Computational Fluid Dynamic

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Lecture – 05
Momentum Equation: Newton's 2nd Law

Welcome back to next lecture that is lecture 3 on Mathematical Modeling. We have already had allocated conservational laws, mathematical preliminaries and one governing equation that which continuity equation. In this lecture, we would focus on momentum equation and we would derive it from first principle starting from the Newton's second law of motion. Let us have a recap what we have covered in the previous lecture.

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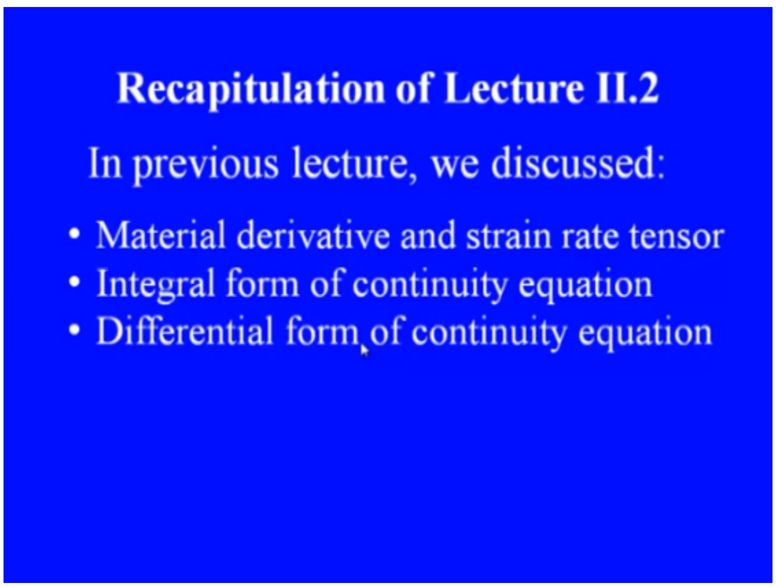

We have defined the concept of material derivative and strain rate tensor, then we obtained integral form of continuity equation using Reynolds's transport theorem. Then we used Gauss's divergence theorem to derive differential forms of continuity equation and it is various simplified forms. We also used a simple Cartesian differential element to derive the differential form of continuity equation.

We would use a similar approach in the derivation of momentum equation starting from Newton's second law of motions. So this is what is the crux of the main topic for third lecture that is momentum equation based on Newton's second law of motion.

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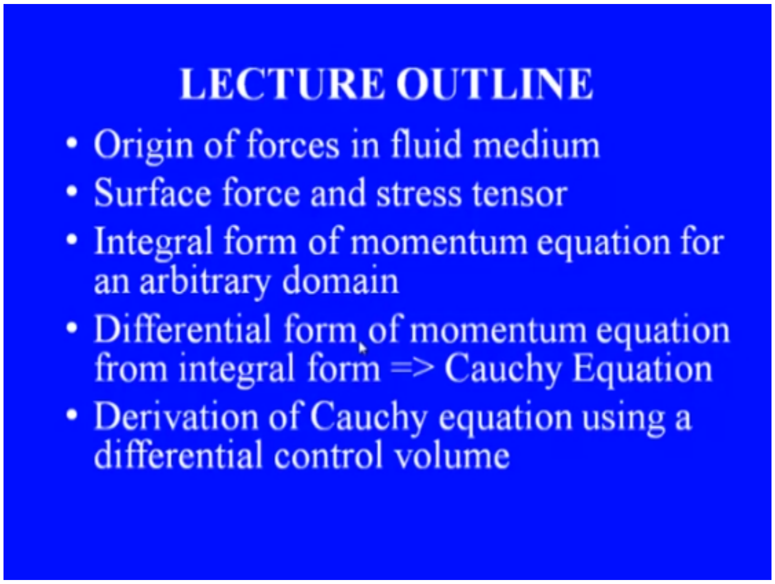

The outline of the lecture will first start with the origin forces in a fluid medium for the different types of forces which we would encounter how do we obtain their mathematical expressions and whatever concepts and notations which are involved in description of forces that is what we would have a brief look at. Specifically, we would see how surface forces and the stress tensor are related and then we would obtain integral form of momentum equation for an arbitrary domain starting from Newton's second law of motion.

And in the end, we would obtain differential form of momentum equation from integral form by invoking Gauss divergence theorem which gives us what we call Cauchy equation of motion. We would also try and derive the Cauchy equation using a differential control volume and then we will have look at few simplified forms specifically the Euler's equation of motion for inviscid fluid.

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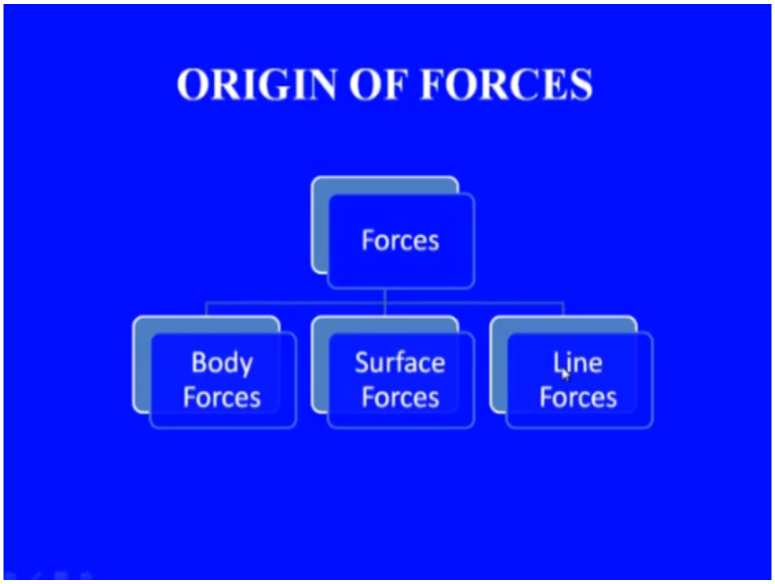

Now let us have a look at the different forces which we will have in a fluid medium, a fluid medium might be acted upon by what we call body forces which are essentially noncontact forces which act at a distance. The next category could be surface forces which are primarily imposed by these contacting surfaces and we might also come across what we call line forces which result at the lines of contact with fluid with certain interfaces a different medium which might be solid or a fluid. Let us have a look at bit more detail about each of these.

