Introduction to Mechanical Vibration
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Lecture – 11
Harmonic Excitations

Welcome to lecture on the forced vibration of a single degree of freedom system. So far, we have discussed the free vibrations of a single degree of freedom system. We discussed the free vibration under viscous damping and coulomb damping. So in free vibrations, we did not apply a force continuously on the system but we just gave some initial disturbance to the system and left the system to vibrate.

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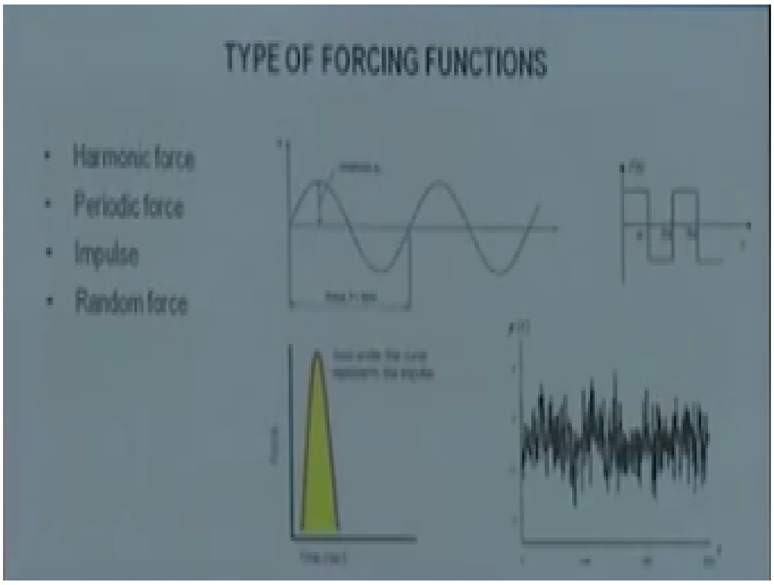

And the system vibrated with its own natural frequency. In case of forced vibration, when we say forced means, that is force applied to the system and what will be the response of the system? What will be the frequency of vibration of the system in these conditions? So this will be the subject of this lecture. So let us discuss the type of forcing functions, so what are different type of force that or loads that can be applied to the system.

So the force could be very simple; is the harmonic one, so already we have discussed what is harmonic motion? So similarly there is harmonic force like F0 sin, omega t, F0 cos; omega t; they have fixed some constant frequency, they have constant amplitude. Then another type of force could be periodic force, as periodic force also have some constant interval after which it repeats.

And from Fourier series, we know that a periodic function can be represented in terms of harmonic series. So the third type of force could be impulse force or some state function or some impulsive function. So in a very short duration of time, a very high amplitude of force could be applied and these force function is called impulse. Then the last that force we discuss is the random one.

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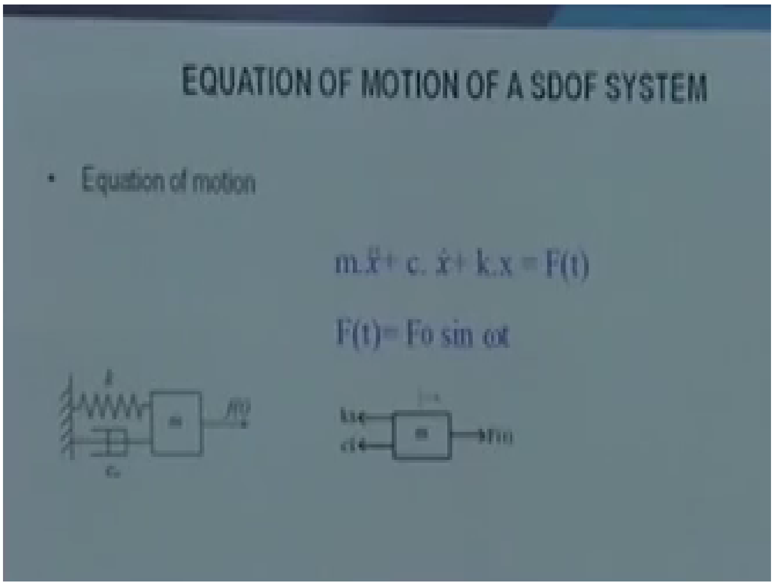

So there could be random force on the system like the force of the wind or some earthquake, they are random. So these are the mainly four type of forces but out of these, we are going to concentrate on the harmonic force. So we have a single degree of freedom system, single degree of freedom system that comprises mass element, stiffness element and damping element.

