Computational Fluid Dynamics
Dr. Krishna M. Singh
Department of Mechanical and Industrial Engineering
Indian Institute of Technology - Roorkee

Lecture - 08
Energy and Scalar Transport Equations

Welcome back to the next lecture in module 02 Mathematical Modeling, let us have a recap of what we have done so far in this module. We started with Statement of conservation laws, then we had a look at notice a mathematical preliminaries. In past 3 lectures we had focused on the derivation of the governing equation of fluid flow. And today in the same series you would focus on derivation of Energy equation and Scalar Transport equation.

And thereafter, we are going to start off with mathematical classification and boundary conditions for the flow problem in the next lecture. Let us have a recap of what we did in the last lecture.

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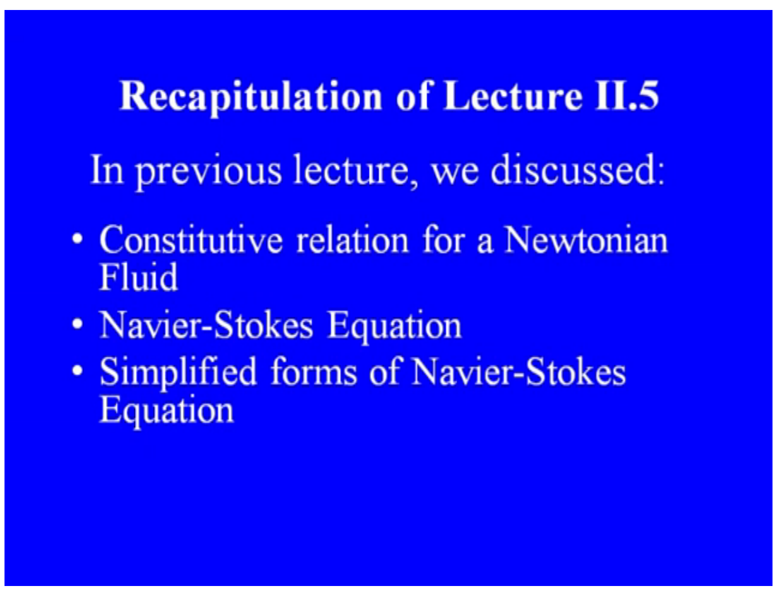

We started off with the discussion on constitutive relations for Newtonian fluid for which stress and strain rate tensor are linearly related, and based on that simplified relation we obtained a famous Navier-Stokes equation. We also derived simplified forms for Navier-Stokes equation for specialized cases for instances incompressible flows, which gives us much simpler equation, and we had a look at the cases for inviscid flow which use us the celebrated Euler's equation.

And we looked at one specialized cases which we called creeping flow equation. In today's lecture that is sixth lecture in the series, we should focus on energy equation and scalar transport equation this would be outline.

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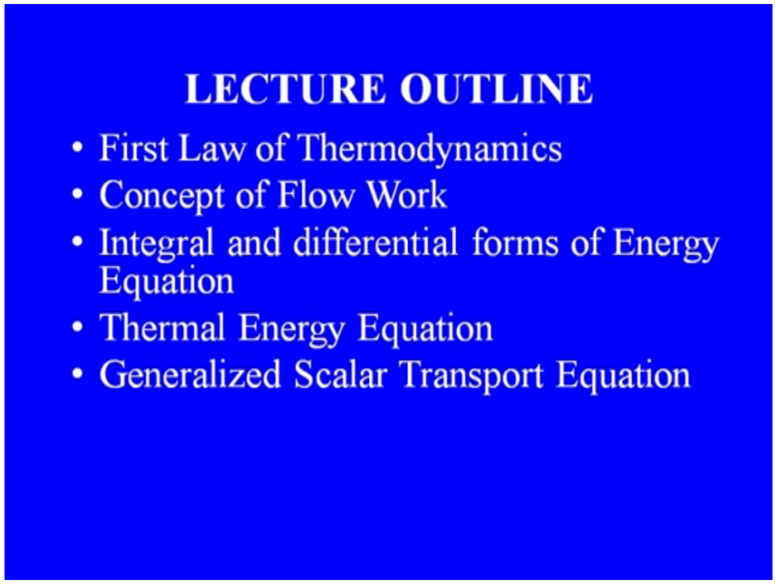

We are start off with first of thermodynamics which is the basic law for energy conservation, and we would have a look at what would you mean by flow work, then based on the first law we would have obtained integral and differential forms of energy equation, then from this equation we would derive a specific equation which we call thermal energy equation, and in the end we would have looked at equations for a generalized scalar transport equation.

