Computational Fluid Dynamics
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Lecture – 06
Momentum Equation: Navier-Stokes Equations

Welcome back to the next lecture in Mathematical Modeling. We had already finished a brief look at conservation laws, notation in mathematical parameters and we started off with the governing equations of fluid flow. We have derived continuity equation and Cauchy equation. The reviewed focus on the specialized form (()) (00:49) to make equation for a Newtonian fluid which we call Navier-Stoke equation. Before we proceed further, let us see what we did in last lecture.

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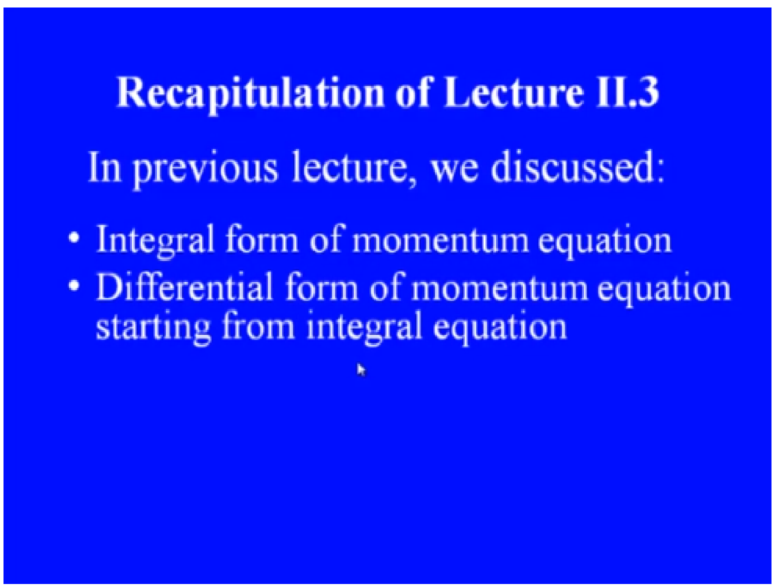

In the previous lecture, we derived the internal form of momentum starting from Newton second law of motion for an arbitrary control volume and thereafter we invoked Gauss divergences theorem and obtained a differential form of momentum equation starting from this integral equation. Now, in today's lecture, we would focus on obtaining an appropriate form for what we call Newtonian fluids that is of a popular Navier-Stokes equations which form the backbone of practical CFD analysis as outlined in today's lecture.

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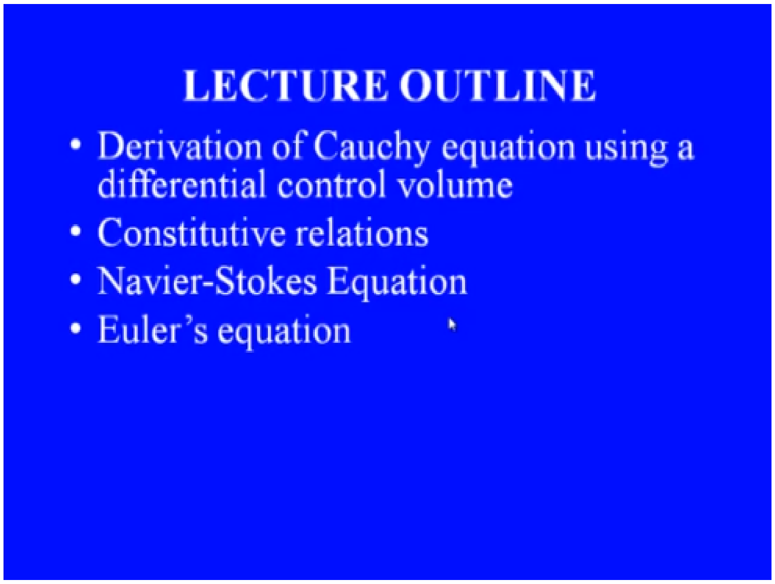

We would finish the unfinished business of previous class. We would derive the Cauchy equation using differential control volume. Then, we would have a look at what we mean by constitutive relations and thereafter we would derive Navier-Stokes equations. We will also have a look at few simplified forms notably the Euler's equation. Now, this is what we did yesterday.

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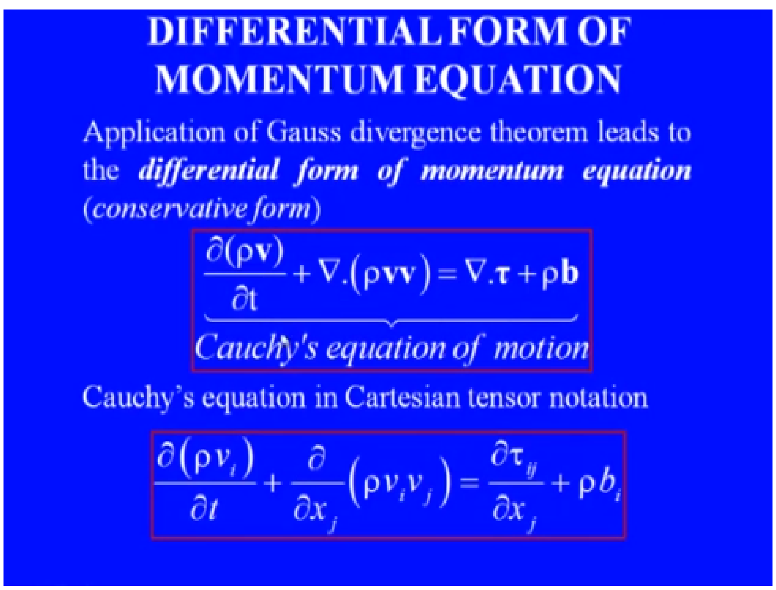

