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Convective Heat Transfer

Lec-17

Rayleigh Benard Convection

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Hello welcome in the 17th lecture of convective heat transfer. In our last few lectures we have discussed about convection inside a duct. Now here we will be discussing, in this lecture we will be discussing about convection between two parallel plates okay, and we will see that due to temperature boundary condition specific temperature boundary condition how convection currents can be generated okay.

So the topic comes under the per view of Rayleigh Benard convection. So in this lecture we will be discussing about Rayleigh Benard convection okay. So let me first tell you that what we will be discussing in this lecture in brief.

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Rayleigh Benard Convection:

Hydrostatic case: $\bar{V} = 0$

Energy equation: $\bar{V} \nabla T = \alpha \nabla^2 T$
 $\nabla^2 T = 0$ as $\bar{V} = 0$
 $T = C_1 \bar{z} + C_2$ (Linear)

Density field:
 $\rho = \rho_0 [1 - \beta (T - T_0)]$
 (Boussinesq approximation)

Case II: $T_1 < T_2$ Heated from bottom

Unstable density field needs to be stratified

After stratification

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So first we will be establishing the convection between two large parallel plates which is essential for Rayleigh Benard convection large parallel plates okay. Then we will be discussing or we will be deriving rather the basic state and disturbed state. So I will show you in case of parallel plate and constant temperature boundary condition there can be stable situation as well as harm stable situation. So whenever we have harm stable situation there can be cases were harm stating can be achieved. So in those cases we need to observe that after basic state disturbed state is going towards in stability or not okay.

So we will be finding out basic state and disturbed state equations along with the corresponding boundary conditions okay. We will also do analysis of stability by using normal mode equations okay, and discuss about temporal as well as special stability okay. And towards the end we will discuss about critical Rayleigh number below which you will be finding out always system is stable and neutral stability okay.

So neutral stability curve will be defining, so first let me give you the idea where Rayleigh Benard convection happens. So this is the case let us say you are having two plates okay in vertical orientation so G is acting in the term what direction, so this is the first plate let us say, situated that $z=0$ $\bar{z}=0$ and here we are having another plate parallel plate a long plate okay in the third direction it is very long having $\bar{z}=d$ so the gap between two plate is actually kept as D .

And for our case let us consider the bottom plate is having constant temperature T_0 and while as the top plate is having constant temperature T_1 okay. Now there can be hydrostatic case that means whenever there is no flow so no flow is here, we can see over here no flow. So in the normal situation there will be hydrostatic case okay. So in hydrostatic case we will be finding out no flow means no velocity, so $v=0$ okay.

And this V is nothing but the vector velocity okay so $u+v+w$ all are 0 actually. Then in the energy equation if we put this $v=0$ we can simply get $\alpha\Delta^2=0$ because the side which is convection side so there is no velocity, so convection will not be present. So you can find out it is dominated by conduction and ultimately the temperature profile comes as linear one C_1z+C_2 okay.

So if you see due to this temperature profile if you find out what is the density field, because as you are having different temperatures in both the plates, so you will be finding out this linear temperature profile in the fluid between two plates we will be giving variable density, so the density field will be having in this fashion $\rho=\rho_0(1-\beta)(T-T_0)$ which is framed as Boussinesq approximation okay.

In a similar fashion here let us see two different cases, first case is let us say the plate is heated from top okay, so let us consider the upper plate which is having temperature T_1 is higher compared to the lower plate, here we have considered T_0 , but let us say here this one is T_2 okay. So this T_0 and T_2 both are same. And so we have considered the upper plate temperature T_1 is higher than lower plate temperature T_2 okay.

So obviously as per this linear profile we will get the linear temperature variation like this, whereas higher temperature will be attached with the upper plate and lower temperature will be attached with the lower plate okay. And if you see this one density fields, so it is having a minus sign over here, so if we are having higher temperature so we will be having lower density and it will be also linear profile as T is linear.

So we can get the density field will also be linear having lower density at the upper plate which is at higher temperature and having higher density at the lower plate which is at the lower temperature okay. So due to this minus sign the Boussinesq approximation the nature of the linearity changes, the slope of the linear curve from T to ρ changes. So this is the case

where we are having hydrostatic situation okay, conduction maybe dominated which is having this kind of temperature profile inside the fluid.

But if the situation alters, situation alters means let us say the case 2 we want to see now where T_1 is actually less than T_2 okay, so upper plate is having lower temperature compared to the bottom plate okay. So in this case you will be finding out once again into sustain this one at the beginning in the hydrostatic case. In order to sustain this linear temperature profile we will be having temperature profile like this.

The lower temperature is attached with the upper plate and higher temperature is attached with the lower plate okay, and it is linear in nature. So to once again satisfy this Boussinesq approximation we will be having density field like this, which is just opposite in nature that means higher density is attached with the lower temperature plate, upper plate and lower density is attached with the higher temperature plate or bottom plate okay. Now this situation is actually not stabled.

Whereas the previous situation where you are having lower density in the attached with the upper plate was actually stabled. But here you will be finding out that no fluid at higher density can stay on top of lower density fluid. So it will be having some tendency to come in the downward side okay, which will be generating some sort of flow velocity in between two plates okay.

So this is actually unstable density field and it would be stratifying okay. So stratifying means there will be current generated from the upper plate to lower plate to accommodate this higher density fluid in the lower side and lower density fluid in the higher side okay. And if you allow this type of flow for a certain amount of period and if the situation is becoming stabled, then ultimately you will be finding out it is being stratified and after stratification density field will be homogenous part of thing like this okay.

So this situation actually we are going to see and this unstable density field will be generating a convection current in between the plates which is actually called Rayleigh Benard convection okay. So let us see this in detail what are the equations of these type of convection and how that can be solved okay.