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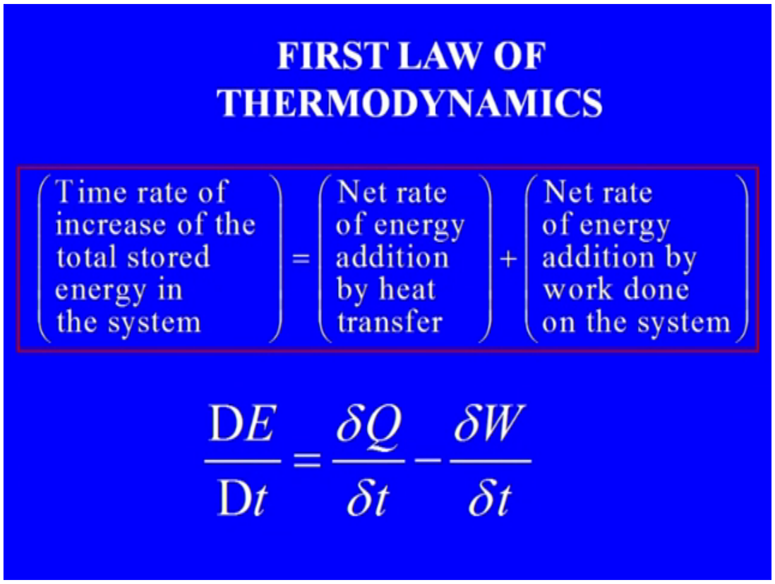

So now let us start off with the fist law of thermodynamics which is the basic equation for energy conservation. What does this law say? Let us recall back what we learnt your thermodynamics class says, the time rate of increase of the total energy stored in the system this = net rate of energy equation by heat transfer from the surroundings + net rate of energy addition by work done on the system.

Let us use our usual symbols we use capital E to indicate the total stored energy in the system, so DE/Dt this represents the time rate of change rather the rate of increase of the stored energy in the system this = del Q/del t where Q is heat transfer we have used symbol delta to indicate that Q is not a state function, it depends on the path, so this is why we use this in adjective frentials on the right hand side.

So del Q/del t this gives us the net rate of energy addition by heat transfer. And then we have followed the typical convention used in thermodynamics, that is to say that work done is considered as positive if it is done by the system on the surroundings and negative vice versa, so here we are looking at the work done on the system that is the reason we have introduced negative sign here, so del Q/del t-del W/del t.

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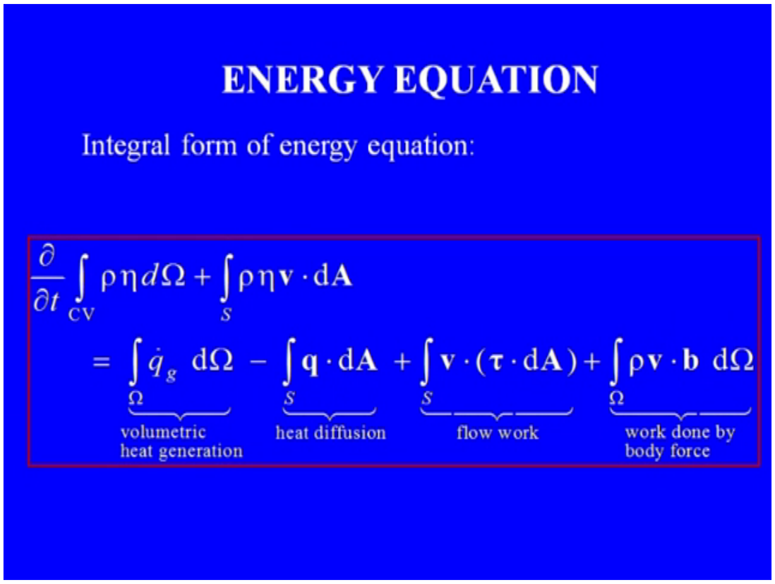

We would look at each of these terms separately and thereafter we would have time our integral form of energy equation, so now let us have a detailed look at each term.