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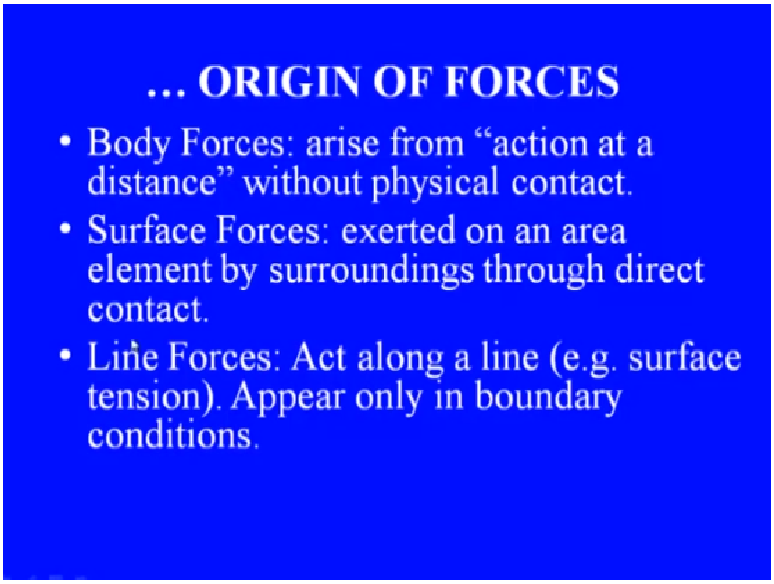

Body forces first, these arise from what we call action at a distance without any physical contact. There is no physical contact between the agency which causes the body force and the fluid medium. Then surface forces, they are exerted on an area element of the fluid by surroundings through direct contact. Now surroundings could be part of another fluid medium, the same fluid medium or a solid surface.

And the line forces, they are primarily involved when we have an interface for example surface tension, now these forces only appear in boundary conditions at the interface of two fluids or at the interface of the solid or liquid. So we will not have any further look at line forces in this lecture, we will first concentrate on the remaining to the first two body forces and surface forces.

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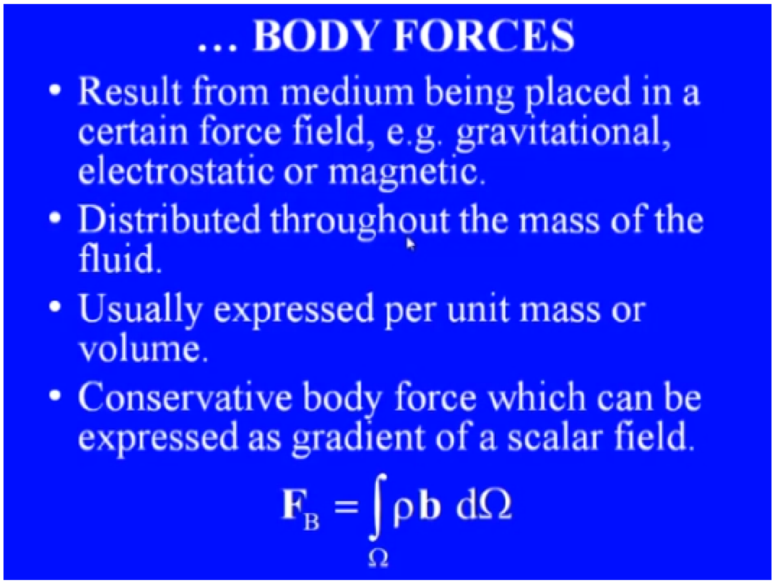

Now let us see body forces. These result from the medium being placed in a certain force field, for example gravitational force field, electrostatic magnetic or electrometric force field. In most of field problems, it is gravitational force field is the one which would be involved and these forces are not confined to specific surfaces or specific areas of the fluid, in fact, they act throughout the mass of the fluid.

That is the main or major feature or major characteristic of body forces and this why we usually expressed them as per unit mass or per unit volume basis. Now let us find out the body forces acting on a fluid medium.

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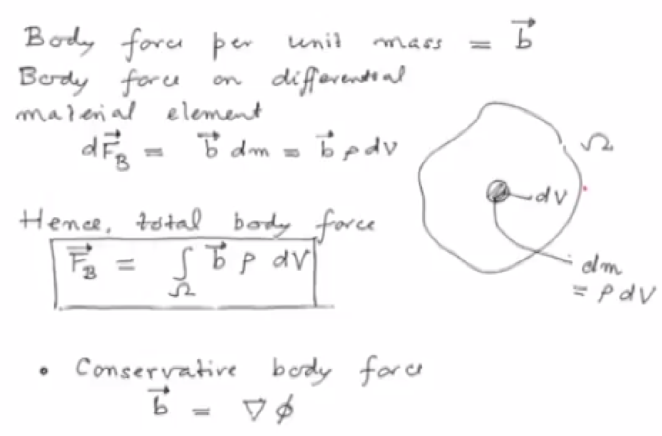

So suppose this body forces per unit mass is represented by symbol small b, this is the vector quantity. Now let us take a very small fluid element of volume dV. Now mass of this differential element dM, this would be the density rho times a differential volume dV. So, the body forces on differential material element, let us call it dF subscript B, this is equal to b times dm or b times rho dV. To obtain the total body force acting on the fluid medium content in the volume omega.

We simply need to integrate it over the entire volume, so a total body force F of B. This is volume integral over domain omega of b rho dV. So in this way, we would be able to obtain the total body force acting on a fluid body if we know the body force per unit mass. Now this body force could be due to gravitational force field, electrostatic force field or magnetostatic force field. In many instances, it is possible for us to express this body force in terms of the gradient of the scalar quantity.

Since such a situation, we call it conservative body force wherein body force per unit mass b, this could b expressible in terms of the gradient of a scalar function phi. For instance, in the case of gravity or phi could be simply the scalar quantity Z. So that completes our description of the body forces that FB can be given as volume integral of rho b D omega. Next let us have a look at the surface forces.