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Continuity:
$$\frac{\partial \bar{u}}{\partial \bar{x}} + \frac{\partial \bar{v}}{\partial \bar{y}} + \frac{\partial \bar{w}}{\partial \bar{z}} = 0$$

Momentum:
$$\frac{\partial \bar{u}}{\partial \bar{t}} + \bar{u} \frac{\partial \bar{u}}{\partial \bar{x}} + \bar{v} \frac{\partial \bar{u}}{\partial \bar{y}} + \bar{w} \frac{\partial \bar{u}}{\partial \bar{z}} = -\frac{1}{\rho_0} \frac{\partial \bar{P}}{\partial \bar{x}} + \nu \nabla^2 \bar{u}$$

$$\frac{\partial \bar{v}}{\partial \bar{t}} + \bar{u} \frac{\partial \bar{v}}{\partial \bar{x}} + \bar{v} \frac{\partial \bar{v}}{\partial \bar{y}} + \bar{w} \frac{\partial \bar{v}}{\partial \bar{z}} = -\frac{1}{\rho_0} \frac{\partial \bar{P}}{\partial \bar{y}} + \nu \nabla^2 \bar{v}$$

$$\frac{\partial \bar{w}}{\partial \bar{t}} + \bar{u} \frac{\partial \bar{w}}{\partial \bar{x}} + \bar{v} \frac{\partial \bar{w}}{\partial \bar{y}} + \bar{w} \frac{\partial \bar{w}}{\partial \bar{z}} = -\frac{1}{\rho_0} \frac{\partial \bar{P}}{\partial \bar{z}} + \nu \nabla^2 \bar{w} + g\beta(T - T_0)$$

Energy:
$$\frac{\partial T}{\partial \bar{t}} + \bar{u} \frac{\partial T}{\partial \bar{x}} + \bar{v} \frac{\partial T}{\partial \bar{y}} + \bar{w} \frac{\partial T}{\partial \bar{z}} = \alpha \left[\frac{\partial^2 T}{\partial \bar{x}^2} + \frac{\partial^2 T}{\partial \bar{y}^2} + \frac{\partial^2 T}{\partial \bar{z}^2} \right]$$

To go ahead first we will be seeing that what are the necessary equations we are having. Remember we had the coordinate planes \bar{x} , \bar{y} and \bar{z} okay I have shown you in the previous slide \bar{x} , \bar{y} and \bar{z} respectively. So this is the continuity equation. Later on let us see the momentum equations, three momentum equations we will be having x direction, y direction and z direction.

So in x and y direction as the plate circuit actually vertically away from each other, so you can find out that no buoyancy term is present in the x and y direction, simply we are having inertia and the pressure value then subsequently viscous terms okay. So if momentum equation is involved with you, why momentum equation is involved with V okay. But in case of Z momentum equation along with inertia, pressure gradient and viscosity term we will be having buoyancy term also, because we are having density gradient okay.

So using Boussinesq approximation we can write down the buoyancy term in this fashion $g\beta(T-T_0)$ okay this you can get once again from your fluid mechanics knowledge okay. And finally if you see the energy equation, so we will be having convection is equals to α into conduction. So left hand side is convection, right hand side is conduction. So these are the sets of equation which we will be using for solving the Rayleigh Benard convection okay.

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Non dimensional parameters:

$$x = \frac{\bar{x}}{d} \quad y = \frac{\bar{y}}{d} \quad z = \frac{\bar{z}}{d} \quad \theta = \frac{T - T_0}{\Delta T} = \frac{T - T_0}{T_0 - T_1} \quad \text{or, } T = T_0 + (T_0 - T_1)\theta$$

$$u = \frac{\bar{u}}{u_0} \quad v = \frac{\bar{v}}{u_0} \quad w = \frac{\bar{w}}{u_0} \quad P = \frac{\bar{P}}{P_0} \quad t = \frac{\bar{t}}{d/u_0}$$

Substituting in energy equation:

$$\frac{u_0 \Delta T}{d} \left[\frac{\partial \theta}{\partial t} + u \frac{\partial \theta}{\partial x} + v \frac{\partial \theta}{\partial y} + w \frac{\partial \theta}{\partial z} \right] = \frac{\alpha \Delta T}{d^2} \left[\frac{\partial^2 \theta}{\partial x^2} + \frac{\partial^2 \theta}{\partial y^2} + \frac{\partial^2 \theta}{\partial z^2} \right]$$

For conduction ~ convection $\Rightarrow \frac{u_0 \Delta T}{d} = \frac{\alpha \Delta T}{d^2} \Rightarrow u_0 = \frac{\alpha}{d}$

Let us also see that how we can non dimensionalize this situations. So as we are having distance between the plates as d so that we can take as the volume scale definitely. So what we have done, x , y and z all are actually non dimensionalized based on d okay. So the distance between two plates okay. For temperature definitely we are having upper plate temperature minus lower plate temperature, so we are taking that $T_0 - T_1$ that the lower plate temperature minus upper plate temperature as our temperature scale.

So temperature is being non dimensionalized to θ as $T - T_0 / T_0 - T_1$ okay. So ultimately we get $T = T_0 + T_0 - T_1(\theta)$ okay. Now let us try to see that what happens for the velocity, now as we do not know what is the predominant velocity which was actually known in case of your ϕ flow or flow in flat plate, so here what is the velocity scale that is not known. So we have to find out what is the velocity scale.

So for the time being let us consider some arbitrary velocity scale we are considering which is nothing but u_0 , so \bar{u} , \bar{v} and \bar{w} all are actually being non dimensionalized by u_0 okay to give rise u , v and w okay. In the same fashion we do not know what is the pressure also, so we actually take some non dimensionalized pressure P as \bar{p} / p_0 okay. So p_0 also we need to evaluate okay.

Now to non dimensionalize the time we are having over here $T = \bar{t}$ by some time scales, so time scale will evaluating from your distance between the plates, what amount of time it takes a fluid takes to cover the distance between the plates with velocity u_0 . So d / u_0 is nothing but the

time scale. So we have obtain $T = T_{bar}/v u_0$ okay. So with this non dimensionalized parameters if you proceed further then obviously first let us look at the energy equation.