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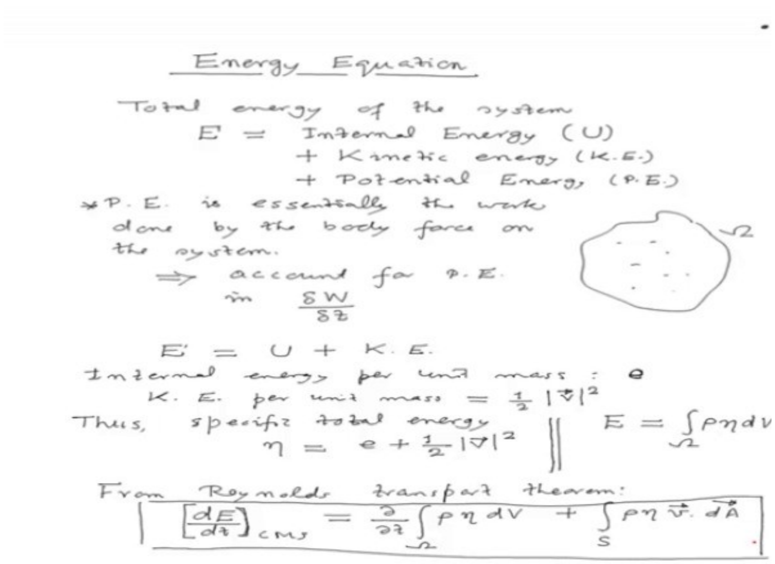

We had total stored energy of the system in those total energy capital E we have got the account for what we call internal energy. In thermodynamics we normally use symbol capital U+ kinetic energy + potential energy P.E. Now if you remember in potential energy arises because of the presence of a field or from force field in which our system is embedded, so this potential energy is essentially the work done by the body force on the system okay.

This bring the case what we will do is we would observe or transfer this term or rather account for this term for potential energy in the work done term that is del W/delta t, so we do not need to put it in as part of a E. So thus, our capital E would consist of only two terms that is our internal energy + kinetic energy, now internal energy per unit mass we will chooses a slightly different symbol that is called it small e to denote the internal energy per unit mass.

Because usually thermodynamics symbol small u we use for velocity component next direction, so let us this that is why we choose a different symbol. Kinetic energy per unit mass this would be given by 1/2 square of a velocity magnitude. So thus, where a specific total energy let us use a symbol eta for it eta = e+1/2 of modulus of v squared. Let us note the relation between capital E that is total energy and eta it is very simple.

Capital energy E is basically integral over the whole up over domain so that is a rho eta dV over domain omega. So now let us invoke Reynolds transport theorem, from Reynolds transport theorem dE/dt which occurs in the first law statement dE/dt for a control mass system, this would be first one with time derivative or this temporal variation of the stored energy and the system is rho eta dV and + the flux of this energy across the control surfaces so rho eta v dot dA.

So this one expression which rather an expansion of the left hand side of a first law of thermodynamics that is dE/dt term in terms of the quantities related to the control volume. Next, we will have a detailed look at 2 other terms on the right hand side, so in right hand side we had 1 term.

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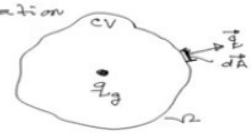
Heat Transfer

Caused by

① Internal heat generation

② Heat transfer from surroundings.



If q_g is volumetric heat generation rate, then its total contribution

$$\boxed{\left(\frac{\delta Q}{\delta t}\right)_g = \int_\Omega q_g \, d\Omega}$$

Heat transfer through control surface

$$\boxed{\left(\frac{\delta Q}{\delta t}\right)_s = -\int_S \vec{q} \cdot d\vec{A}}$$

Thus, net rate of heat transfer into the c.v. (or system occupying CV)

$$\boxed{\frac{\delta Q}{\delta t} = \underbrace{\int_\Omega \dot{q}_g \, d\Omega}_{\text{Internal heat source}} - \underbrace{\int_S \vec{q} \cdot d\vec{A}}_{\text{Diffusion of heat through c.s.}}}$$

Now we had heat transfer now there are 2 principal components of heat transfer for our fluid body omega, there would be some amount of heat which is generated internally let us call this volumetric heat generation at qg. So heat transfer is caused by 2 mechanism, number one is our internal heat generation now this internal heat generation could be due to chemical reaction or it could be let us say a nuclear reaction or maybe an embedded heated throughout the volume.