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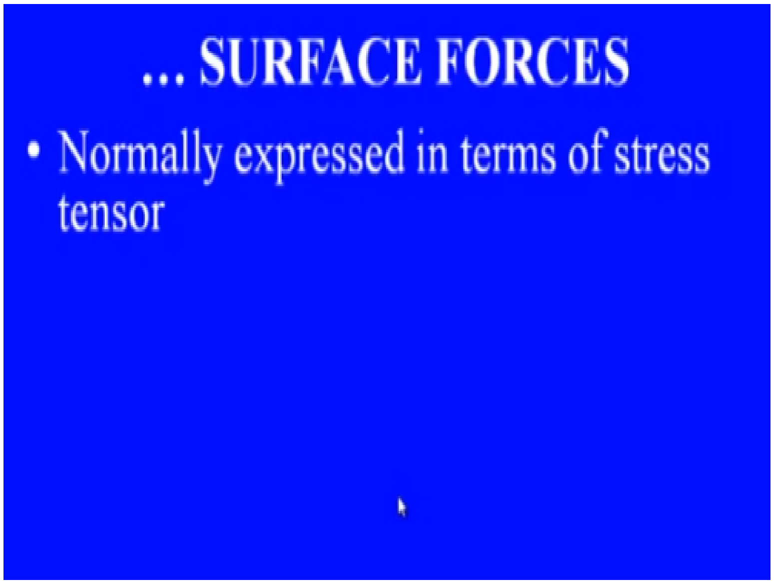

Now the surface forces normally expressed in terms of stress tensor. So how do we obtain or how to express the surface force.

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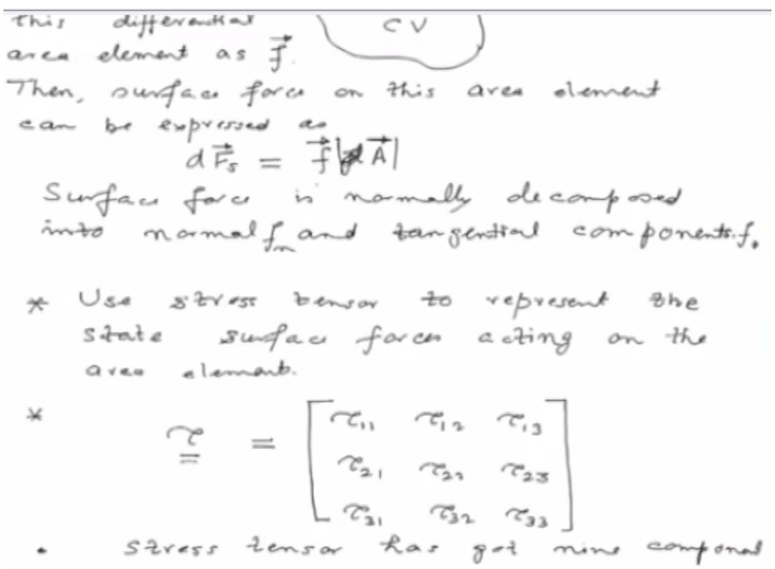

Let us take a finite control volume and let us consider a small differential area element, identified by the vector aA. Due to X and F surroundings, there is a force which will act on this small unit area and let us call, let us denote force per unit area acting on this differential area element as F. So then surface force on this area element can be expressed as the dFs=F times dFA.

Now this surface force can be broken into two components, normally decomposed into normal and tangential companies. For a general case, instead of using these only two components, normal component f of n and tangential component f of t, what we instead do is we use stress tensor to represent the state of surface forces acting on the area element.

Now this stress tensor tau in 3-D cardigan components is given by a 3x3 matrix and we can represent it as tau=tau 11, tau 12, tau 13, tau 21, tau 22, tau 23, tau 31, tau 32, tau 33. So this stress tensor has got nine components.

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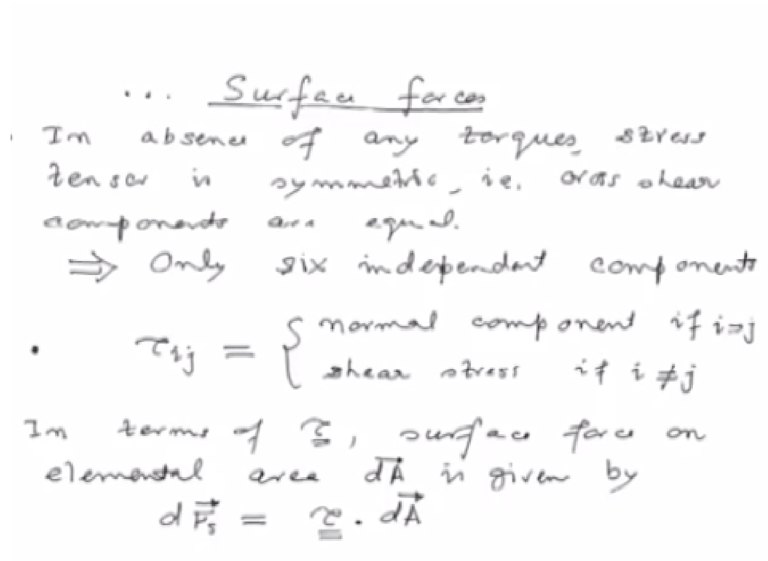

Now if we had a fluid medium in which no torques are present, so in absence of any torques, stress tensor is symmetric, that is, a cross shear components are equal and this tells us that we have in the case only six independent components. Now these components let us say tau ij, this is called a normal component, if i=j that is all diagonal elements of this stress tensor, there are normal components of stress and shear stress if i not equal to j.

Now, using stress tensor it is very easy for us to represent the surface force acting on the elemental area. So in terms of tau, surface force on elemental area dA is given by dFs. So dot product of the stress tensor with the area vector. So if you want to find out the total surface force on the fluid body, total surface force on fluid body F subscript s. All that we need to do is integrate this differential component of force over the entire surface area that is Fs=area integral of tau.dA.