We derived the differential form momentum equation starting from the integral form and the first one was so-called conservative form given by del rho V/del T+ divergence of rho VV=divergence of tau+rho*B where tau is stress tensor B is a body force and this equation is popularly referred to as Cauchy equation of motion. Same equation we can write in Cartesian component form using tensor notation del of rho Vi/del T+del/del X rho ViVj=del tau Ij/del

Xj+rho Bi.

So this represents the momentum equation or Cauchy equation for (()) (02:47) component of velocity. So, we have got a total of three components for three-dimensional space.

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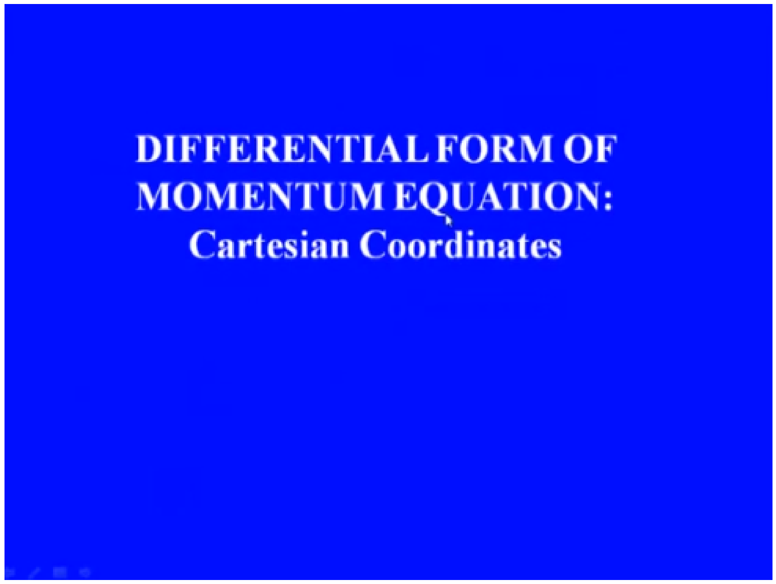

Now today, we would focus on obtaining the same equation in differential form using a differential control volume in Cartesian coordinates. So, to derive this equation, let us first have a brief look at the conventions used for the stress components.

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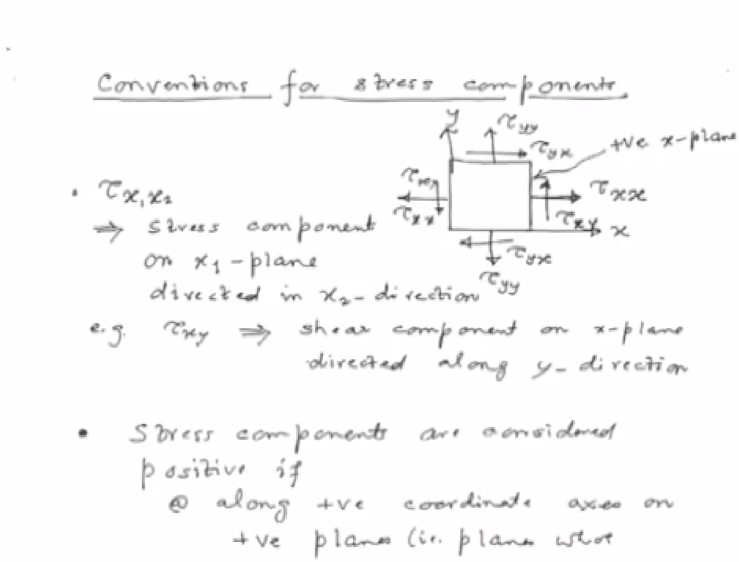

Conventions for this stress components. For the sake of clarity, let us draw a two-dimensional element with X direction, with Y direction and let us identify the stress components which we work on the small element. The first one on the right face which you would call this as your positive explain. Because normal to this area element, points in positive X direction. So, the stress component here is referred as tau XX normal component on this phase and it would be considered to be positive if it is aligned in X direction.

Similarly, you have a sheer component on the surface directed in Y direction, it is called tau XY. Similarly, on the negative X phase tau xx and we can draw the stress components on Y plane. This is tau yy. This is normal component, tau yx, tau xy and this is tau yx. So, what is the Convention which you have adopted. Suppose we use symbol tau x1, x2, this denotes a stress component on X1 plane directed in X2 direction.

For example, if you see this tau xy, this denotes sheer component on X plane directed along Y direction. So, this is about the subscript notation which we used to identify different components. The next one is when will these components be considered positive. The stress components are considered positive if the two possibilities that if they are along positive coordinate axis on positive planes, i.e., the planes whose normal are in positive axis direction and B on negative planes aligned along negative axis.