And the second would be the heat transfer from surroundings okay. So can we find out the total amount of each of these terms, so if qg is our volumetric heat generation rate then its total contribution we can express in terms of a volume integral, so let us use the symbol delta q/delta tg the subscript g which denotes internal heat generation so this is integral of qg over volume.

Now how do you obtain the next term that is the heat transfer from the surroundings, let us look at one small boundary here and if the q where the heat flux normally we say that q would be directed outside this heat flux vector across an area element dA, so q dot dA that will give us the energy or rather heat which is leaving our control volume, but we are interested in finding out how much heat is being transferred from the surroundings to a system or to the control volume.

So heat transfer through control surface let us call it del q/ del t s we have to put a negative sign because q dot dA are integral service integral of that term will give us the amount of heat which is going out per unit time and what we are interested in finding out what how much energy which is being transferred inside the system, so this would be q dot dA, so if you combined these two terms we can find out the net rate of heat transfer into this control volume or rather the system which instantaneously occupying control volume.

So del q/del t this we can express as a first term of volume integral qg or if you want to make the rate process involved we can put qg dot, so qg dot d omega-q dot dA, and nature of these two terms is very clear. The first term this qg dot d omega this is due to internal heat sources, and the latter term this surface integral this tells us the diffusion of heat through control surface. So now we have obtained the mathematical form for the rate of energy transfer due to heat.

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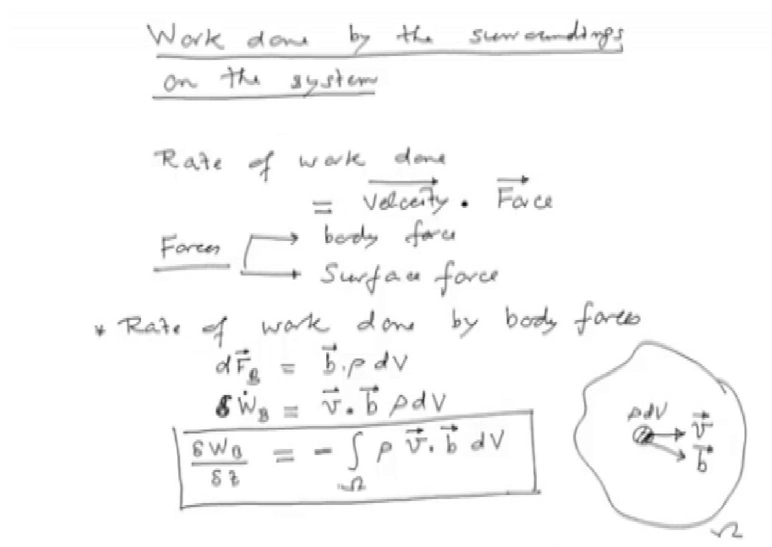

Next, we need to have a look at the rate of work done by this surroundings on the system, so work done by the surroundings on the system, so if you can go back to the mechanics we know the rate of work done can be expressed in terms of rate of work done is dot product of velocity with force vector. So now in our case we will in general have 2 set of forces we have body forces and surface forces, so let us treat them separately and find out what is the rate of work done by each component.

So the forces involved are a body force and surface force. So let us take the first case that is the rate of work done by body forces, let us take a small differential element rho*dV that will give us the mass of this small element, it has got the velocity v and the body force acting on it per unit mass is b. So this total body force on this small differential element this force dFB that is b times rho dV.

So differential rate of work done delta WB dot, this would be v dot B rho dV, so to find out a total rate of work done by the body forces let us call that is del WB/del t this would be negative of volume integral over the control volume rho v dot b dV okay. The next we have to look at what is the rate of work done by surface forces.