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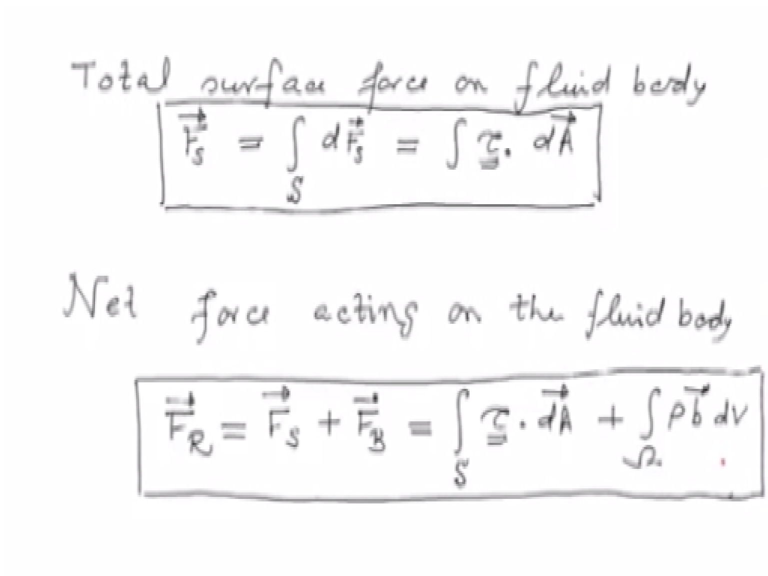

Now we can combine the two force components which we have discussed so for, the surface forces and the body forces and we can write that net force acting on the fluid body FR=FS, the surface forces+body forces or in terms of the integral quantities this is =tau.dA over the surface of the fluid body+rho b.dV of the total volume of the fluid body. So now we have obtained the resultant force, expression for the resultant force acting on a fluid body.

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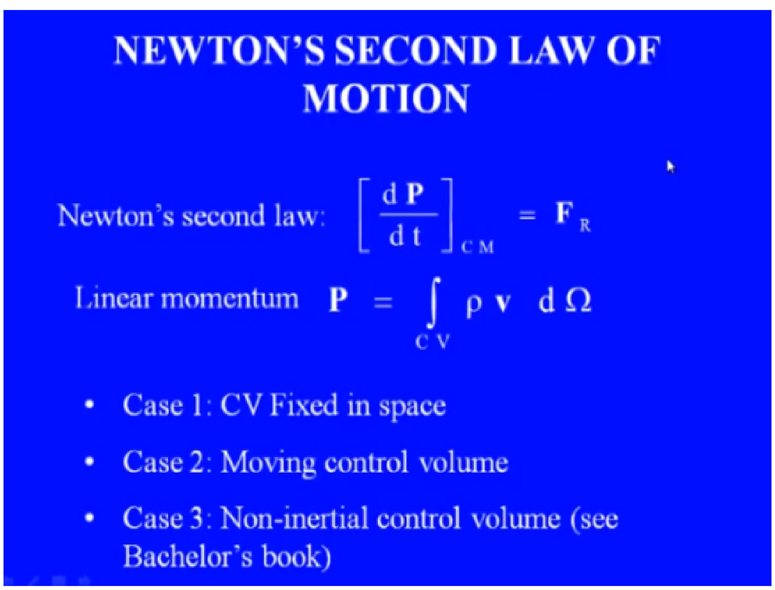

Next, now, let us now move onto the Newton's second law of motion and the Newton's second law was initially stated for a particle. A p article of mass dm and this law said that if we had a particle of mass dm and if it is acted by a force that resultant force is essentially given by the rate of change of momentum of the particle.

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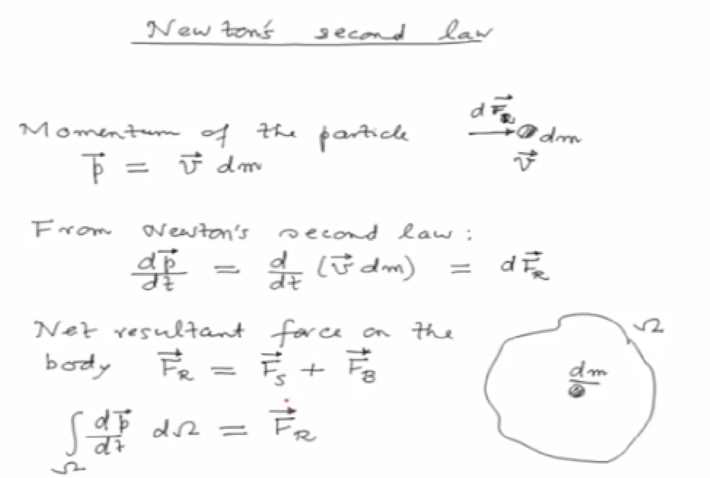

So suppose you have got a particle of mass dm, it is easily moving with the velocity V and there is a force which is acting on this particle, let us call that force is dF. So momentum of the particle, if you call it a p, this will be given by V and to dm. So from Newton's second law, the rate of change of momentum that is d/dt of momentum p = d/dt of Vdm. This should be equal to the resultant force acting on this particle.

Now if in case of a continuous media, we will not have a particle in fact we have got a collection of particles which make up our finite fluid body occupying a finite region of space called omega. So now in this case how do we implement Newton's second law? Newton's second law can be extended in a straightforward fashion, by finding out what is the total or net resultant force acting on this particular body.

So net resultant force on the body which we have already calculate earlier, let us call it as FR, this could be found out in terms of the surface forces on the body, fluid medium and the body forces which act on a fluid body. Now what would be the total momentum or rather rate of change of the momentum of the entire body?

This body could be thought of as if it is composed of many particles having a mass dm and so we can integrate if you want to find of the rate of change of momentum of the entire body, we can integrate the momentum of the particle. So what we have got is d/dt integrated over the entire body. Now this should be equal to the resultant force which acts on this fluid body. Now, we are dealing with a fixed material region.

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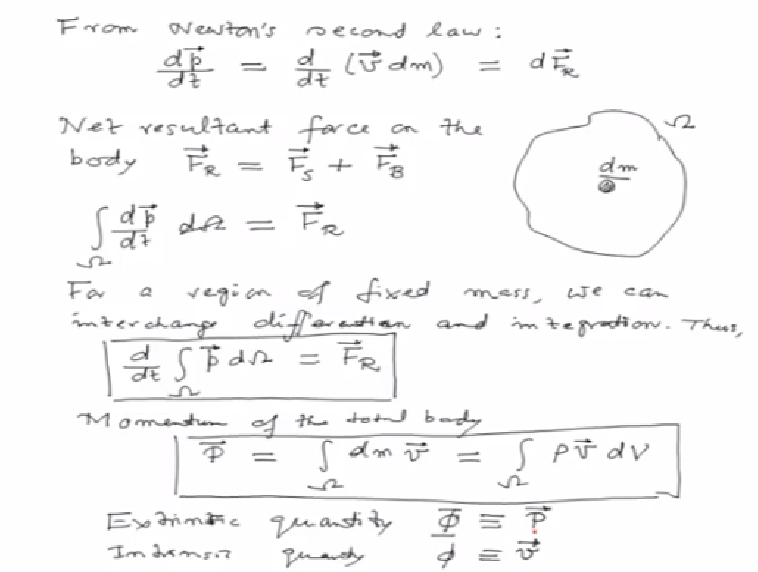

So for a region of fixed mass, we can interchange differentiation and integration, so then what we get is simply this, that we got d/dt of integral pd omega. Now from here we can identify one extrinsic quantity which he can call momentum of the total body. So this P would be given by integral over the entire volume, dm times V or integral or omega rho VdV.

