, let us write this expression, so here, we will have amplitude = under root a square + b square + 2ab cos delta t, now we put a = b = X, so it will be X square + X square + 2X into X into cos delta t, so it is equal to two X square + two X square cos delta t, so = under root 2X square (1 + cos delta t).

So we can write, equal to under root 2X square (1 +, cos 2 theta = 2 cos square theta - 1, so here we write cos - 1. So we will use this, this formula to open to break this term, so (1 + 2 cos square). Here 2 theta is delta t, so theta is delta t by two. So here, we can write delta t by two - 1. So, this one and one will cancel out. And we will get 4X square cos square delta t by 2.

So here we can have, this equal to + - 2X cos delta t by 2. So here, we see that the amplitude varies, as the function of cos delta t by 2. And we can see these from, clearly from these diagram.
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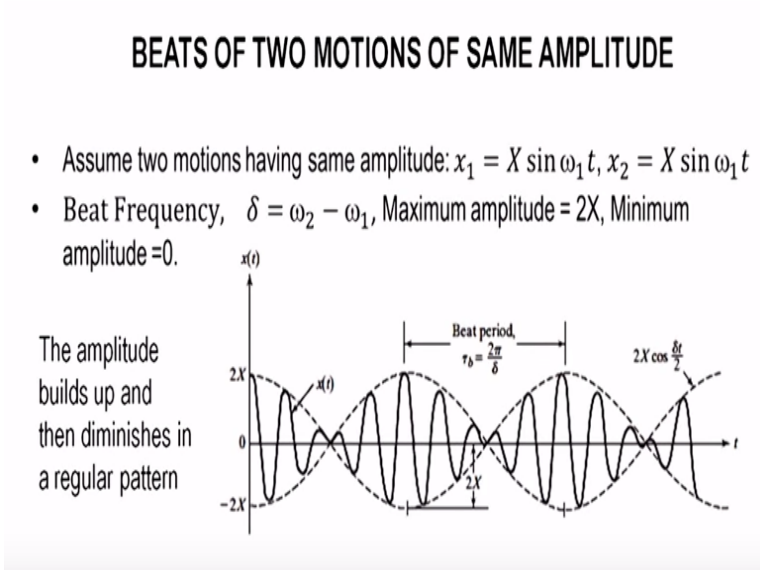

So here, this is the resultant wave that is moving. However, the amplitude is moving in a cos delta t minus, because it is a function of cos delta t by two, so this is a cosine, so here it is maxima and this way it is moving and then the next maxima is coming here. In between these, we can see that there is the beat. So, beat can be considered as the time period between these two nodes, or two maximers.

So, therefore the beat frequency, beat time period is half the time period of the cosine curve or the beat frequency is twice the frequency of the cosine curve that is two times delta by two that is equal to delta and delta is, we assumed omega two minus omega one that is the difference between the two frequencies. So here, when we have same amplitude, so we see that the maximum is 2X then minus 2X and here this minimum is zero here.

And so, the amplitudes build up, so amplitude is builds up, so here it is building and then diminishes in a regular pattern. So therefore, we whenever the two motions, two harmonic motions, with these close frequencies are there, they will show the beats phenomenon, where the amplitude will build up and then it will diminish in a regular pattern. So we would stop this lecture and we will meet again. Thank you for your attention.