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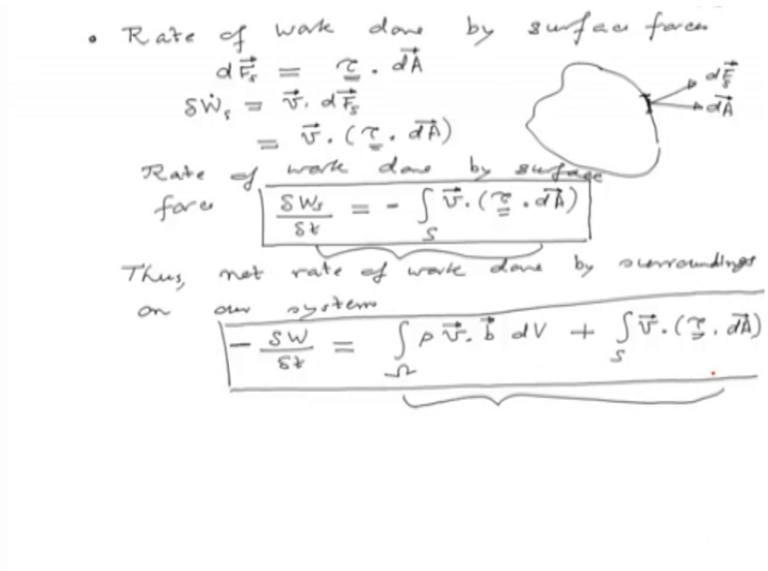

So rate of work done by surface forces, so let us consider a small area element on the surface dA so we obtain the surface force on it dFs that we can obtain by taking the dot product of this stress tensor with our area element, so this gives us the surface force acting on a differential area element dA. So now rate of work done due to surface forces that would be v dot dFs=v dot tau dot dA, so we can easily find out total work done by the surface forces.

So this rate of work done by surface forces let us call it as delta Ws/delta t-v dot tau dot dA, please remember this we have putting this thing the negative sign to take care of the sign convention using thermodynamics okay. So thus, net rate of work done by surroundings on our system this - delta W/delta t that we saw in the 2 integrals integral reasons which we derived earlier, the first one which corresponds to our worked in a body force.

The second one which corresponds to body worked in surface process, so combine these two together so we get rho v dot b dV+ the area integral v dot tau dot dA. So now you have got detailed expressions for all the terms, the 3 terms which we started off in our first law of thermodynamics we have obtained the detailed expressions for dE/dt, we have obtained a quantitative expressions for delta Q/delta t that is heat transfer into the system from the surroundings.

And we have also obtained quantitative expressions for the work done by the surroundings on the system, let us combine all these together and if you do that this is what we get.

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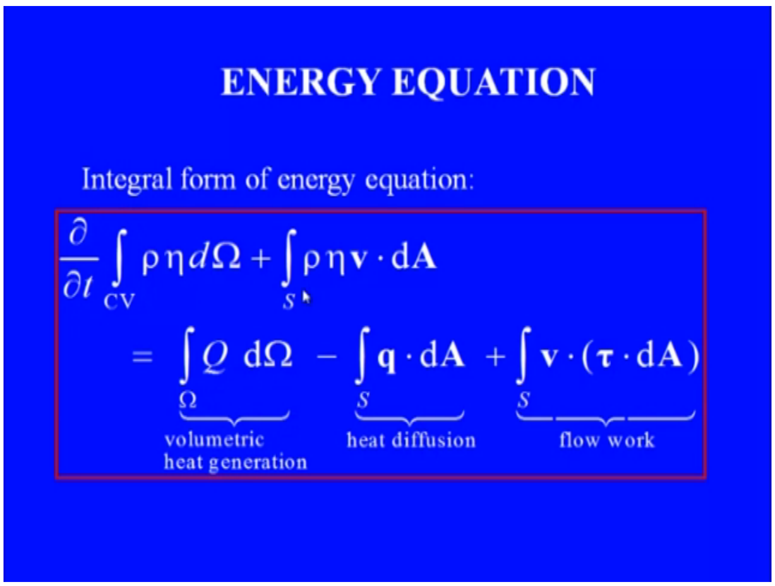

We get the integral form of energy equation, from the left hand side we got del rho/del t rho eta d omega, so this is the time rate of change of this stored energy into the control volume + second term uses the flux rho eta v dA the flux of energy which is going out from this control volume. And on the right hand side the first 2 terms which account for the heat transfer Q*d omega is the volumetric heat generation and second one is heat diffusion q dot dA + the next term is v dot tau dot dA.