Now if you look at this expression we can easily identify that extrinsic property or extrinsic quantity, capital PHI, which we using Reynolds transport theorem. This can be identified with the total momentum of the body and intrinsic quantity, small phi, this could be identified with the velocity field. Now we have used these two identifications to find out the momentum equation for the fluid medium.

To summaries what we have discussed just now is that Newton's second law we have got d, dp/dt for the control mass system = to FR and here capital P is the linear momentum of the fluid body which can be defined as rho V d omega. Now there are different possible cases, different ways in which we can choose our control volume depending on problem.

The first case would be CV fixed in space and the second case could be a moving control volume and third case could be we have got a non-inertial control volume that is control volume which is moving with some velocity, accelerating and rotating at the same time. Now the third case, we are not going to discuss it here, interested readers might refer to the Bachelor's book on fluid dynamics. We will just concentrate on the fixed CV in space primarily.

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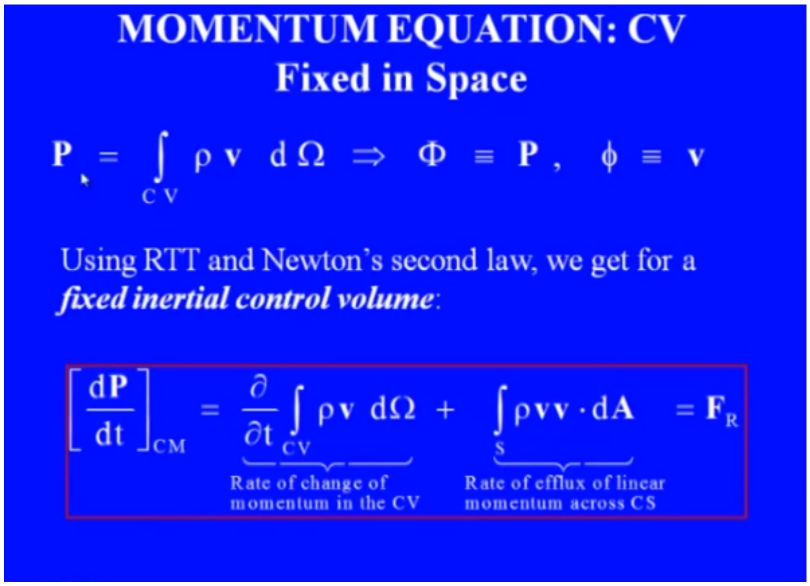

$$\mathbf{P} = \int_{CV} \rho \mathbf{v}\, d\Omega \Rightarrow \Phi \equiv \mathbf{P}, \quad \phi \equiv \mathbf{v}$$

$$\left[\frac{d\mathbf{P}}{dt}\right]_{CM} = \frac{\partial}{\partial t}\int_{CV} \rho \mathbf{v}\, d\Omega + \int_{S} \rho \mathbf{v}\mathbf{v}\cdot d\mathbf{A} = \mathbf{F}_R$$

So if CV is fixed in the space we have already seen bold is seen that P is, that is our linear momentum of the system is defined by this relation, P=rho v d omega and in size that extrinsic properties momentum, P itself and corresponding intrinsic quality is v. Now we can use Reynolds transport theorem to obtain what is dp/dt, dp/dt for the control mass system would be the local time derivative that is del t of rho v d omega integrated over the control volume+rho vv.dA.

So this is the expression for the time rate of change of momentum for a control mass system which occupies our control volume instantaneously. So this is particular del/del t term, it represents the rate of change of the momentum in control volume and this term rho vv.dA it represents the rate of the efflux of linear momentum across the control surfaces. And by Newton's second law, combination of these two terms should be equal to the resultant force which acts on our fluid body.

Now we have come across a very strange term here in the surface integral rho vv.dA, now, what is this vv? We had a brief look at it earlier when we had discussed the tensor products. Now let us have a recap what we had discussed, what we mean by this vv, there is no dot or cross product here between these two vectors, so what we mean by a simple outer product of velocity vector with itself.

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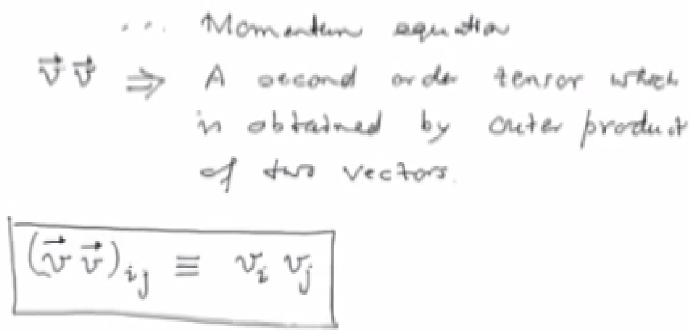

So this particular term which we have got in momentum equation, this actually represents a second order tensor which is obtained by outer product of two vectors. So if you want to compute a particular component of this particular tensor, let us vv, is ij-th component, in fact this is nothing but the i-th component of vector v multiplied by the j-th component of vector v.

So that is how we obtain the components of this second order tensor and the product of the second order tensor, dot product with the area element that will give us a vector, this is what we need in this particular equation which is a vector equation for rate of change of momentum. So if we look closely, actually the single equation represents the three equations in 3-dimensional space.