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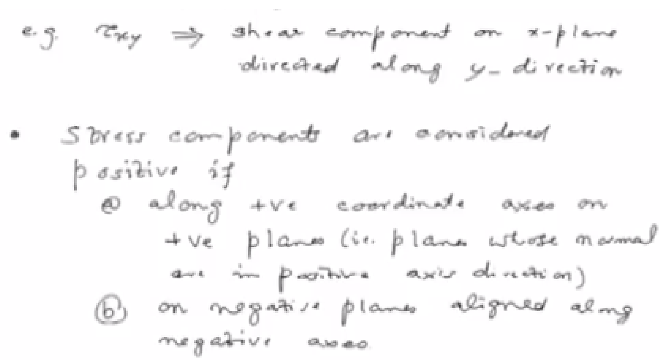

So, that is why if you look at the way had drawn this tau yy on a Y plane, this is a positive signs of the stress component and it is pointing negative Y direction. Same holds for the sheer component tau yx on this negative Y plane. It will be considered positive if it is aligned in negative X direction.

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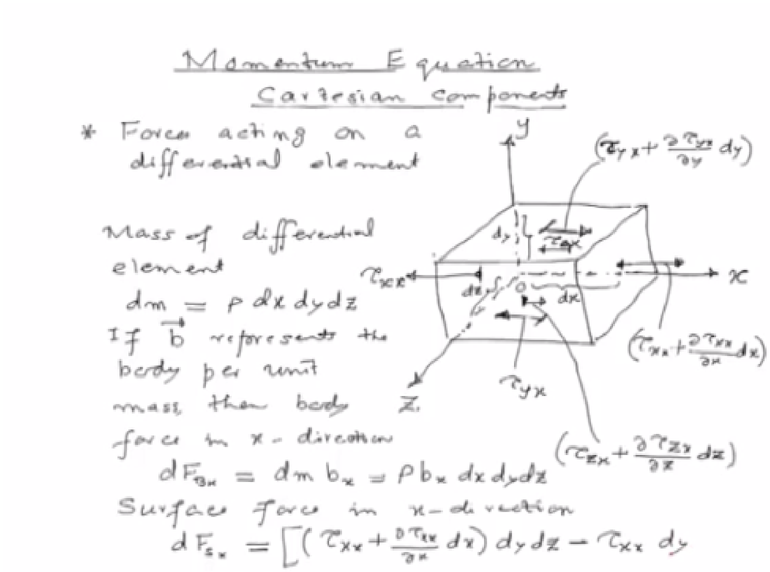

Now, let us move on to our momentum equation and we are going to derive it in Cartesian components. First, let us find out forces acting on a differential element. So, let us draw a simple differential volume in three-dimensional space. This labeled access properly X, Y and Z. So, origin for the sake of clarity, now let us identify the stress components which are aligned in X direction.

Each side of this element, it has got Dx in X direction, Dy in Y direction and Dz in Z direction. Now, let us identify the stress components first on the positive X plane which will act on the centroid of the space. On negative X phase while this is tau xx. Now, with reference to the origin, this tau xx component acting on the positive X phase that can be denoted by Taylor series expansion.

This can be written as tau xx+del tau xx/del XDx and if you want to find the forces on these phases, we need to multiply by the respective areas, we will do that later. Now, what are the sheer components which would work along X direction. Let us focus on positive Y plane. So, we would have one sheer component which will be along positive X direction. Similarly, on a negative Y phase, it would be aligned in negative X direction.

So, this component is tau yx and this tau yx component on positive Y phase, this would be given by tau yx+del tau yx/del YDy. Similarly, we can identify the sheer components which work along X direction on Z phases. On positive Z face, it will act in positive X direction and on negative Z phase it will act in negative X direction. So, it is tau zx. On the positive Z phase, we can represent in terms of Taylor series expansion and is tau zx+del tau zx/del ZDz.

What would be the body force in X direction. So, this mass of this differential element DM is rho DxDyDz. So, if you denote our body force by symbol B, the body force per unit mass, then body force in X direction. Let us call that as DFBX, this will be simply DM*Bx or rho BxDxDy. Next, let us find out the surface forces in X direction, let us call this as DFSX. So, let us see, let us find out all the components. First, let us write the components due to the normal stress tau xx in X direction.

So, on positive X phase, we have got tau xx+del tau xx/del XDx*by area DyDz. The force acting on the negative X phase due to this normal component, i.e., negative X direction and this given by –tau XxDyDz.