And we apply this harmonic force like F0 sin omega t, so we time, it is sin curve and at each instant of time, it is working on the system, acting on the system and varying as the sin curve with time. So when this kind of force is applied, what will be the response of the system? That is the important point of investigation, how the system will be have. Because if we see in case of free vibration, the system was vibrating.

When there was undamped, it was vibrating with its own natural frequency, when it was viscous damped, it was vibrating with some damped frequency that was omega t = omega n root 1 - zeta square. In case of coulomb damping, free vibration case it was again vibrating with omega n, but when we have the forced condition, then what will be the frequency of vibration? What will be the amplitude of vibration?

Means, the amplitude of the response of the system. These are the important aspects of to the study. So let us make again, we will follow the similar analysis procedure that we discussed in the previous lectures that okay we have a system, we isolate the system, isolate, make free body diagram, show all the internal and external forces and inertia of forces and we apply Newton's law of motion or D'Alembert's principle.

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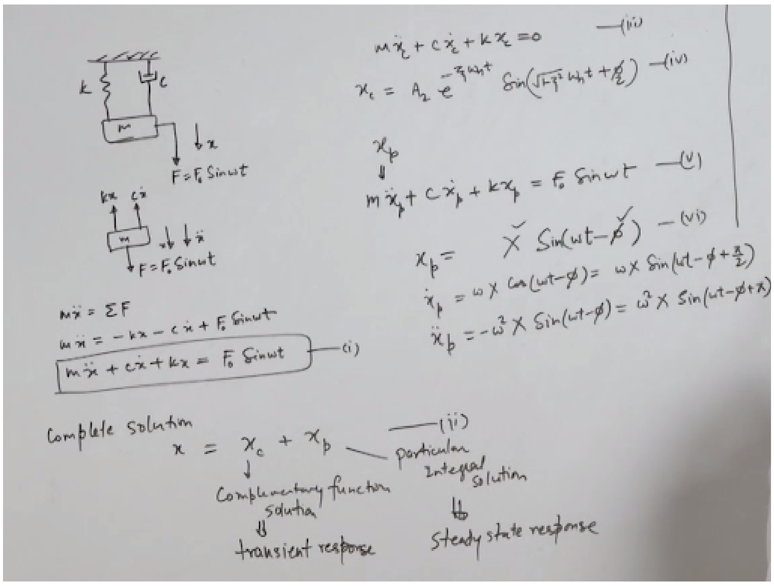

And we find a differential equations representing the equation of motion of that system and then we go for the solution of the system and the solution will give us the response of the system and based on the response, we can get some insight to the characteristics of the system, okay; so we follow the procedure; same procedure. So we have single degree of freedom system.

It has only one degree of freedom in this direction, a vertical direction we can say and here a force is applied on the system F; that is harmony, so it is F0 sin omega t and we know that this system has a natural frequency omega n, that is root k/ m. So we make free body diagram of the system. So this is our mass. So to make a free body diagram, we have to let us assume that when the system is vibrating under the application of this force.

And at a particular instant of time t, it is at certain displace position x, so here; let us say here, it is at some position x, so when this is displaced at some position x, there is some spring force working upward that is kx and it has certain velocity with respect to the dampers, so here we will have some damper force cx dot and already here this force F is acting; F0 sin omega t and we assume the direction of expression and x here in the positive x.

So this is the free body diagram. We showed these forces and now we will apply the Newton's law of motion. So Newton's law we said that the Sigma F = mx double dot. Okay this is the Newton's law and so mx double dot is equal to; so force is the spring force and that is acting opposite to x double dot direction, so it is - kx. Then there is the damping force that is again acting opposite, so cx dot.

And there is this force, it is in the direction of this excitation F0 sin omega t. So here we write mx double dot + cx dot +kx = F0 sin omega t. So this equation one, is representing the equation of motion of the single degree of freedom system under forced condition. Now we have to solve the differential equation. So earlier differential equation that we have solved in the right side, it was 0, because they were the free vibration case.