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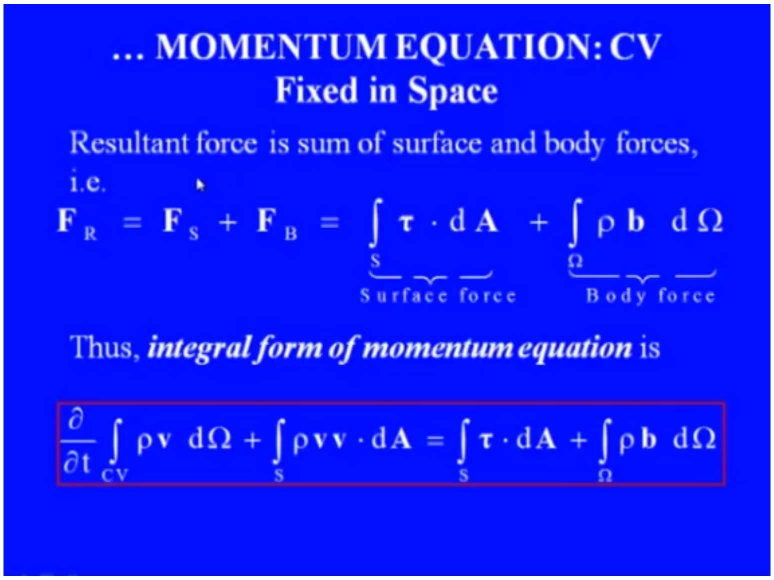

Now as we have already seen that the resultant force which act on the fluid body is sum of surface and body forces, so we can say FR=FS+FB or we can represent in terms of stress tensor that FS is tau.dA, the surface integral over S and FB is the volume integral of rho b d omega. Now let us substitute this expanded form for the resultant force in our previous equation and then we obtain the integral form momentum equation given by del rho/del t integral over CV of rho v d omega+ surface integral of rho vv.dA = to tau.dA+rho b d omega.

So this integral form is very useful if you want to obtain a finite volume approximation for solving a flow problem and it is also very commonly used in integral analysis or overall analysis of complicated fluid system, in the analysis of a turbine system.

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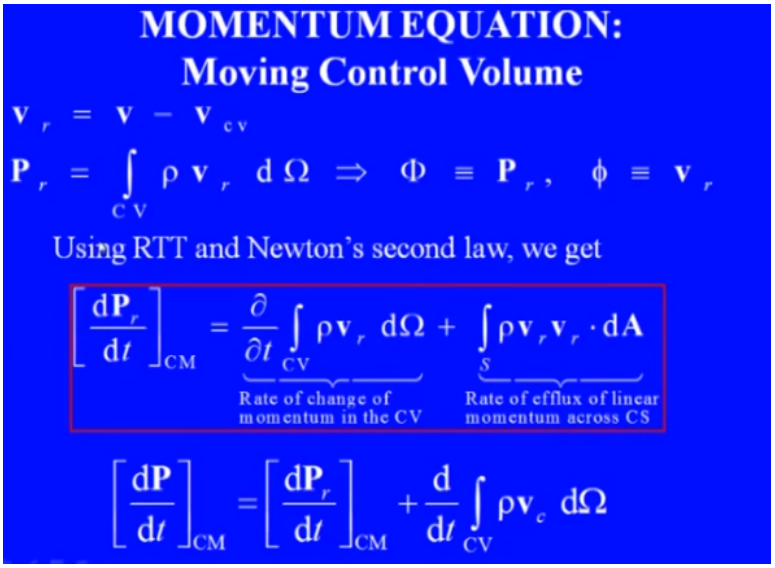

Now in case of instead of a fixed one if we had a moving control volume which is moving with a fixed velocity or with a velocity Vcv so we can defined a relative velocity Vr=V – Vcv and we can define relative momentum of this control volume Rr = rho Vr d omega was CV and in this case our extrinsic property would not be P but instead Pr and extensive quantify would be Vr and then we can use RTT using RTT we get dPr by dr that is time derivative of this relative momentum for a control moss system is equal to del/del t of rho vr d omega integrated over the control volume which gives us the rate of change of momentum in the control volume.

Plus the surface integral of rho Vr, Vr . dA which gives us the rate of the efflux of the momentum across the control surface. Now this rate of change of the relative momentum Pr for the control moss system it is related simply by this expression, dP/Dt of CM = dPr/dt+d/dt of rho Vc d omega. So we now have got these two equations, we know what is the expression for dP/dt using Newton's second law, we can replace this by the resultant force and thereby we can easily obtain an integral equation for momentum for the case, where they have chosen a moving control volume.

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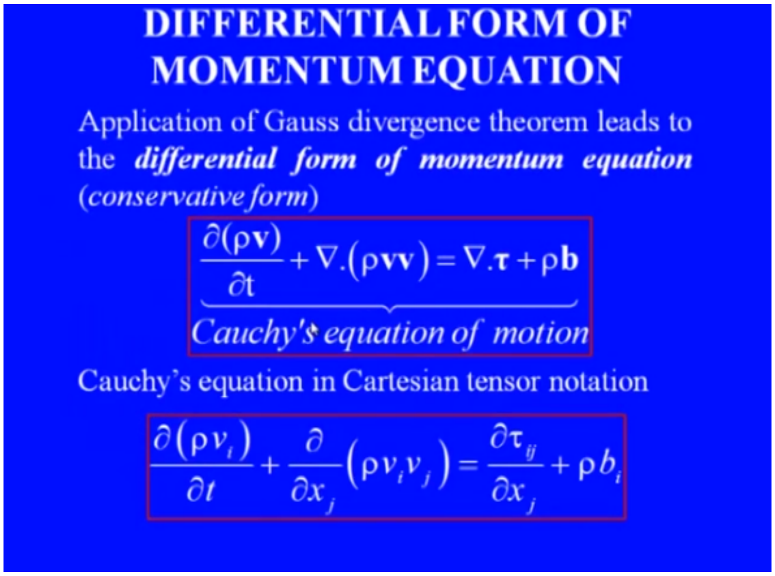

Now suppose we want to derive a differential form, so we would follow the same approach which we have used earlier in obtaining the differential form of continuity equation starting from the integral momentum equation. So now let us have a look at integral form of momentum equation.