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Surface force in x-direction

$$dF_{sx} = \left[\left(\cancel{\tau_{xx}} + \frac{\partial \tau_{xx}}{\partial x} dx\right) dy\,dz - \cancel{\tau_{xx}}\, dy\,dz\right] + \left[-\cancel{\tau_{yx}}\, dx\,dz + \left\{\cancel{\tau_{yx}} + \frac{\partial \tau_{yx}}{\partial y} dy\right\} dx\,dz\right] + \left[-\cancel{\tau_{zx}}\, dx\,dy + \left(\cancel{\tau_{zx}} + \frac{\partial \tau_{zx}}{\partial z} dz\right) dx\,dy\right]$$

$$\Rightarrow \quad dF_{sx} = \left[\frac{\partial \tau_{xx}}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z}\right] dx\,dy\,dz$$

Thus resultant force in x-direction:

$$dF_x = \left[\rho b_x + \frac{\partial \tau_{xx}}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z}\right] dx\,dy\,dz$$

Newton's second law: $\vec{F} = m\vec{a}$

$$dm\, \underline{a_x} \equiv \rho\, dx\,dy\,dz \underline{\frac{Du}{Dt}} = dF_x$$

$$\Rightarrow \quad \boxed{\rho \frac{Du}{Dt} = \rho b_x + \frac{\partial \tau_{xx}}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z}}$$

Non-conservative form of momentum eqn.

Next, the force due to the sheer component, first once on the Y plane. So, on negative Y plane we have got –tau YxDxDz plus the component which is acting on the positive Y plane, i.e., tau YX+del tau YX/del YDy*DxDz.

The last term would be the contribution coming from sheer stress on Z planes. So, on the negative Z plane, we have got –tau ZxDxDy and on positive Z plane, this stress component multiplied by area and we get the force. So, tau ZX+del tau ZX/del Z DzDxDy. So, we can clearly see the first on this component gets cancelled by this, tau YX gets cancelled. Similarly, here tau ZX gets cancelled with tau Z.

We get a very simplified form for this DFSX and this is given by del tau XX/del X+del tau YX/del Y+del tau ZX/del Z DxDyDz. So, finding the resultant force in X direction is now very simple. So, thus resultant force in X direction DFX which is sum of surface and body forces that will become rho BX+del tau XX/del X+del tau YX/del Y+del tau ZX/del Z DxDyDz. Now, we can invoke Newton's second law and we have got one form of Newton's second law which says F=MA.

See, if we apply this particular form and apply it to a small differential element, so what do we get DM*AX which is given in to rho DxDyDz*D of U/DT. D/DT is metal derivative of X component which gives us the acceleration in X direction. This will be equal to D of FX which you have derived previously. If we substitute these two equations and simplify, we get rho*DU/DT=rho BX+del tau XX/del X+del tau YX/del Y+del tau ZX/del Z.

So, this is the momentum equation derived in X direction or for X velocity component. We can repeat the same exercise and we can obtain these differential equations for two other velocity components as well and you can easily identify what we have derived just now. This represents non-conservative form of momentum equation. So, to derive the conservative form of momentum equation, we have already found out what are the forces acting on this differential element.

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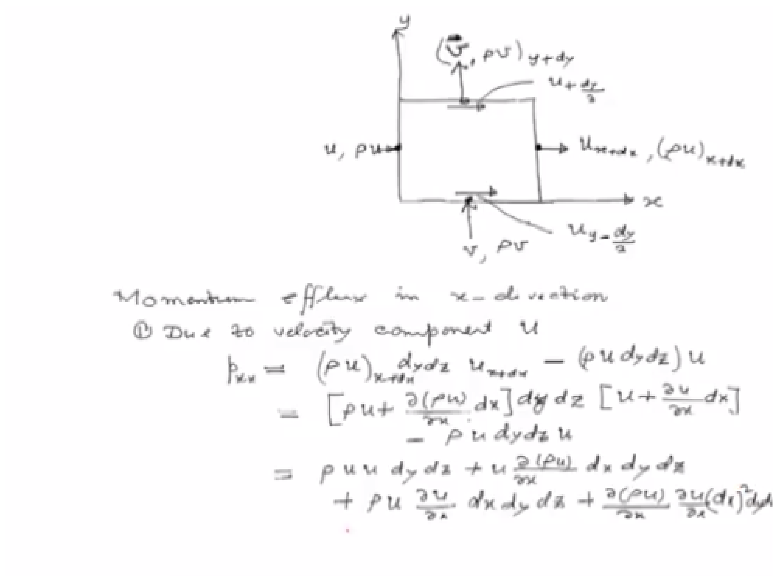

Now let us find out the rate of momentum efflux and once again we would focus only on momentum exchanges in X direction and to simplify the visual representation, let us first take a two-dimensional element or rather let us have a look along XY plane and let us see the momentum efflux or influx which is coming from different areas. At the centroid of the net X plane, we have the velocity component given by U and rate of mass that will be given by rho*U.

On the positive X phase, here this U has to be obtained from Taylor series expansion, UX+delta X. Similarly, this rho U, this has to be obtained again in terms of Taylor series expansion. Same thing about the Y velocity components. So, V component rho V which will represent influx with this velocity. On the positive Y plane, we will have the fluid going out with the velocity V. Now, these quantities must be evaluated at Y+DY.

Now if you look at these Y planes, the contribution of these velocities in X direction momentum efflux would not come directly from this but in a speed what is the momentum transfer along X direction that would be given by finding out what would be the X velocity components at these locations. So, with reference to the centroidal planes at negative Y plane, we have X velocity which should be evaluated at Y-DY/2 and the top phase, it has to be evaluated at +DY/2.