So in the energy equation using the convection side $u_0 \Delta T / d$ will come out okay. So this t will give you once again v/u_0 so this u_0/v will become θ will give ΔT which is nothing but $T_0 - T_1$ and the spatial terms in the convection side obviously takes Y and Z they will be giving $D U, V$ and W they will be giving U_0 and θ will be giving out ΔT okay. So in the convection side we get $u_0 \Delta T / d$.

In the similar fashion in the conduction side all the T_s in the conduction terms gives rise $\Delta t \theta$ okay x, y and z those give Δ^2 okay d^2 rather d^2 . So here you can find out that if you equate the conduction and convection order then we can write down $u_0 \Delta T b = \alpha \Delta T / d^2$, so these are the order matching and if you do this type of order matching we can get the velocity scale u_0 okay, so velocity scale u_0 is coming out to be after cancellation α / d . So you have obtained what will be the velocity scale u_0 for non dimensionalization of u, v and w okay.

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Non dimensional parameters:

$$x = \frac{\bar{x}}{d} \quad y = \frac{\bar{y}}{d} \quad z = \frac{\bar{z}}{d} \quad \theta = \frac{\bar{T} - T_0}{T_0 - T_1} \quad \text{or } \bar{T} = T_0 + (T_0 - T_1)\theta$$

$$u = \frac{\bar{u}}{u_0} \quad v = \frac{\bar{v}}{u_0} \quad w = \frac{\bar{w}}{u_0} \quad P = \frac{\bar{P}}{P_0} \quad t = \frac{\bar{t}}{d/u_0}$$

Substituting in energy equation:

$$\frac{u_0 \Delta T}{d} \left[\frac{\partial \theta}{\partial t} + u \frac{\partial \theta}{\partial x} + v \frac{\partial \theta}{\partial y} + w \frac{\partial \theta}{\partial z} \right] = \frac{\alpha \Delta T}{d^2} \left[\frac{\partial^2 \theta}{\partial x^2} + \frac{\partial^2 \theta}{\partial y^2} + \frac{\partial^2 \theta}{\partial z^2} \right]$$

For conduction ~ convection $\Rightarrow \frac{u_0 \Delta T}{d} = \frac{\alpha \Delta T}{d^2} \Rightarrow u_0 = \frac{\alpha}{d}$

Next let us try to see now the form of energy equation, so as we have equated these two sides, so this will be coming 1 that means u_0 is chosen as a fashion that these two coefficients will become 1, so here in this side we have actually total derivative of θ and in the right hand side we are having $\partial^2 \theta$ so this is the energy equation we get okay, final form of the energy equation. Let

us now see the momentum equations and to start with we will be seeing first the momentum equation in the vertical direction that means Z direction.

So in the Z direction inertia term will be giving coefficient u^2/d once again u^2/d is coming from the uw or vw or ww side which is giving u^2 and xyz those can be vowe d, and the same passion w is giving one in note and t is giving d/ u node the coefficient comes out has u^2/d okay. In the pressure gradient term we are having p is giving out p_0 and this z is giving d okay, so here we have seen that this pressure gradient terms gives $-1/\rho_0 \times p_0 /d$ okay. Beyonce it will be giving you nothing but $g \beta \Delta T \theta$ because we had over here t - let me show you the equation $t-t_0$, okay so $t - t_0$ will be giving you Δt which is nothing but $t_0 - t_i \times \theta$, so this beyonce term we have written in this passion and obviously viscous term will be giving you $\eta u_0 /d^2$, remember the equation whatever I have shown for the momentum equations here we have divided all the equations by ρ okay.

So this is actually a part of during a short terms so $\rho \times$ this in actually inertia so we have divided so here it got ρ_0 and here μ/ρ_0 it is actually μ okay.

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$\Rightarrow \frac{D\theta}{Dt} = \nabla^2 \theta$

Substituting non-dimensional parameters in z-momentum equation:

$$\frac{u_0^2}{d} \left[\frac{\partial w}{\partial t} + u \frac{\partial w}{\partial x} + v \frac{\partial w}{\partial y} + w \frac{\partial w}{\partial z} \right] = -\frac{1}{\rho_0} \frac{P_0}{d} \frac{\partial P}{\partial z} + g \beta \Delta T \theta + \frac{\nu u_0}{d^2} \left[\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 w}{\partial z^2} \right]$$

$$\frac{u_0^2}{d} \frac{u^2}{\nu u_0} = \frac{u_0 d}{\nu} = \frac{\alpha}{\nu} = \frac{1}{Pr} \quad \left[\text{as } u_0 = \frac{\alpha}{d} \right]$$

Equating coefficients of pressure and viscous term:

$$\frac{P_0}{\rho_0 d} = \frac{\nu u_0}{d^2} \quad \text{or,} \quad Ra = \frac{\rho_0 \nu \alpha}{d^2}$$

Dividing coefficient of buoyancy by viscous term coefficient:

$$\frac{g \beta \Delta T}{\frac{\nu u_0}{d^2}} = \frac{g \beta \Delta T d^2}{\nu u_0} = \frac{g \beta \Delta T d^3}{\nu \alpha} = Ra$$

So we have obtained this z momentum equation and if we try to now equate actually the this $u_0^2 d \times d^2 / \eta u^2$ which is nothing but inertia coefficient and viscous coefficient then we will be finding out it turns out to be after cancelation nothing but α / μ which is $1/Pr$ number okay so here in this side we get $1/Pr$ number if we equate this two okay. Here this u_0 scale we have used if you see

this u_0 scale we have used which we have just determined by equating convection and conduction orders okay.

So if you put the value of u_0 then only it will be turning out to be $1/Pr$ number okay. Now let us try to find out what is p_0 so to find out the p_0 we equate the pressure and viscous term coefficient so if you see here the pressure term coefficient works $p_0 / \rho_0 \times d$ and viscous term coefficient $\mu u_0 / d^2$ so if you equate this two then we get the term p_0 which is nothing but $\rho_0 \mu \alpha / d^2$ okay. So we have got p_0 already here I have shown you what was the u_0 so the scales for non dimension we have clearly identified okay.