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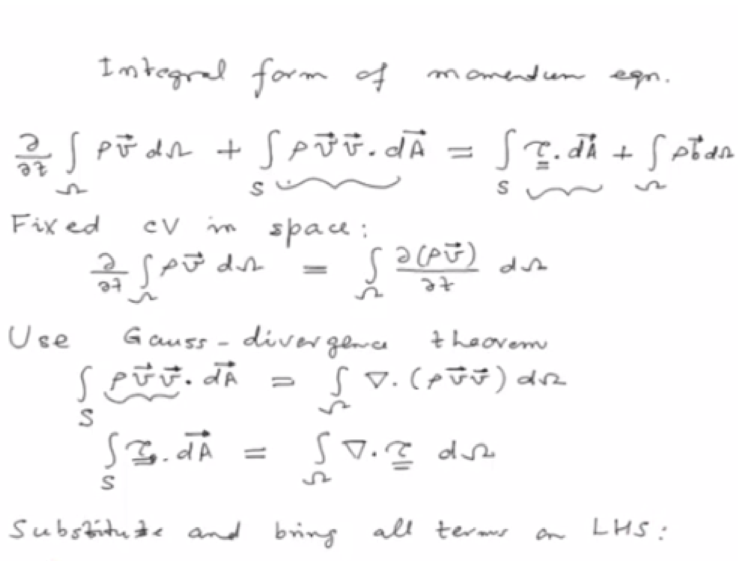

Integral form of momentum equation, on the left hand side we had the first term was a time derivative del/del t of rho v d omega, plus we had a surface integral rho vv.dA. On the right hand side, we had the first one was a surface integral in terms of the stress which represents a surface sources tau.dA + the last term was a volume integral in terms of the body force per unit moss rho b d omega.

Now let us focus on the first case that is, we have got a fixed CV in space. Then, we can interchange the temporal derivative with integration, so del/del of rho v d omega. This can be written as the volume integral of del of rho V/del t d omega. We would like to transform the two surface integrals which we get here using, Gauss divergence theorem.

So let us use the Gauss divergence theorem to both of these terms separately and the application is very simple rho vv . dA. So surface integral of rho vv . dA = the volume integral of what divergence of whatever we get to the left for dot operator. So rho vv d omega and similarly, the surface integral of the stress tensor multiply by area element, this is tau . dA, that becomes a volume integral of divergences of tau d omega.

So now we have converted all the terms in our integral momentum equation, in terms of volume integrals. So let us substitute and bring all the terms on one side, all terms on LHS. So that gives us a big volume integral, the first term becomes del of rho v/del t+the second term is divergences of rho vv-divergences of tau-rho times b d omega=0.

Now once again we have got an integral which vanishes in our arbitrarily chosen control volume omega.

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$$\int_{\Omega} \left[\frac{\partial (\rho \vec{V})}{\partial t} + \nabla \cdot (\rho \vec{V}\vec{V}) - \nabla \cdot \underline{\underline{\tau}} - \rho \vec{b} \right] d\Omega = 0$$

Above integral would vanish if and only if its integrand is zero everywhere. ⇒

$$\boxed{\frac{\partial (\rho \vec{V})}{\partial t} + \nabla \cdot (\rho \vec{V}\vec{V}) - \nabla \cdot \underline{\underline{\tau}} - \rho \vec{b} = 0}$$

So, we can argue similarly that look maybe then some parts with positive, some other parts is it is negative which makes the whole integral=0. Then, we realize that look our choice was arbitrary, so we can choose a smaller control volume in the positive part. Once again the same equation must hold good, so that leads us to the same conclusion that look, the above integral would vanish, if and only if it is integrant is 0 everywhere.

And this implies that del of rho v over del t++divergences of rho vv--divergence of tau-rho times b, this should be=0. Now, we can just redistribute the terms and obtain the differential form of momentum equation, okay, so that is what we get in a summary that application of Gauss divergence theorem leads to a differential form of momentum equation and we call it as a conservative form.

There is a specific reason why we call it conservative form, means del rho v/del t+divergences of rho VV=to divergence of tau+rho times b. So on the left hand side, we have got the quantities which are linked to the material and the velocity field. On the right side, we have got the quantities which result from the external effects that is applied forces. Now this particular equation is popularly known as Cauchy's equation of motion.

And the way we have derived, this was derived from arbitrary fluid medium. It does not matter whether a fluid is Newtonian, non-Newtonian, compressible, incompressible, it does not matter, this particular equation is applicable for fluid of any type. Now, we can also write this equation in expanded forms, in specified coordinate system. Let us have a look at forms in Cartesian tensor notation. So, it is compact form in tensor notation would be del/del t of rho Vi++del/del xj of rho vi, vj=del tau ij/del xj+rho bi.

So this particular equation gives us the momentum equation for i(th) component of velocity. So, please remember this equation of motion is a vector equation, so it essentially represents a set of three equations.

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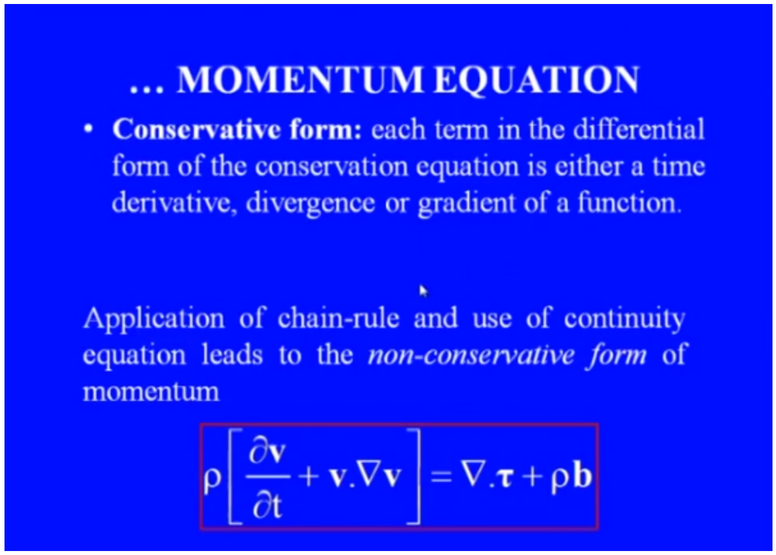

Now we should look at this previous form, is called conservative form, why we had called it conservative? As if you look at each term carefully the first term is a time derivative, del rho v/del t, the second term is a divergence, divergence of rho vv. The first term in RHS is the second divergences of the stress tensor+rho times b, b would normally be in the case of gravity, it would be expressible in terms of gradient of some scalar function.