Here again if we put F0 is 0, so it will be a free vibration, so if there is no any force. Then to solve this kind of system, okay, we have; at the forced vibration case we have the solution that will comprise the 2 parts; the theory of differential equations says that this kind of system will have the solution; so the complete solution will comprise 2 parts, so x= xc + xp; so xc is the complementary function and this is particular integral solution okay.

So these 2 solutions will add up and give the complete solution of this system. So this to get the complementary function solution, we will have; this side will be 0, so we will have, mx double dot + cx dot + kx = 0, so this is the case of free vibration and already we have got the response of this system, so we already got that this kind of system has the response like A2 e power - zeta omega n.

Assuming under damped system; - zeta omega n t into sin, here we have omega t or we write 1 - zeta square omega n t + phi 2, so this is the response of an under damped system. We have already got this in the free vibration, so x is this one. Now we have to find xp. So to find xp, let us use some general science and vector method. So the general science says that to find xp, we have this mxp double dot + cx dot p + kx p = F0 sin omega t.

So this equation, we have to solve and the physics says that if we have a force; harmonic force acting on a system that has the frequency omega. The system will vibrate with the same forced frequency, so it means the response xp; first thing it will also be harmonic, so it will be harmonic sign omega t and here it will be x because here it is forced amplitude, here will be some displacement amplitude.

And it will also the response; will also be of the same frequency omega but some phase lag because we apply first the force and then the response will come late after the; response is the result of the application of the force, so there will be certain phase lag between the force and the response. So response will lag the force; let us say omega t - phi, okay. So the response is; so this is the; we assume this is the solution of these equation.

So now we have to find x? we have to find phi, if we want the solution of this equation. Okay, so now here we will use the vector method, so vector method first we put in these equation, so this is equation one. In fact, these complementary function solution, these side also, this keeps the transient part, transient solution, transient response and this particular integral solution gives the steady state response.

So now, we have equation one, let us say this is equation 2, this is equation 3, 4, this is equation 5, this is equation 6. Now we have xp, we put, we calculate xp dot and we calculate xp double dot and we put these values in this equation 5, so xp dot = x cos omega t - phi into omega, because here is omega, so this we can also write omega x sin omega t - phi + pi/ 2, because sin theta + pi/ 2 is cos theta.

Similarly, we get; to get xp double dot, we have to differentiate these, so we have, we differentiating this, so we get omega x square x and this is - sin omega t - phi. Because differentiation of cos is - sin and one omega again coming to square this, so it is omega square and because this is - we can write these in another form like omega square x sin omega t - phi + pi.

Because sin theta + pi = sin theta. So we have these 3 terms now we put these terms in this expression, number 5. So if we are putting this is in expression number 5.

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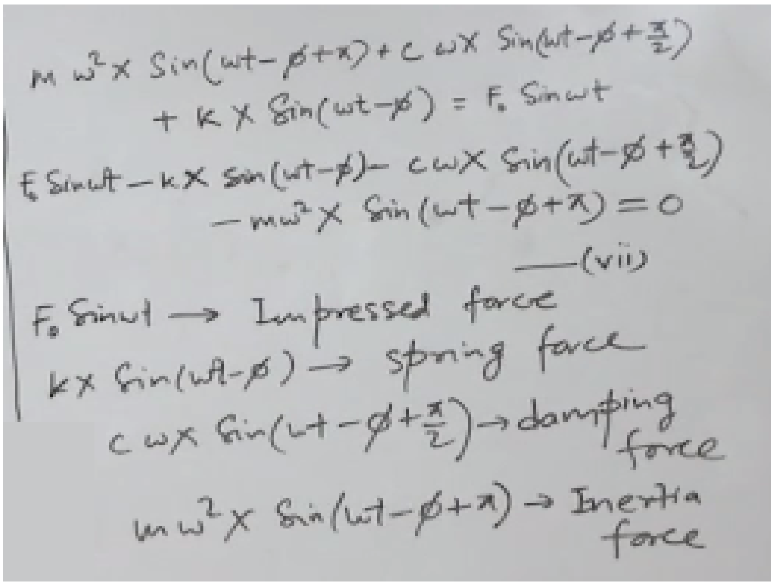

So here is m xp double dot; xp double dot here, m omega square x sin omega t - phi + pi + c xp dot; xp dot is omega x sin omega t - phi + pi / 2 + k xp; xp is x sin omega t - phi and this is equal to F0 sin omega t. Now we arrange these terms and if we arrange these terms, we can write F0 sin omega t - kx sin omega t - phi - c omega x sin omega t - phi + pi/ 2 - m omega square x sin omega t - phi + pi =0.