As well as we have the equations already the energy equation I have shown and we are now deriving the z momentum equation. Now on the z momentum equation lastly let us see that what will be the buoyancy term becoming so in the buoyancy term we are having $g \beta \delta t$ as your coefficient and as throughout this equation we are dividing by $\mu u_0 / d^2$, so buoyancy term coefficient will become if you reduce further by putting the value of u_0 over here once again from the velocity scale.

So it will come out to be $g \beta \delta t d^3 / \mu \alpha$ which is nothing but our Rayleigh number which we have discussed in the first lecture in the non dimensional number field okay, so this equation throughout we will be dividing by $\mu u_0 / d^2$, so here we get coefficient one here we get once again coefficient one as p_0 will be chasing in this passion to cancel out all the terms okay. Here we will be getting Rayleigh number and here obviously we will be getting $1/Pr$.

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Dividing throughout by coefficient of viscous term and substituting modified coefficients: $\frac{1}{Pr} \frac{Dw}{Dt} = -\frac{\partial P}{\partial z} + \nabla^2 w + Ra \theta$

Similarly x and y momentum can be written as: $\frac{1}{Pr} \frac{Du}{Dt} = -\frac{\partial P}{\partial x} + \nabla^2 u$

$\frac{1}{Pr} \frac{Dv}{Dt} = -\frac{\partial P}{\partial y} + \nabla^2 v$

Boundary conditions: at $z = 0$ $u = 0$ $v = 0$ $\theta = 0$ $w = 0$

at $z = 1$ $u = 0$ $v = 0$ $w = 0$ $\theta = -1$

Basic state: Hydrostatic: $u = v = w = 0$

Steady: $\frac{\partial \theta}{\partial t} = 0$ $\theta = \theta(z)$ only

So let me show you so this actually here we get pr okay so this equation comes out to be $1/pr \frac{Dw}{Dt}$ which is the total derivative of w okay, so this whole term will be becoming total derivative of the w and then $-\frac{\partial p}{\partial z} + \delta^2 w +$ rally number in to θ this is the beyonce term okay. So this becomes my z momentum equation obviously other momentum equation x and y equations those will be very simple in the similar way we can derive that these will be the x momentum equation and y momentum equation only the beyonce term will be actually absent in this cases okay.

So after equations let us see the boundary conditions so at $z = 0$ that means bottom plate obviously it will be giving rise to sleep velocity condition so u and v is actually 0 and no penetration is over here $w = 0$ and for θ which is non dimensional temperature you will find out this becomes $t_0 - t_0 / t_0 - t_1$ so which is 0, and at $z = 1$ once again no sleep no penetration gives me uvw all are 0 and θ gives me $t_1 - t_0 / t_0 - t_1$ which is nothing but -1 okay.

So this boundary conditions and these equations we have to solve. Now at the beginning let us see the basic state that means whenever there is hydro static situation so in case of hydro static situation we are having $vw = 0$ so this is the initiation of the instability if there is existing okay. So you we get uv and $w = 0$ and from the steady state things we get the energy equation is being changing let me show you the energy equation once again, so this was the energy equation so right hand side we are having your conduction term and in the left hand side we are having the convection term okay.

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Dividing throughout by coefficient of viscous term and substituting modified coefficients: $\frac{1}{Pr} \frac{Dw}{Dt} = -\frac{\partial P}{\partial z} + \nabla^2 w + Ra \theta$

Similarly x and y momentum can be written as: $\frac{1}{Pr} \frac{Du}{Dt} = -\frac{\partial P}{\partial x} + \nabla^2 u$

$\frac{1}{Pr} \frac{Dv}{Dt} = -\frac{\partial P}{\partial y} + \nabla^2 v$

Boundary conditions: at $z = 0$ $u = 0$ $v = 0$ $\theta = 0$ $w = 0$

at $z = 1$ $u = 0$ $v = 0$ $w = 0$ $\theta = -1$

Basic state: Hydrostatic: $u = v = w = 0$

Steady: $\frac{\partial \theta}{\partial t} = 0$ $\theta = \theta(z)$ only

So in this case whenever we are having hydro static situation you will be getting that we are having $\partial \theta / \partial t = 0$ okay rest other terms will become 0 okay, so whenever you have this one so we get θ is actually a function of z okay. So this is the steady state basic state temperature profile okay.

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$$\begin{aligned} \text{x momentum: } & \frac{\partial P}{\partial x} = 0 \\ \text{y momentum: } & \frac{\partial P}{\partial y} = 0 \\ \text{z momentum: } & \frac{\partial P}{\partial z} = Ra \theta \\ \text{Energy equation: } & \frac{\partial^2 \theta}{\partial z^2} = 0 \quad \text{or, } \theta = C_1 z + C_2 \end{aligned}$$

Putting $\theta(0) = 0 \quad \theta(1) = -1$ We get $\theta = -z$

$$\text{z momentum: } \frac{\partial P}{\partial z} = Ra \theta = -Ra z \quad \text{Integrating: } P = -\frac{Ra z^2}{2} + C$$

Then from x momentum y momentum and z momentum equation as we are having these equations so you can find out that if u v and w those are 0 then ultimately in this equations we will be finding out x momentum y momentum equation will be giving you only the pressure gradient terms are 0, and z momentum will giving you pressure gradient term is equals to rally number in to θ okay.

So once we integrate this one okay as energy equation has given θ is actually a function of z okay, from this side as it is steady state so in case of steady state if we consider $\partial\theta/\partial t = 0$ so in the energy equation we will be finding out total convection term actually goes to 0. So if total convection term goes to 0 only we are left to with the right hand side which is nothing but your conduction sides when the conduction side let us consider that cross wise and team wise conductions are actually 0 only the ventricle conductions exist so and we have already shown that θ is the function of z in case of steady state so only the last term will be remaining $\delta^2\theta/\delta z^2 = 0$ okay.