So all the terms in this particular equation, they are either a time derivative, divergences or gradient of a function, such a form is termed as conservative form of the equation. We can also obtain what is called as non-conservative form of the equation. That is very easy, we can use the chain rule of differentiation to expand the first term this del rho v/del t, expand in terms of two terms and similarly expand the second term also divergence of rho vv and then apply your continuity equation and we should be able to get a non conservative form of momentum equation. Now let us have a look at how to we obtain this non-conservative form.

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Non-conservative Form

$$\frac{\partial(\rho v_i)}{\partial t} = \frac{\partial \rho}{\partial t} v_i + \rho \frac{\partial v_i}{\partial t}$$

$$\frac{\partial}{\partial x_j}(\rho v_i v_j) = v_i \frac{\partial}{\partial x_j}(\rho v_j) + \rho v_j \frac{\partial v_i}{\partial x_j}$$

Thus,

$$\frac{\partial(\rho v_i)}{\partial t} + \frac{\partial(\rho v_i v_j)}{\partial x_j} = v_i \underbrace{\left[\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x_j}(\rho v_j)\right]}_{0} + \rho\left[\frac{\partial v_i}{\partial t} + v_j \frac{\partial v_i}{\partial x_j}\right]$$

Continuity eqn.:

$$\boxed{\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x_j}(\rho v_j) = 0}$$

So let us just start with the conservative form of equation and first term del rho vi/del t. This can be straightway broken into two parts, del rho/del t times vi++rho times del vi/del t. So we want to obtain the non-conservative form. Now let us have a look at the second term, del/del xj of rho vi vj. So we can couple the two terms together, so this could be written as vi times del over del xj of rho vj+rho VJ times del/del xj of vi.

So thus some of these two terms del of rho vi/del t+del/del xj rho vi vj. This can be expressed as, let us first take vi common, so we get del rho/del t by looking at the first terms of the expansion, del rho/del t+del/del xj of rho vj. So as we have gathered the first two terms on the RHS of expansions+rho times del of vi/del t+vj times del of vi/del xj. Now let us have a look at the first-term on RHS. What does continuity equation say, continuity equation in Cartesian tensor notation was given by del rho/del t+del/del xj of rho vj=0.

So, hence the first term the RHS vanishes and what we get that del of rho vi/del t+del/del j rho vi vj=rho times del vi/del t+vj del vi/del xj. We can also write it in tensor notation, so dyadic form.

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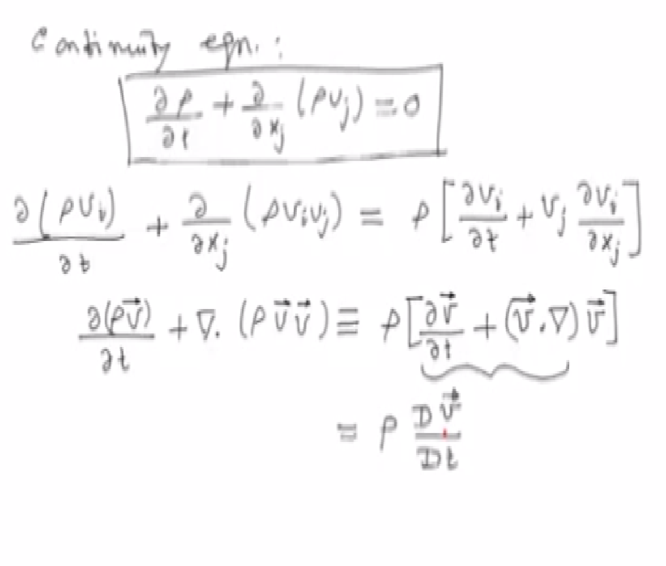

So dyadic form in the left hand side, we had del of rho v/del t+divergences of rho vv, this is same as rho times del v/del t+v . del v. Now this right hand side bracketed term that can be identified as the material derivative of velocity field, so we can also write this as rho Dv/Dt. We do not have to do anything with the right hand side that remains as such in conservative form or non-conservative form.

So that is the summary how do we obtain the non-conservative form of momentum equation rho del v/del t+v . del v=to divergences of tau+rho b. This conservative form is useful in some of the numerical schemes, though most of the time specifically infinite volume formulations of CFT, we prefer conservative form of momentum equation.

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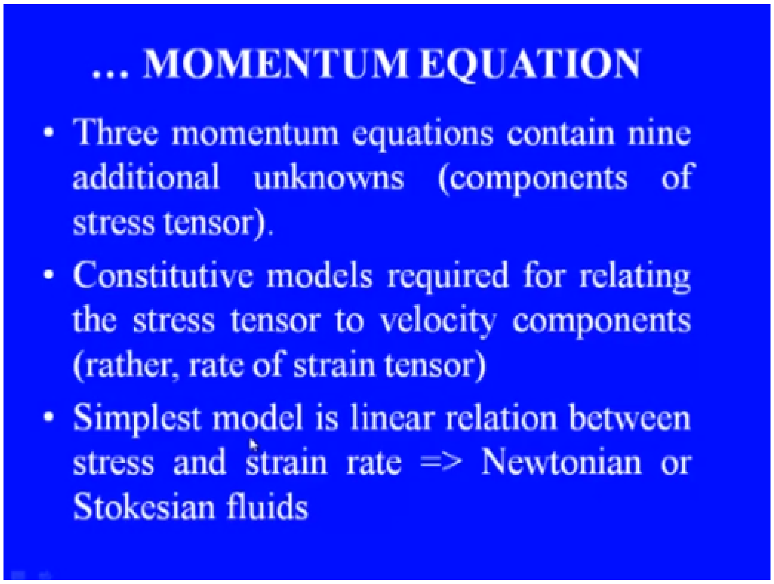

Now let us have a careful look at the right hand side momentum equation. Left hand side, we had the density rho and three-velocity components but what we have on the right hand side? We have got the stress tensor, body force field that is known to us, so b is absolutely no problem, b is known to us but as far as stress tensor tau is concerned, it has got at least six unknown components or nine unknown components if we do not account for the symmetry.

We see there are nine additional unknowns that is the components of stress tensor and they must be somehow related to the velocity field and for that we need what we call constitutive models which are required for relating the stress tensors to the velocity components rather we would relate the stress tensor to the rate of strain tensor.

And this aspect, we would look at in next lecture when we derive (()) (48:37) structure equation and that equation is obtained by using the simplest model which is a linear relationship between stress and strain rate which is applicable to what we call Newtonian or Stokesian fluids. Why these two terms are used that we will have a look at when we come to the next lecture.