So this equation let us say equation 7. These equation is of quite importance because this equation is showing us 4 terms and they 4 are the forced terms, they are the forces. For example, here F0 sin omega t; this is showing the impressed force or externally applied force then kx sin omega t - phi; it is showing the spring force, then c omega x sin omega t - phi + pi / 2; this is showing a damping force.

And then m omega square x sin omega t - phi + pi; this is showing the inertia force. So these 4 forces together or making 0 means, they are in equilibrium, that is the D'Alembert’s principle.

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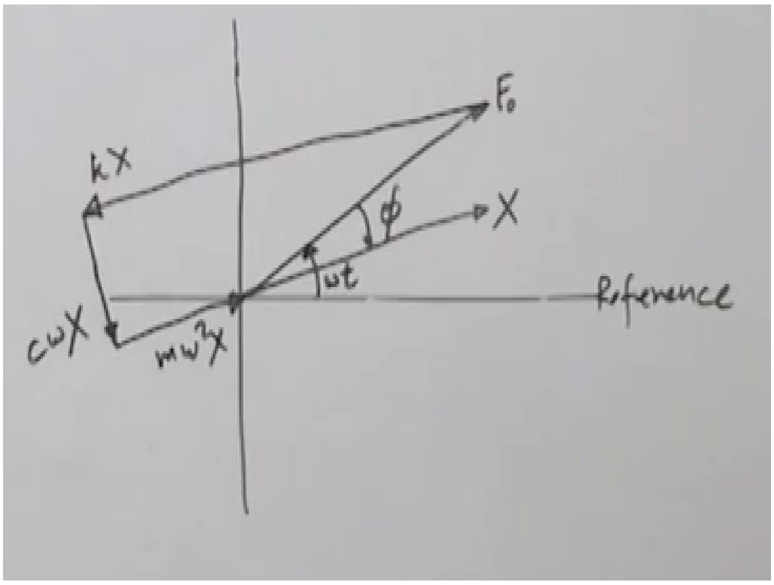

So these 4 forces, we can represent on a vector diagram, because they are the forces, they are vectors, we can represent them on a vector diagram. So let us make the vector diagram. So when we want to make the vector diagram, first we plot the x in the direction of x; first we plot F0, because F0 sin omega t, so from here it is omega t, this is the reference axis, so it is omega t F0.

Then phage lag of phi, we plot x, because x is lagging with phi, here. Now we have to plot the spring force, because here F0 sin omega t - kx, so - of these x means, it is opposite to the direction of x. So x direction is this; it is parallel to x direction and opposite, so this is coming from here; it is a spring force so it is this kx and its phase is omega t - phi, because this is phi.

Now we have to damping force; c omega x sin omega t - phi + pi / 2, so from x it is making angle this - pi by 2 lag, so from here it is pi by 2. So from kx, it is + pi by 2, so we have this; so this is; c omega x. Then m omega x square x that is + pi, here with respect to kx, so it is like this. So m omega x square x and so the vector force diagram is start from 0 here from this and going these and it is going to close, so these 4 forces are in equilibrium.