And once you integrate it two times then you will be getting $\theta = c_1 z + C_2$ okay, so with steady state as well as hydrostatic this two consideration we have actually neglected the convection term okay and as we have got also in steady state θ is the function of z the $\partial^2 x$ and $\partial^2 y$ term is becoming 0, so only we are left with this $\partial^2\theta/\partial z^2$ okay. Now if you put the boundary conditions

as we are having at the bottom plate and at the top plate then we will get this profile the value of c_1 and c_2 we can evaluate and we get θ is nothing but $-z$ okay.

So if we get $\theta = -z$ so immediately we can put it over here in the z momentum equation reduce z momentum equation so we get $\partial p / \partial z = Ra \times -z$ okay, and once if you integrate you can get the pressure which is nothing but $-Ra z^2 / 2 + c$ okay. So this is the pressure field we have obtained and this is the temperature field for the basic state.

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For basic state: $\theta_0 = -z$ $P_0 = -\frac{Ra z^2}{2} + C$ $\rightarrow \frac{\partial P_0}{\partial z} = Ra \theta_0$

Let us now add perturbation:
 $u = 0 + u'$ $v = 0 + v'$ $w = 0 + w'$ $\theta = \theta_0(z) + \theta'$ $P = P_0(z) + P'$

Disturbance equations:

Continuity: $\frac{\partial u'}{\partial x} + \frac{\partial v'}{\partial y} + \frac{\partial w'}{\partial z} = 0$

x momentum: $\frac{1}{Pr} \left(\frac{\partial u'}{\partial t} + u' \frac{\partial u'}{\partial x} + v' \frac{\partial u'}{\partial y} + w' \frac{\partial u'}{\partial z} \right) = -\frac{\partial P'}{\partial x} + \nabla^2 u'$

y momentum: $\frac{1}{Pr} \left(\frac{\partial v'}{\partial t} + u' \frac{\partial v'}{\partial x} + v' \frac{\partial v'}{\partial y} + w' \frac{\partial v'}{\partial z} \right) = -\frac{\partial P'}{\partial y} + \nabla^2 v'$

So for the basic state we have obtain this two and here from we can easily get once again that $\partial p_0 / \partial z$ is nothing but $Ra \times \theta_0$, so this will be using for the later cases so this is very important relationship okay. Now let me show you that is there is requirement of stratifications so definitely there will be some convection call ingenerated and we have to actually part of the flow okay, so flow will be perturbation.

Now if the flow is perturbation you need to study whether it is becoming steady once again onward to achieve that let us add some perturbation in the situation and let us try to see that what is the stability criteria for this okay. So if this flow situation is satisfying that stability criteria then we will be tending towards the stratification as I have shown in the figure or otherwise there will be unstable convection cells okay.

Let us see in the basic state was the $u = 0$ $v = 0$ $w = 0$ remember that the basic state θ is actually hydrostatic okay, for the θ the basic state which the $\theta = \theta(z)$ have shown the hydro states is function of the z this is the function $\theta = \theta(z)$ in the similar fashion let us in case of the pressure portion the pressure parting involves the factor of the we have shown over here so now we are giving in the portion $-p$ in the gear of that the part of the innovation. We are finding out we have the separate source of this equation so the disturb equation we are replace the W/W in the momentum equating we are the energy and then subtract the ρ the equation we will called as the disturb equation

This will almost involved with the, so it will be always involved with the a part of the quantity a part of the quantity is the dashed quantity so if you get the continuity equation coming out like this like this we are having the X momentum in the equation that the term is you see in the left hand side everywhere we have the quantities okay.

In the right hand side we are having $-p$ quantities and in the P there p is the function of the T θ is the function of the Z so the in the p θ in the θ is the vector we are not added in the that are in the in the added over here. Similarly we are the direction so the target is not coming over here X and the Y momentum it is the ratios so this is the equation next this equation we will see so left hand side we will have in the term in the part of the quantity and in the right of the Z of P not and $-\rho$ of the z .

So this is term cannot cancelled now in the function of z okay mind the rally number it will having in the θ so the θ will be the θ_0 and θ' it will be replace in the next and your discussed in the giving in you w okay. So now let us move in the alley number is $=$ to the $\theta +$ then odd number θ . But in the θ number is added in the over here in case of that we are in the ∂ and the Z will put in the R A ∂ not and we have already shown over here and the z will not but in the nothing and the derivative of this equation respected Z okay.

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$$\begin{aligned}
& z \text{ momentum: } \frac{1}{Pr} \left(\frac{\partial w'}{\partial t} + u' \frac{\partial w'}{\partial x} + v' \frac{\partial w'}{\partial y} + w' \frac{\partial w'}{\partial z} \right) = -\frac{\partial P_o}{\partial z} - \frac{\partial P'}{\partial z} + Ra\theta + \nabla^2 w' \\
& \rightarrow \frac{1}{Pr} \left(\frac{\partial w'}{\partial t} + u' \frac{\partial w'}{\partial x} + v' \frac{\partial w'}{\partial y} + w' \frac{\partial w'}{\partial z} \right) = -Ra\theta_o - \frac{\partial P'}{\partial z} + Ra(\theta_o + \theta') + \nabla^2 w' \\
& \text{or, } \frac{1}{Pr} \left(\frac{\partial w'}{\partial t} + u' \frac{\partial w'}{\partial x} + v' \frac{\partial w'}{\partial y} + w' \frac{\partial w'}{\partial z} \right) = -\frac{\partial P'}{\partial z} + Ra\theta' + \nabla^2 w' \\
& \text{Energy: } \frac{\partial \theta'}{\partial t} + u' \frac{\partial \theta'}{\partial x} + v' \frac{\partial \theta'}{\partial y} + w' \frac{\partial \theta'}{\partial z} + w' \frac{\partial \theta_o}{\partial z} = \nabla^2 \theta' \\
& \frac{\partial \theta'}{\partial t} + u' \frac{\partial \theta'}{\partial x} + v' \frac{\partial \theta'}{\partial y} + w' \frac{\partial \theta'}{\partial z} - w' = \nabla^2 \theta'
\end{aligned}$$

So if you add that, that will become in z and $RA * \theta$ okay so if we cancelled in the RA in the then my Z is not term of the equation and with this and in the energy equation in the other had if you see in the left hand side we are having in the part of the n the equation for there and the, but here we are having in the q is actually found out the Z , we will be having this will be in the and the term also present and that is apart from the Z in the okay.