This term we also referred to as flow work + we will have we have got include here the work done by body forces which is missing in this expression at the moment.

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Energy equation

$$\frac{dE}{dt} = \frac{\delta Q}{\delta t} - \frac{\delta W}{\delta t}$$

$$\boxed{\begin{aligned} &\frac{\partial}{\partial t}\int_{\Omega} \rho \eta \, dV + \int_{S} \rho \eta \, \vec{v} \cdot \vec{dA} \\ &= \int_{\Omega} \dot{q}_g \, d\Omega - \int_{S} \vec{q} \cdot \vec{dA} \\ &+ \int_{\Omega} \rho \vec{v} \cdot \vec{b} \, d\Omega + \int_{S} \vec{v} \cdot (\underline{\underline{\tau}} \cdot \vec{dA}) \end{aligned}}$$

Integral form for total energy

So complete expression would be this is what we had we had dE/dt=delta q/delta t-delta W/delta t, dE/dt we had obtained its expanded form using Reynolds transport theorem + del/del t of rho eta dV+ the surface integral which gives us the rate of efflux of energy from the system rho eta v dot dA. Now this equal to first let us put that expanded form for del q/del t, del q/del t contains first one is a heat generation term qg d omega so volumetric integral of that -q dot dA.

So this q dot da that represents the heat which comes in, q is the heat flux vector so negative sign here that gives us that how much heat is being transferred into this control volume from the surroundings, let us have work done term work done by the body forces rho v dot d omega, the last one is work done by the surface forces v dot tau dot dA. So this is our integral form for energy equation for total energy.

Now sometimes you might be interested in only to transport of thermal energy, so in that case what we need to do is take out the mechanical component which can be easily done if we multiply or we take dot product with v of our Cauchy equation, so that will give us the rate of mechanical energy transport subtract that from our integral form of total energy and we will get an integral form for thermal energy, but for the time being we will stop here as far as the integral forms are concerned.

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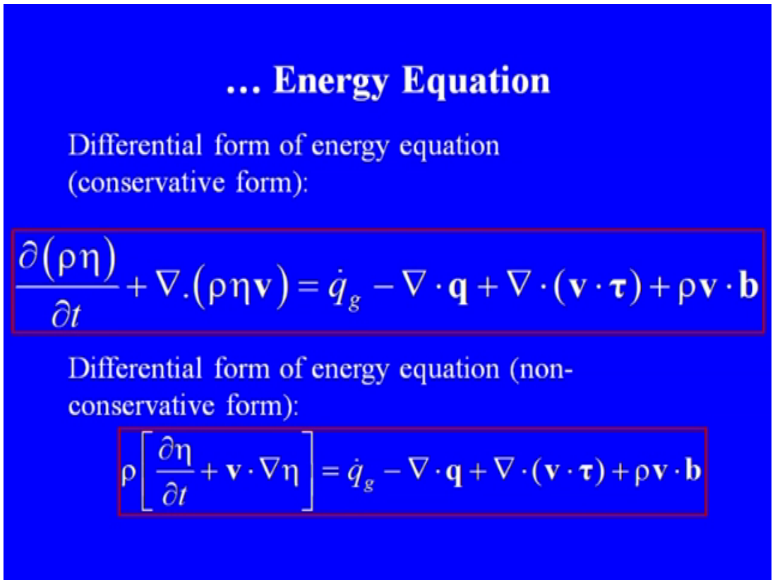

Next, we would like to derive a differential form for energy equation, so how do we obtain the differential form for energy equation, once again we will follow the same approach which we had followed earlier in the derivation of different differential forms like Cauchy equation momentum equation. First, we need to transfer all the terms which we had in integral equation in to corresponding volume integrals, so now let us transform each one of them separately.