Okay, now let us find out this momentum exchanges arising from each velocity component. So, momentum efflux in X direction, number one due to velocity component U which is in X direction, let us call this as PXX. So, what is the momentum going out, it is rho U at X+DX*by the area DyDz, that gives us the mass flow rate multiplied by the velocity here which is evaluated at XDx-the incoming momentum which would be given by rho UDyDz*U.

Expand the first term on the right hand side, so you will have rho U+del of rho U/del X Dx*DyDz*U+del U/del X Dx-rho UyDz*U. Let us carry out the multiplications and find out which terms we have to retain. So, first multiplying two terms on the first side, so you get rho UUDyDz+U times del/del X of rho U DxDyDz+rho U times del U/del X DxDyZ+del of rho U/del X, del U/del X, DX square DYDZ-rho UU DxDyDz.

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① Due to velocity component u

$$p_{xx} = (\rho u)_{x+dx}\, dy\, dz\, u_{x+dx} - (\rho u\, dy\, dz)\, u$$

$$= \left[\rho u + \frac{\partial(\rho u)}{\partial x} dx\right] dy\, dz \left[u + \frac{\partial u}{\partial x} dx\right] - \rho u\, dy\, dz\, u$$

$$= \rho u u\, \cancel{dy\, dz} + u \frac{\partial(\rho u)}{\partial x} dx\, dy\, dz + \rho u \frac{\partial u}{\partial x} dx\, dy\, dz + \frac{\partial(\rho u)}{\partial x} \frac{\partial u}{\partial x} (dx)^2 dy\, dz - \rho u u\, \cancel{dy\, dz}$$

$$= \left[\underbrace{u \frac{\partial(\rho u)}{\partial x} + \rho u \frac{\partial u}{\partial x}} + \frac{\partial(\rho u)}{\partial x} \frac{\partial u}{\partial x} dx\right] dx\, dy\, dz$$

$$= \left[\frac{\partial(\rho u u)}{\partial x} + \underbrace{\frac{\partial(\rho u)}{\partial x} \frac{\partial u}{\partial x} dx}\right] dx\, dy\, dz$$

$$\approx \frac{\partial(\rho u u)}{\partial x} dx\, dy\, dz$$

② Due to velocity component v

$$p_{yx} \approx \frac{\partial(\rho u v)}{\partial y} dx\, dy\, dz$$

So, last term gets cancelled with the first term and let us take out DxDyDz outside. So, we are left with U times del of rho U/del X+rho U times del U/del X+del of rho U del X del U/del X*Dx and this whole thing gets multiplied by the differential volume DxDyDz. These two terms can be combined as a single derivative by using chain rule, so this can be written as del of rho UU/del X+del of rho U/del X del U/del X DxDyDz.

Now, let us have a look at the second term in the bracket, i.e., this particular term. It is being multiplied by the differential area element Dx which is finitely small quantity. So, compared to the first term, this can be easily neglected. So, we can approximate Pxx as del of rho UU/del X DxDyDz. We can repeat the similar exercise and we can find out that momentum efflux in X direction due to velocity component V, we call this as Pyx. This can be obtained as del of rho UV/del Y DxDyDz and similarly we can obtained the contribution of Z velocity component.

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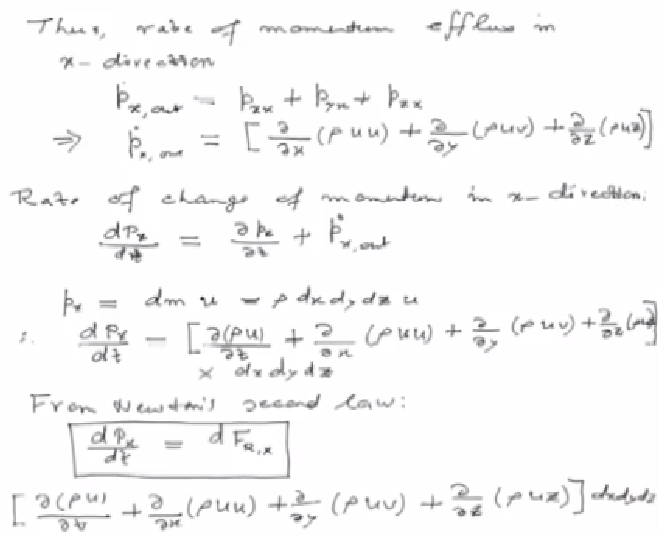

Not let us combine these together. So, this rate of momentum efflux in X direction, let us call it small Px out. This is sum of three terms which you have done earlier, Pxx+Pyx+Pzx. P dot X out, this can be written as del X of rho UU plus partial derivative with respect to YF rho UV plus partial derivative with respect to Z of rho Uz. So, now what is the rate of momentum change in X direction.

For this differential element, rate of change of momentum in X direction DPx/DT. We can invoke this Reynolds transport theorem. So, this will be given by del of PX/del T+P dot X out. Now, what is Px. Px is nothing but DM*U which is rho DxDyDz. So, therefore DPx by DT is del of rho U/del D+del X of rho UU, del Y of rho UV+del of del Z of rho UZ*DxDyDz. Now, from Newton's second law, this DPx/DT, this would be equal to the resultant force in X direction which we have denoted by DFRX which we have derived earlier.