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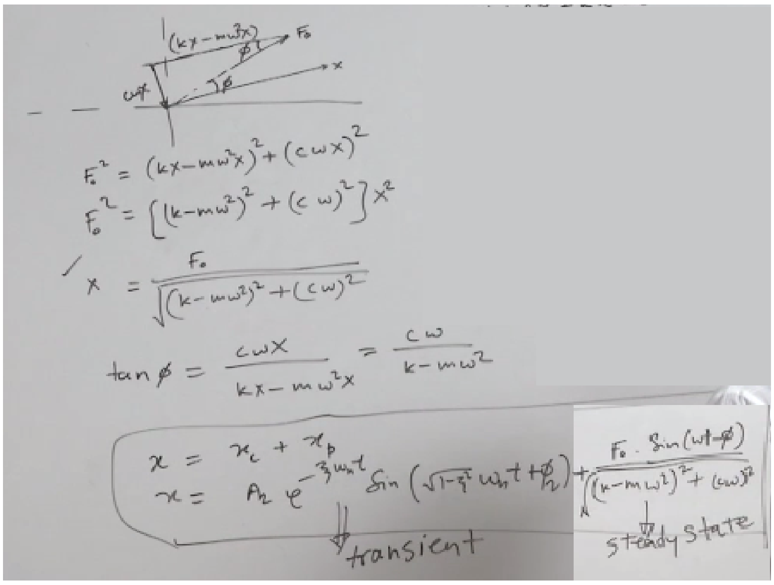

Now we can write these; so we can make this vector diagram even better like, so we have x, this is F0. So we can write, like this is kx and this is m omega square, so this quantity is kx - m omega x square x and this is c omega x, so here we can write F0 square = kx - m omega square x is square + c omega x square. So here we can write k - m omega is square, whole square + c omega whole square x square = F0 square.

So here we can write x = F0 upon k - m omega square, whole square + c omega square and this is under root. Moreover, here we can also write time phi = c omega x upon kx - m omega square x that is equal to c omega upon k - m omega square. Okay. So we have got x, we have phi, because phi is equal to tan inverse this, so the complete solution x = x c + xp that is equal to A2 e power - zeta omega n t sin root 1 - zeta square omega n t + phi 2 + xp.

So here xp is x sin omega t - phi and x is given by this equation, so it is F0 upon k - m omega square, whole square + c omega whole square under root. So this is x into sin omega t - phi. So this is the complete solution and we see that these part is the exponential decaying, so this is the transient part and this will decay after certain time, certain small instant of time.

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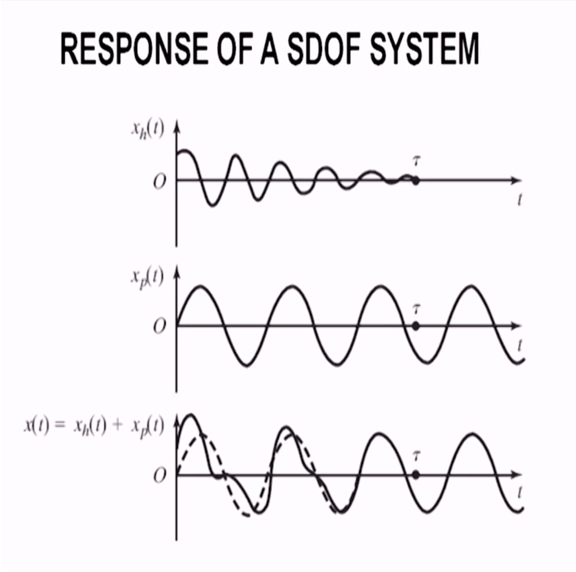

And these part is a harmonic function with sin function and this is the steady state part and these part will prevail even after this part is 0. So we can show the response of a single degree of freedom system, so here; we see that the first part is shown the complementary function xc or xh here it is, so it is going to decay with time so here it is like free vibration, so it is going to decay with time and this is the particular integral part that is harmonic sinusoidal.

So it is continued with time and the resultant response the total response completely responses the super position of the 2, so we superimposed the 2 and we see that initially due to these transient part, the way from it is started but when the transient part becomes 0, the response completely is the harmonic response and with the same frequency as the force frequency and therefore it is the steady state part of the response.

So the total response is this one. So we understand that in case of the forced vibration of a single degree of freedom system, the complete solution comprises the 2 parts; the transient and a steady state part and the transient part will decay after some initial instant of time and the steady state part that is the harmonic with the same frequency as the forced frequency will prevail for the rest time, the rest amount of time.

And here we derived the amplitude x of the response and the phase phi of the steady state response, so thank you for attending this lecture and see you in the next lecture. Thank you.