Now here we know that the θ will be ordered if you see n here and the Q was the Q not was nothing but z and in the nothing but and in by 1 . so if you put you over here soon will get -1 this is one in the 1 , so this my energy equation we have obtain and we see the obtain here and the discard for the part of the equation and we will be the and the part of the equation assuming in the linearism the disturb and in the problem.

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Linearized disturbance equation (For small amplitude):

Continuity: $\frac{\partial u'}{\partial x} + \frac{\partial v'}{\partial y} + \frac{\partial w'}{\partial z} = 0$

Momentum: $\frac{1}{Pr} \frac{\partial u'}{\partial t} = -\frac{\partial P'}{\partial x} + \nabla^2 u'$

$\frac{1}{Pr} \frac{\partial v'}{\partial t} = -\frac{\partial P'}{\partial y} + \nabla^2 v'$

$\frac{1}{Pr} \frac{\partial w'}{\partial t} = -\frac{\partial P'}{\partial z} + Ra\theta' + \nabla^2 w'$

And we are disturb in theory there and the small amplitude and were ever we are having the terms in the neglected as do that this is the community equation first in the momentum equation in the term of and the multiplication of the dashed d term in the actually in this is and its is finally and in the convection term and in the energy and you see in the Q dash term and in the see in and the W 'very small neglected in the and the rest equation we sar X Y and Z and in the term and its and this will be in the mud energy equation and in the boundary condition and obviously at the atom in the water animation velocity.

But here Q was the -1 and already absorb the θ 0 is not for the and it is 0 and it will be the condition and will be linear zed now let us a consider to have and this is the equation and this is the equation and it will be the equation let us consider this is 2 and in the second equation and we are ratio and in that a we arte4 the three d dis approximant of this matrix.

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2D disturbance equation:
 $v' = 0$ Any 3D disturbances can be converted into addition of many 2D disturbances

Normal mode analysis:
 $u'(x, z, t) = \hat{u}(z)e^{iKx + \sigma t}$
 $w'(x, z, t) = \hat{w}(z)e^{iKx + \sigma t}$
 $\theta'(x, z, t) = \hat{\theta}(z)e^{iKx + \sigma t}$ $\sigma = \sigma_R + i\sigma_I$
 $p'(x, z, t) = \hat{p}(z)e^{iKx + \sigma t}$ $K = K_R + iK_I$

Temporal stability:	Spatial stability:
Unstable $\sigma_R > 0$	$\sigma_I = 0$ or σ real
stable $\sigma_R < 0$	$K_I > 0$ stable
neutral $\sigma_R = 0$	$K_I < 0$ unstable

So I am making the simplification over here and this is 2 d digital and then it is equation and we were assuming that any 3 d disturbance can be converted into the meaning to the sequence with this allocation the 3 d allocation is as an alternative actually we are converting into two d construct electrons, here let us consider the 2 d disturbance we are having u' and v' and the w' it is +to 0.

So once we are considering that then let us try to do in the normal mode analysis and that is the normal mode analysis in the any disturbance of that in the quantity ni that we remember that we actually we have convert into 0 can we retain which is nothing but in the function of X and T can be retained U Z and the E to the power the X +and this the module Ad that we can write down so here is the Z dependents X dependents come over here and the I will be multiplication and the t dependents come so here with the \sum and in the ratio were this key and the \sum can be real complex and composition if you see these things if the so this can be like this and in the form of real imaginary. So okay.

So obtain the temporal stability \sum compensation and it is actually becomes in the part that is the respective to the and this will be taking in exponential serious and in the exponential growth. so that means it will be in the going towards in the unstable zone and in the obviously the temporal stability .till that real term in this they are in the normal molding and this is the require to be less than 0 for becoming neutral if we had in the \sum and it = to 0 on the time ,so here for the stability guide here we have to \sum or less than 0 in the similar fashion in familiar stability we should have in $\sum I =$ to 0 or that is real and having in the I multiplication over here okay.

As well as for the Σ we have to have in the Σ $I =$ to 1 and real definitely and in the stress stability we have to $K I$ and greater than to 0 then the multiplication of $I * I$ negative. So the negative or the positive becoming once again negative then in it will be ending towards 0 .so this is will damp down with the respective place ,will be having the special stability . So for the special stability we will require than the 0 .so let us try to put all this normal mode analysis in our equations.

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Linearized disturbance equation (For small amplitude):

Continuity:
$$\frac{\partial u'}{\partial x} + \frac{\partial v'}{\partial y} + \frac{\partial w'}{\partial z} = 0$$

Momentum:
$$\frac{1}{Pr} \frac{\partial u'}{\partial t} = -\frac{\partial P'}{\partial x} + \nabla^2 u'$$

$$\frac{1}{Pr} \frac{\partial v'}{\partial t} = -\frac{\partial P'}{\partial y} + \nabla^2 v'$$

$$\frac{1}{Pr} \frac{\partial w'}{\partial t} = -\frac{\partial P'}{\partial z} + Ra\theta' + \nabla^2 w'$$

Energy:
$$\frac{\partial \theta'}{\partial t} - w' = \nabla^2 \theta'$$

Boundary conditions: at $z = 0$ $u' = 0$ $v' = 0$ $w' = 0$ $\theta' = 0$

at $z = 1$ $u' = 0$ $v' = 0$ $w' = 0$ $\theta' = 0$

So in order to those we will so what is the determination ,so far what I s the infinity in termination we will be going for vertices and in the determination what we are having ∂ and the $'$ and this nothing but ,if you see but the normal mode analysis and do the derivatives so it will become in the $(DU _I K W)eikx+ot$ and E so this is becoming or what is the velocity and if you to in the vertices in this fashion , so it will Σ of $Z * 2 * I$ to the $Kx+ot$. So this Ω is becoming nothing but actually Ω bar is nothing but becoming $Du - ikw$ okay, competition between this two okay then let us write down from continuity so if yu remember our continuity equation so continuity equation was like this, so if you put the value of u_0 u' and w' in the form of your normal mode analysis.