Let us substitute the expressions of both of them. So, on our left hand side, we get del of rho U/del T+del of del X of rho UU+del of del Y of rho UV+del of del Z of rho UZ*DxDyDz.

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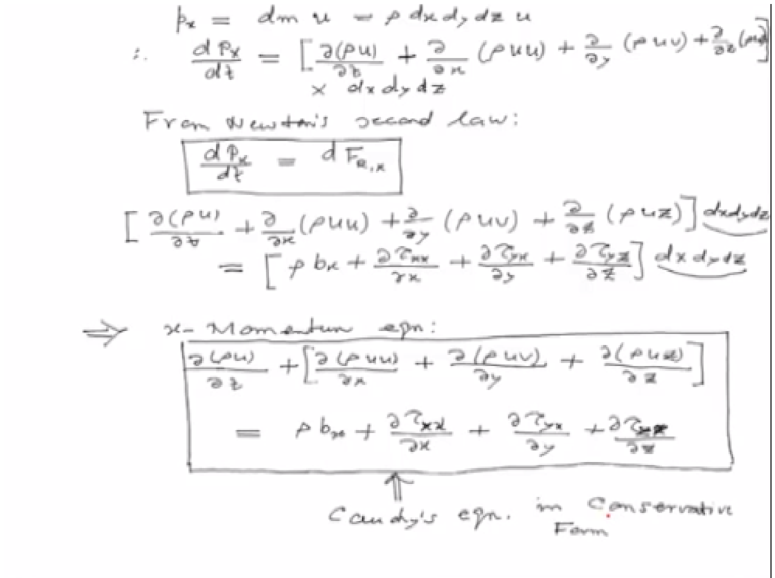

This is equal to our force term rho BX+del tau XX over del X+del tau YX/del Y+del tau YZ/del Z DxDyDz. We can divide by this volume of differential element DxDyDz and thereby we get X momentum equation, del rho U/del T+del of rho UU/del X+del of rho UV/del Y+del of rho Uz/del Z=rho BX+del tau xx/del X+del tau yx/del Y+del tau zx/del Z.

This is precisely the expanded form of Cauchy equation in X direction. So, this is Cauchy's equation in conservative form, okay. So, we have now derived only one component or equation for only one momentum component. The similar equations can be derived for the momentum equations in other two directions. The whole purpose of doing this exercise was that how we can obtain the desired governing equations starting from the first principles.

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$$\left[\frac{\partial(\rho u)}{\partial t} + \frac{\partial}{\partial x}(\rho uu) + \frac{\partial}{\partial y}(\rho uv) + \frac{\partial}{\partial z}(\rho uz)\right] \underbrace{dxdydz}$$
$$= \left[\rho b_x + \frac{\partial \tau_{xx}}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z}\right] \underbrace{dx\,dy\,dz}$$

$\Rightarrow$ x- Momentum eqn:

$$\boxed{\frac{\partial(\rho u)}{\partial t} + \left[\frac{\partial(\rho uu)}{\partial x} + \frac{\partial(\rho uv)}{\partial y} + \frac{\partial(\rho uz)}{\partial z}\right] = \rho b_x + \frac{\partial \tau_{xx}}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z}}$$

$\Uparrow$

Cauchy's eqn. in Conservative Form

Exercise: Derive the momentum eqn. for a 2-D case using a polar differential element (i.e. using r-θ coordinates).

As an exercise, you can try and derive momentum equation in planar polar coordinates or let us say cylindrical polar coordinates. So, derive the momentum equation for a 2D case using a polar differential element, i.e., using R theta coordinates. So, once we have done this exercise, this would give us the confidence that for a specific application, this would be able to derive the governing equations ourselves which is the first step in our numerical simulation.

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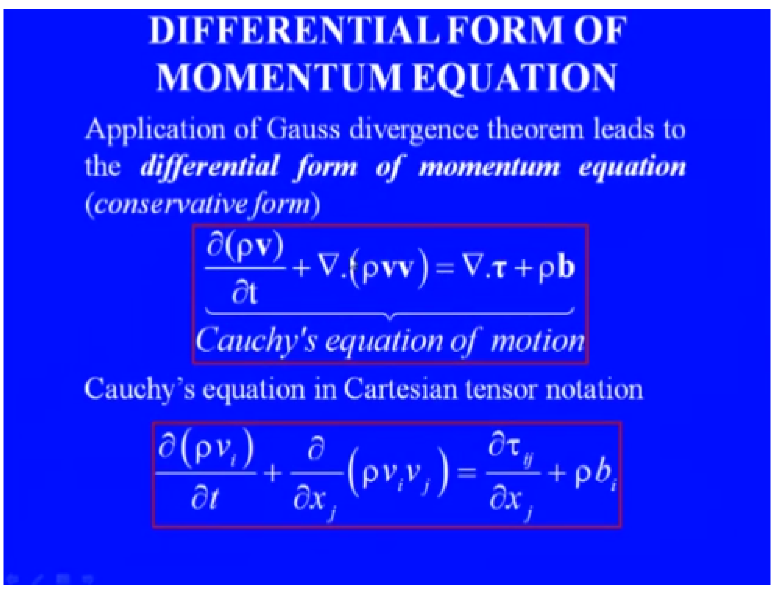

Now, let us get back to the formal form of these equations. So, what we had on the left inside, the first term in momentum equation that is what we call temporal derivative which was in terms of the density rho and velocity components, velocity vector V. The second component what we call the connector term which again involves rho and V. On the right hand side, we got two terms, the first one is divergence of tau where tau is the second (()) (40:18) and the last term is in terms of rho*B, B is our body force.