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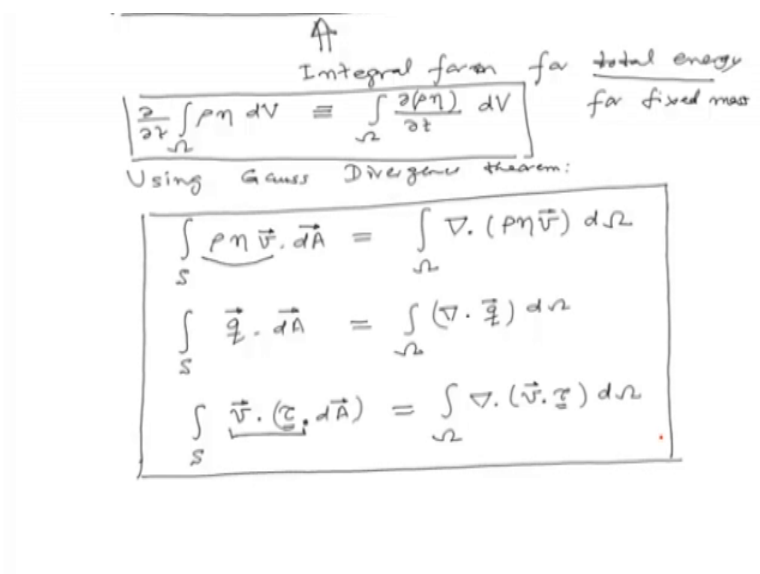

The first one is del/del t of rho eta dV this can be written as volume integral over omega of del rho eta/del t dV for fixed mass, so for fixed mass what we can do is that we can assume that time derivative or the temporal derivative and the integration they can be easily interchanged. Next, let us use Gauss divergence theorem to transform all the surface integrals which we had into volume integrals, so using Gauss divergence theorem let us see.

Let us have a look at each one of the surface integral, this a full surface integral on the left hand side, so see what do we get for it rho eta v dot dA this will be given by the volume of what divergence of whatever terms which we have got before the dot operator that is rho eta*v d omega, so if now transformed the surface integral which we had on the left hand side into a volume integral.

What we will have next, we have got 2 surface integrals in the rhs that is q dot dA and v dot tau dot dA, now let us transform these ones also into respective volume integrals, so q dot dA this simply becomes divergence of q d omega and our last surface integral that is our v dot tau dot dA once again while using divergence theorem all that we need to note down as the terms which occur before dot dA we have to take divergence of those terms.

So that this will give us divergence of v dot tau d omega, so now we have got the corresponding forms or volume integral forms for all the surface integral and we will collect all the terms on left hand side.

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Substitute and re-arrange the integral form of energy eqn.:

$$\boxed{\int_{\Omega} \left[\frac{\partial(\rho\eta)}{\partial t} + \nabla\cdot(\rho\eta\vec{v}) - \dot{q}_g + \nabla\cdot\vec{q} - \rho\vec{v}\cdot\vec{b} + \nabla\cdot(\vec{v}\cdot\underline{\tau}) \right] d\Omega = 0}$$

The integrand would be zero for an arbitrary cv if and only if the integrand is zero everywhere. Thus,

$$\frac{\partial(\rho\eta)}{\partial t} + \nabla\cdot(\rho\eta\vec{v}) - \dot{q}_g + \nabla\cdot\vec{q} - \rho\vec{v}\cdot\vec{b} - \nabla\cdot(\vec{v}\cdot\underline{\tau}) = 0$$

$$\Rightarrow \boxed{\frac{\partial(\rho\eta)}{\partial t} + \nabla\cdot(\rho\eta\vec{v}) = \dot{q}_g - \nabla\cdot\vec{q} + \rho\vec{v}\cdot\vec{b} + \nabla\cdot(\vec{v}\cdot\underline{\tau})}$$

So let us substitute and re-arrange the integral form of energy equation, so what we will get we will get a very big integrand inside or volume integrand omega let us put a big bracket so first is [del rho eta/del t the second term is divergence rho eta*velocity vector, then now terms which have come from the right hand side, so first term will give us -qg dot next we will get divergence of q and -rho v dot b let let us called it as body force and the last term is on this integrand would be divergence v dot tau] let us close a big bracket d omega which = 0.

So if we apply the same argument here we have got an integral which is 0 for an arbitrary control volume omega, so we can argue that integrand would be 0 for an arbitrary volume control volume if and only if the integrand is 0 everywhere. So thus, what would we get let us set our integrand to 0, so we get del rho eta/del t+ divergence of rho eta v-qg dot+ divergence of q-rho v dot b sorry that is a – divergence of v dot tau= 0.