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$$\text{Let, } \hat{\Omega}' = \text{Vorticity} = \frac{\partial u'}{\partial z} - \frac{\partial w'}{\partial x} = (D\hat{u} - iK\hat{w})e^{iKx + \sigma t}$$

$$\text{Let us say } \hat{\Omega}' = \hat{\Omega}(z)e^{iKx + \sigma t}$$

$$\text{So, } \hat{\Omega} = D\hat{u} - iK\hat{w}$$

Now from continuity,

$$iK\hat{u} + D\hat{w} = 0 \longrightarrow iK\hat{u} = -D\hat{w} \longrightarrow \hat{u} = -\frac{1}{iK} D\hat{w}$$

$$\hat{\Omega} = D\hat{u} - iK\hat{w} = -\frac{1}{iK} D^2\hat{w} - iK\hat{w} = -\frac{1}{iK} [D^2\hat{w} + (iK)^2\hat{w}] = -\frac{1}{iK} [(D^2 - K^2)$$

$$\text{or } -iK\hat{\Omega} = (D^2 - K^2)\hat{w}$$

$$\frac{\partial \theta'}{\partial t} - w = \nabla^2 \theta'$$

$$\sigma \hat{\theta} - \hat{w} = (D^2 - K^2)\hat{\theta} \quad (1)$$

So then ultimately we will be getting equations like this from the continuity equation $iku^- + dw^-$ let us call this \hat{u} so $iku^{\hat{}} + dw^{\hat{}} = 0$ remember the radius term we have neglected okay and then for 2d cases. So we get over here what is u that is nothing but $-1/ik d u^{\hat{}}$ okay so $u^{\hat{}} = -1/ik dw^{\hat{}}$. Now if we put this back in the value of Ω over here okay so we get $\Omega^{\hat{}}$ was actually $du^{\hat{}} - ik w^{\hat{}}$ so if we put this $u^{\hat{}}$ value over here then we can get after simplification this $\Omega^{\hat{}}$ becomes $-1/k d^2 - k^2 x w^{\hat{}}$ okay.

So $\Omega^{\hat{}}$ has been found out in terms of $w^{\hat{}}$, so little bit of side change will be giving you $-ik \Omega^{\hat{}} = d^2 - k^2 x w$ okay now let us see in my energy equation so this was my energy equation in the energy equation if I try to put the normal mode analysis so read become $\sigma x \theta^{\hat{}} - w^{\hat{}} = \partial^2$ will be always giving you $\partial^2 k^2$ okay so $d^2 - k^2$ that we have shown over here, so $d^2 - k^2 x \theta^{\hat{}}$ okay so this becomes my energy equation.

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Let, $\Omega' = \text{Vorticity} = \frac{\partial u'}{\partial z} - \frac{\partial w'}{\partial x} = (D\hat{u} - iK\hat{w})e^{iKx + \sigma t}$

Let us say $\Omega' = \hat{\Omega}(z)e^{iKx + \sigma t}$

So, $\hat{\Omega} = D\hat{u} - iK\hat{w}$

Now from continuity,

$$iK\hat{u} + D\hat{w} = 0 \longrightarrow iK\hat{u} = -D\hat{w} \longrightarrow \hat{u} = -\frac{1}{iK} D\hat{w}$$

$$\hat{\Omega} = D\hat{u} - iK\hat{w} = -\frac{1}{iK} D^2\hat{w} - iK\hat{w} = -\frac{1}{iK} [D^2\hat{w} + (iK)^2\hat{w}] = -\frac{1}{iK} [(D^2 - K^2)\hat{w}]$$

$$\text{or } -iK\hat{\Omega} = (D^2 - K^2)\hat{w}$$

$$\frac{\partial \theta'}{\partial t} - w = \nabla^2 \theta'$$

$$\sigma \hat{\theta} - \hat{w} = (D^2 - K^2)\hat{\theta} \quad (1)$$

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And my momentum equations x momentum equation we will be x momentum equation and z momentum equation.

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Now,

$$\frac{1}{Pr} \frac{\partial u'}{\partial t} = -\frac{\partial p'}{\partial x} + \nabla^2 u' \quad (2)$$

$$\frac{1}{Pr} \frac{\partial w'}{\partial t} = -\frac{\partial p'}{\partial z} + Ra\theta' + \nabla^2 w' \quad (3)$$

$\frac{\partial}{\partial z}(2) - \frac{\partial}{\partial x}(3) \rightarrow$

$$\frac{1}{Pr} \frac{\partial}{\partial t} \left[\frac{\partial u'}{\partial z} - \frac{\partial w'}{\partial x} \right] = -\frac{\partial^2 p'}{\partial x \partial z} + \frac{\partial^2 p'}{\partial z \partial x} + \nabla^2 \left[\frac{\partial u'}{\partial z} - \frac{\partial w'}{\partial x} \right] - Ra \frac{\partial \theta'}{\partial x}$$

$$\frac{1}{Pr} \frac{\partial \hat{\Omega}'}{\partial t} = \nabla^2 \hat{\Omega}' - Ra \frac{\partial \hat{\theta}'}{\partial x}$$

$$\frac{1}{Pr} \sigma \hat{\Omega} = (D^2 - K^2) \hat{\Omega} - iKR a \hat{\theta}$$

Thus Y momentum equation we need to be considered for 2D cases. So those things if we do little bit of simplification so first let us do this $\partial/\partial z$ of first equation X momentum equation $-\partial/\partial x$ of second momentum equation to reduce some terms. So you will be getting this type of equation where this is nothing but your λ okay. So we can put $1/Pr \partial \lambda' \partial P - \Delta^2 \lambda'$ to this term these two terms are cancelling minus of $Re(\partial \theta') / \partial x$ okay. So once you try to put the non dimensionalization over here in this equation it comes out $1/Pr \sigma \lambda_{cap} = (D^2 - K^2) \lambda_{cap} - iKR a \theta_{cap}$ okay.