In the given problem, we have to obtain what would the density be. We would like to find out what are the velocity components at each point in the flow domain. B would normally be specified but how about tau. If we combine Cauchy equation with our continuity equation, continuity equation gives us equation for transport of density, so that is one equation and the Cauchy equation of motion represent a set of three equations.

So, we have got four equations in terms of how many tensor; density is one, three components of velocity that makes it four and we have got nine components of this stress tensor. So, this system equation is still not closed.

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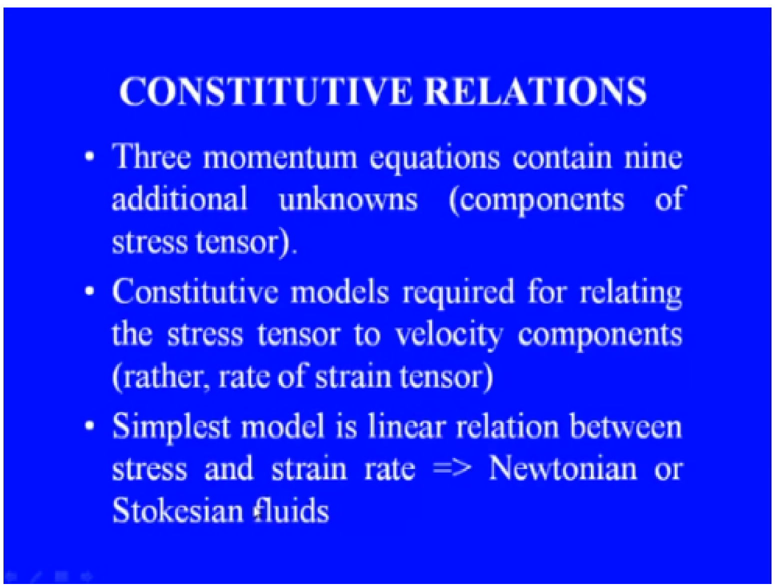

We have got more number of unknowns than the number of equations and that brings us to the topic of what we call constitutive relations, i.e., we have got nine additional unknowns which are components of stress tensor in our momentum equation. Now, these nine unknowns they must be modeled or they must be expressed in terms of a primary unknowns which are velocity pressure and density.

So, we need to use what we call constitute models for relating the stress tensor to the velocity components. In fact, we will try to relate the stress tensor to the rate of the strain tensor. So, relationship between stress tensor and rate of strain tensor is what we would call a constitutive relation. The simplest model which was proposed long time ago by Newton is a linear relation between the stress and strain rate and this model is valid for what we call Newtonian or Stokesian fluids.

Why do we call these fluids as Stokesian that would become clearer when we look at the formal derivation for our Navier-Stokes equations.

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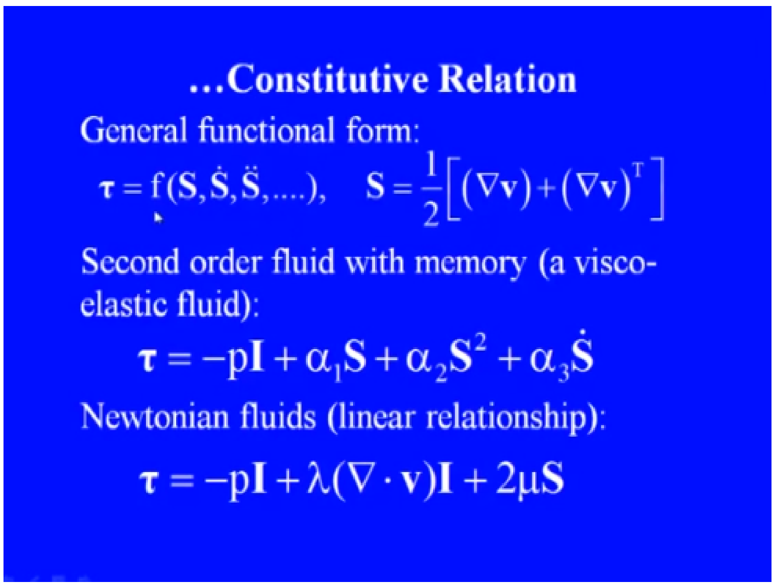

Now, let us look at a general constitutive relation. Now, when we want to relate the stress tensor to the strain rate tensor, we can write a general functional form tau is equal to a function of the strain rate tensor S. The first derivative of S the second derivative with respect to time of S and so on where the strain rate tensor S is given as 1/2 gradient of V+ gradient of V transpose. Now, you might wonder why you have just taken the strain rate tensor and why not rotational component.