Re-arrange, so that we have terms in a specific energy on the left hand side and remaining terms on the right hand side, so del rho eta/delta + divergence of rho eta v =qg dot- divergence of q+rho v dot b+ divergence of v dot tau, so this gives us the differential form of energy equation and that is what we have summarized here. So differential form of energy equation conservative form del rho eta/del t+ divergence of rho eta v, and this term missing in this equation ppt here that body force term.

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Equation for thermal energy, subtract the contribution of mechanical energy [which can be obtained by taking dot product with velocity $\vec{v}$ of momentum eqn.]. Final eqn. is

$$\boxed{\frac{\partial(\rho e)}{\partial t} + \nabla\cdot(\rho e \vec{v}) = \dot{q}_g - \nabla\cdot\vec{q} + \nabla v : \underline{\underline{\tau}}}$$

$$\boxed{\nabla v : \underline{\underline{\tau}} = \frac{\partial v_j}{\partial x_i}\tau_{ij}} \Leftarrow \text{work done by pressure and viscous dissipation}$$

We have obtained the differential form for total energy equation. Now if you want to obtain the equation for thermal energy we simply need to subtract the contribution of mechanical energy, on this later part this could be obtained by taking dot product with v or velocity vector of momentum equation okay. Now the final equation is in terms of internal energy e del rho e/del t+ divergence of rho e*v = heat generation term q dot g- divergence of q+ gradient of v double dot.

This last term has been obtained from some algebraic manipulations, so the last time in our total energy equation and its expanded form is del v dot dot tau this is basically del vj/del xj tau ij for usual summation convention supply to give us a scalar term, this del v double dot tau, del v that is a gradient of velocity vector which is say against our tensor, tau which is stress tensor which against our tensor.

So we have to take doubly contracted product of these two tensors to obtain this scalar contribution and this term represents the work done by pressure as well as viscous dissipation. Now let us note that if you are dealing with low speed flows which are very common with incompressible flows in particular, the contribution of the viscous dissipation would be very small, the viscous dissipation is important only for high speed compressible flows.

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Low speed flow, viscous dissipation is negligible. Contribution from pressure work can be combined with internal energy e. ⇒ Resulting energy eqn. becomes:

$$\boxed{\rho c_p \frac{DT}{Dt} = \dot{q}_g - \nabla \cdot \vec{q}}$$

Thermal energy eqn. for low speed flows. ($T \equiv$ Temperature)

$c_p \equiv$ Specific heat at const. p.

For the substance obeying Fourier's law:

$$\boxed{\vec{q} = -\underline{\underline{k}}\,\nabla T} \qquad \underline{\underline{k}} \Rightarrow \text{Thermal conductivity}$$

then:

$$\boxed{\rho c_p \frac{DT}{Dt} = \dot{q}_g + \nabla \cdot (\underline{\underline{k}} \cdot \nabla T)}$$

So for low speed flows we can simply draw this term off and combine the contribution coming from pressure into a right hand side in terms of enthalpy, so for low speed flows viscous dissipation is negligible contribution from pressure work can be combined with internal energy, and this gives us the resulting equation for such flow that a low speed flows resulting energy equation becomes rho CP DT/Dt the capital T represents the temperature and this capital D by Dt this gives us material derivative this =q dot g- divergence of q.

So this is the thermal energy equation for low speed flows, here capital T stands for temperature and CP is specific enthalpy at constant pressure sorry a specific heat at constant pressure, so CP stands for a specific heat at constant pressure. Now for some substances specifically for heat thermal energy transfer in solids we can invoke Fourier's law. So for the substance obeying Fourier's law they have got a very special relationship between heat flux q and the gradient of temperature.

So the heat flux vector is given by -K*gradient of T where K is thermal conductivity for an anisotropic substance this K would have 9 components it will be shaken our tensor but for simple isotropic medium it will become a scalar quantity. So now if you substitute the Fourier's equation in the thermal energy equation which we have obtained earlier, so then we get on the simplified form rho CP D of capital T/Dt this =qg dot+ divergence of K times gradient of T.

So this is a form of thermal energy equation for substances which obey the Fourier's law, so in this lecture we stop here. In the next lecture we will have a look at conservation of scalar quantities we would obtain at transport equation for generalized scalar transport, and then we will have a look at mathematical classification and the boundary conditions which we require for the fluid problem.