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Multiplying with $-iK$, $\frac{\sigma}{Pr}(-iK\hat{\Omega}) = (D^2 - K^2)(-iK\hat{\Omega}) - K^2 Ra\hat{\theta}$

$$\frac{\sigma}{Pr}(D^2 - K^2)\hat{\omega} = (D^2 - K^2)(D^2 - K^2)\hat{\omega} - K^2 Ra\hat{\theta}$$

$$(D^2 - K^2)^2 \hat{\omega} - K^2 Ra\hat{\theta} = \frac{\sigma}{Pr}(D^2 - K^2)\hat{\omega}$$

$$(D^2 - K^2)\hat{\theta} + \hat{\omega} = \sigma\hat{\theta}$$

Input Ra, Pr, K

Neutral curve is a function of Pr

If $Ra < Ra_{critical}$ Hydrostatic state is stable

So if you try to multiply this equation with $-i(K)$ so ultimately you will be getting $\sigma/Pr(-iK\lambda_{cap}) = (D^2 - K^2)(-iK\lambda_{cap}) - K^2 Ra\theta_{cap}$ okay so from this equation let us multiply both sides with $-iK$. So once you see this $-iK\lambda_{cap}$ already we have proved this is nothing but actually $\Delta^2 - (D^2 - K^2)\omega_{cap}$, so let us know that term over here okay. So this $-iK\lambda_{cap}$ is nothing but $(D^2 - K^2)\omega_{cap}$, so in this side also we can put the same thing, so this will become $(D^2 - K^2)\omega_{cap}$ okay.

So finally if this equation is coming out to be $(D^2 - K^2)^2$ so this whole square will be coming $\omega_{cap} - (K^2 Ra\theta_{cap})$ this term is equals to $\sigma/Pr(D^2 - K^2)\omega_{cap}$ so this is becoming my momentum equation now. And already the energy equation I have showed you, energy equation comes out to be like this, so that equation let us write down over here, so this is my energy equation okay. So we have two equations now which needs to be solved for known value of Ra prandtl number and your K okay.

So Ra prandtl number and K if those are known then ultimately you can get this type of graph okay, so here you can find out that below is starting level of Rayleigh number everything will be stabled okay. So once you have the quarter equation that will come back to the stratification, but above a certain Rayleigh number you will find out that these situations are actually getting towards in stability, so the zone will become unstable okay.

So prandtl number in stability in Rayleigh number in stability will pickup okay, and this is the boundary between these two which is nothing but neutral curve okay of linear stability analysis. So remember here we have done the linear stability analysis. So this is the neutral curve as I have

said and this is the neutral curve actually depends on the value of prandtl number what you are entering okay.

And as I have said that below a certain critical Rayleigh number always we are having hydrostatic state or stable state we will come back to hydrostatic state or stable state, above which you will be getting the in stability.

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Summary

- Governing equation for Rayleigh Benard convection:

Basic state: $u = v = w = 0 \quad \theta_0 = -z \quad P_0 = -\frac{Ra z^2}{2} + C$

Disturbed state: $(D^2 - K^2)^2 \hat{w} - K^2 Ra \hat{\theta} = \frac{\sigma}{Pr} (D^2 - K^2) \hat{w}$

$$(D^2 - K^2) \hat{\theta} + \hat{w} = \sigma \hat{\theta}$$

- Proposition of stability criterion:

<p>Temporal stability:</p> <p>Unstable $\sigma_R > 0$</p> <p>stable $\sigma_R < 0$</p> <p>neutral $\sigma_R = 0$</p>	<p>Spatial stability:</p> <p>$\sigma_I = 0$ or σ real</p> <p>$K_I > 0$ stable</p> <p>$K_I < 0$ unstable</p>
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So this is related to Rayleigh Benard convection. So let us summarize, so what we have understood in this lecture we have understood what is the governing equations for Rayleigh Benard convection, so for the basic state, so basic state θ_0 and P_0 in terms of θ_0 and P_0 in terms of z okay, and these two state we have got two equations okay, two equations like this many terms of w_{cap} and θ_{cap} , so w_{cap} and θ_{cap} this will lead to eigenvalue problem okay, where I can find out what is the value of σ_R and K_i for stable situation.

If σ_R is actually less than 0, then we will be having temporal stability and if K_i is greater than 0 we will be having spatial stability okay. So this we have seen for the stability criteria okay. So after this let me test what you have understood in this lecture. So which configuration has stable density field we are having four answers.

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Test your understanding ?

1. Which configuration has stable density field:

(a) Heated from top	(b) Heated from bottom
(c) Both are stable	(d) None
2. System will be always stable:

(a) above a critical Ra	(b) Below a critical Ra
(c) at any positive value of Ra	(d) for all Ra
3. Basic state pressure gradient in axial direction:

(a) $\frac{-z}{Ra}$	(b) $-Ra$	(c) $\frac{-Ra}{z}$	(d) $-Raz$
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Heated from top, heated from bottom, both are stable or none are stable okay. So this we have shown in the second slide of this lecture, so obviously the answer is heated from top, we will be having always hydrostatic case, so there is no in stability majority okay. Then next one is system will be always stable for above a critical number, below a critical number at any positive value of Rayleigh number and for all value number okay.

So critical or above the critical value, below a critical value at any positive value of Rayleigh or for all Rayleigh the system will be stable. So obviously in the last slide we have shown you the Rayleigh number curve and we have shown that criteria that below a critical value always the system will become stable, so this is the correct answer. Then third one basic state pressure gradient in axial direction will be becoming $-z/Ra$, $-Ra$, $-Ra/z$, $-Raz$, Rayleigh number into z .

So this already we have shown pressure gradient basic state is P_0 , so P_0 actually is equal to $0Raz$ so this is the correct answer okay. So I think all of you note down the right answers in these three questions. So with this I am ending this lecture in our next lecture we will be discussing about heat transfer with phase change okay. If you have any query related to this lecture or any other lecture in general please keep on posing the discussion forum thank you.

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