Normally, what we would think that in a moving fluid, the stress arises due to velocity gradients, but if you look carefully the rigid body rotations which were represented by the rotation tensor, they do not contribute to the stress generation and thus why the stress is normally related only to the strain rate tensor. To give a simple example of this particular functional form for a second order fluid with memory which is typically referred to as viscoelastic fluid.

Tau could be given as –P*I where this I represents Kronecker delta or identity matrix which is a second order tensor with nine components +alpha 1 times S+alpha 2 times S square +alpha 3 times S dot. Now, here these P, alpha 1, alpha 2, and alpha 3, these will depend on the thermodynamic state of the fluid. For Newtonian fluid where we have a simple linear relationship between stress and the rate of strain, we get a fairly simplified relationship.

How do we obtain this relation is tau=-Pi+lambda times divergence of V*I+ twice mu times S, we will find it out. How do we say that if we assume a linear variation, we would get this particular component and the term if you look carefully we have got tau=-P*I, from where this P crops up, what is justification of having this scalar term in a momentum equation. So, now let us look at that justification.

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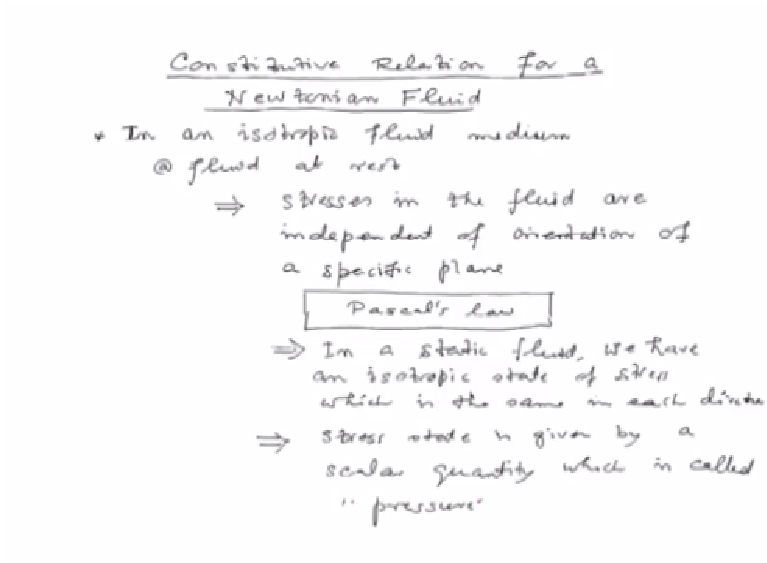

This is constitutive relation for a Newtonian fluid. What do we mean by Newtonian fluid here. We would focus only on the fluids for which stress and strain rate relationship is linear. You will not consider the so-called non-Newtonian fluids for which the stress-strain relationship is non-linear. Now, if we are dealing with this isotropic fluid, in an isotropic fluid medium whatever constitutive relation we come up with that should satisfy few constraints.

So, let us consider two specific cases. The first case is fluid at rest. When the fluid is at rest, what do we say in this case that stress is in the fluid or independent of direction of orientation of a specific plane. In fact, this is nothing but what we call as Pascal's law which says that in a static fluid we have an isotropic state of stress which is the same or it can be reworded what we say. We have got a single scalar component or a scalar quantity which describes the state of stress in a static fluid, i.e., stress state is given by a scalar quantity which is called pressure.

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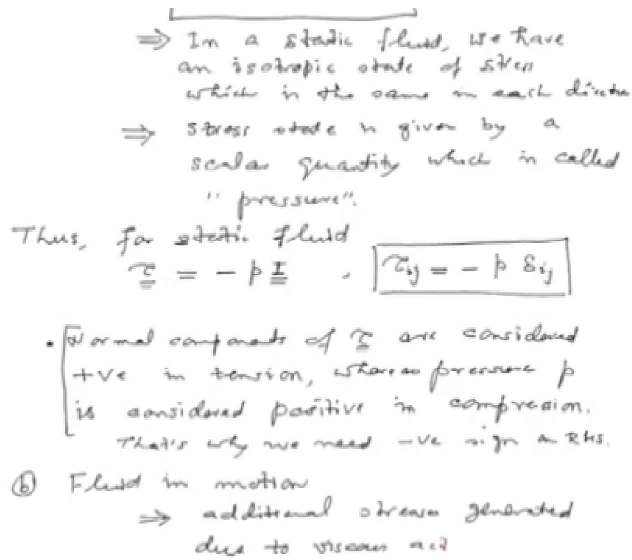

So, if that were the case for static fluid, we must represent the stress tensor as -P times I where I is our Kronecker delta or identity matrix or in component forms, we can write tau IJ=-P times delta IJ. Now, the presence of negative sign before P that is due to the convention which we follow for P and tau.

The normal components of tau are considered positive intention whereas pressure P is considered positive in compression. So, that is the reason why we need a negative sign on RHS. So, we have seen the state of stress how we are going to represent for a static fluid. The remaining part at what happens for fluid in motion. Here we will have additional stressors, stressors generated due to viscous action, okay.

So, we are going to stop here in this lecture and what could be the additional forms or additional terms which we need to add to obtain the stress tensor for removing fluid is what we will look at in the next lecture.