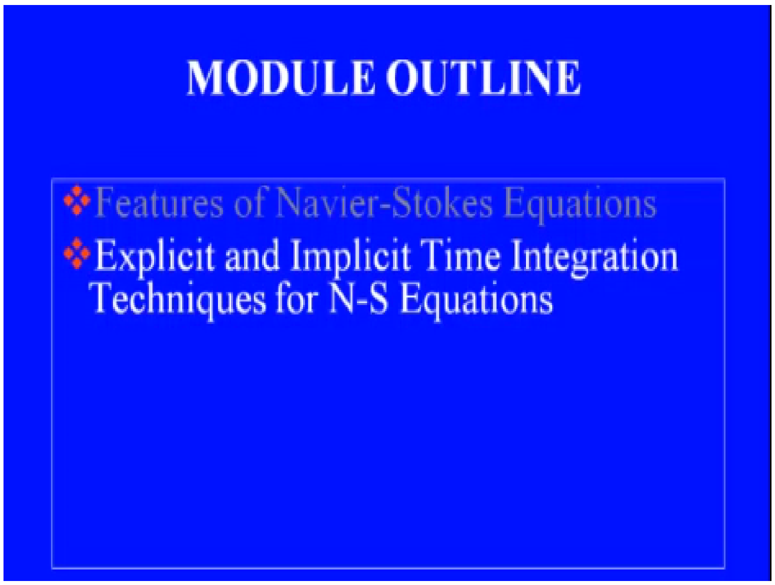
Computational Fluid Dynamics
Dr. Krishna M. Singh
Department of Mechanical and Industrial Engineering
Indian Institute of Technology – Roorkee

Lecture - 36
Time Integration Techniques for Navier-Stokes Equations

Welcome to the second lecture in module 8 on Numerical Solution of Navier-Stokes Equations.

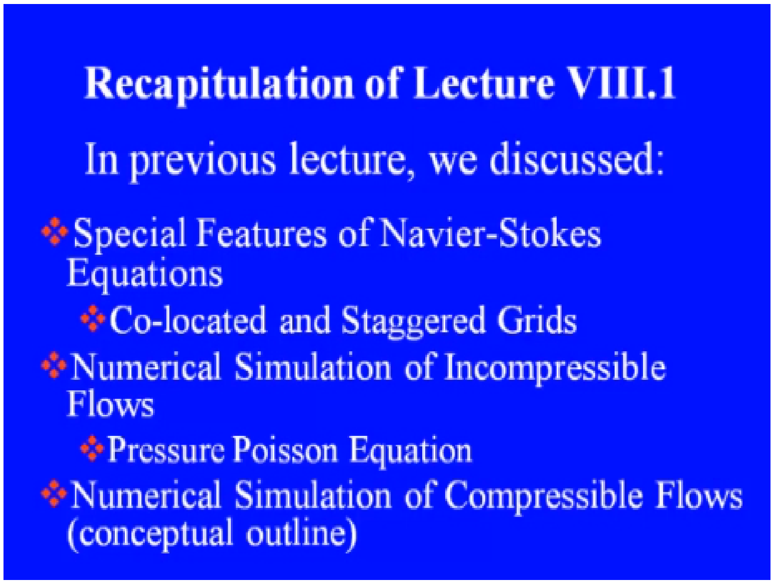
(Refer Slide Time: 00:32)



In the previous lecture we had discussed the features of Navier-Stokes equations and we also discussed some important points which we need to keep in mind in steady state compressible as well as incompressible Navier-Stokes equations. We will next discussion application of explicit and implicit time integration schemes for Navier-Stokes equation. And then we would look at one familiar methods which are called implicit pressure correction methods which are widely used for solution of steady state problems.

And we will also have a brief look at familiar method called Fractional Step methods. Let us have a recap of what we discussed in the previous lecture.

(Refer Slide Time: 01:17)



We discussed the special features of Navier-Stokes Equations then, Vector equations, it has got special properties we have got flexibility with respect to the choice of the grid and (()) (01:27) that is what we discussed at previous lecture. We also discussed the mix nature and due to that would or the implications that we would rarely have a, a steady state solution, we would rather try to obtain a steady state solution as an end result of and time dependent simulation.

And we briefly discussed these special things which are involved in numerical simulation of incompressible flows. In particular, that this no independent equation for pressure so we must find out or we must derive a Poisson for pressure which would ensure the satisfaction of continuity.

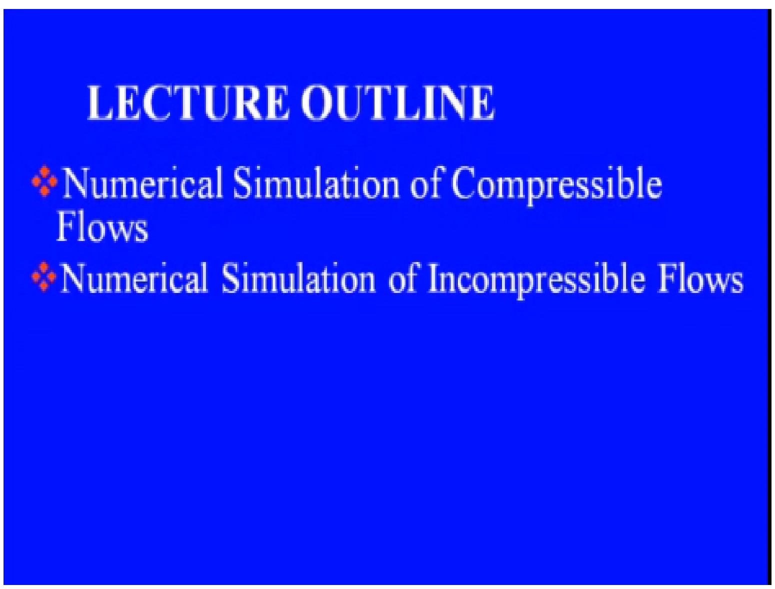
We also had a brief look at numerical simulation of compressible flows and we have had brief conceptual outline as to how we can solve compressible flow problems and what we saw that conceptually this simulation of compressible flows is appears to be lot simpler because here we have got independent equation for each variable; we have got continuity equation with represent transport equation for density.

A momentum equation has transport equation for velocity components. And energy equation as transport equation for temperature or total energy and we have got equation state to supply just the relationship between pressure, density and temperature so from which we can obtain pressure. And similarly, we have got certain constitutive equations which give us the material properties in terms of the flow variables.

So since everything is available which we require for the compressible flow, all the equations available and we to just have to use a proper set of discretization techniques in special time to obtain our numerical solution. So this stoke and (()) (03:19) compressible flows where we have to contrive, we contrive to derive an equation for pressure using both momentum and continuity equations.

Now in this lecture we would focus primarily on a time integration techniques for Navier-Stokes equations. We would not discuss in detail about this special discretized in use. So we will presume our starting point would be what we call the Semi-discrete system equations wherein we have applied our favorite special discretization scheme it could be finite differences finite volume or finite element method and we have obtained a discrete algebraic in terms of time. And how do we integrate that equation that would be the feature of focus of the lecture today.

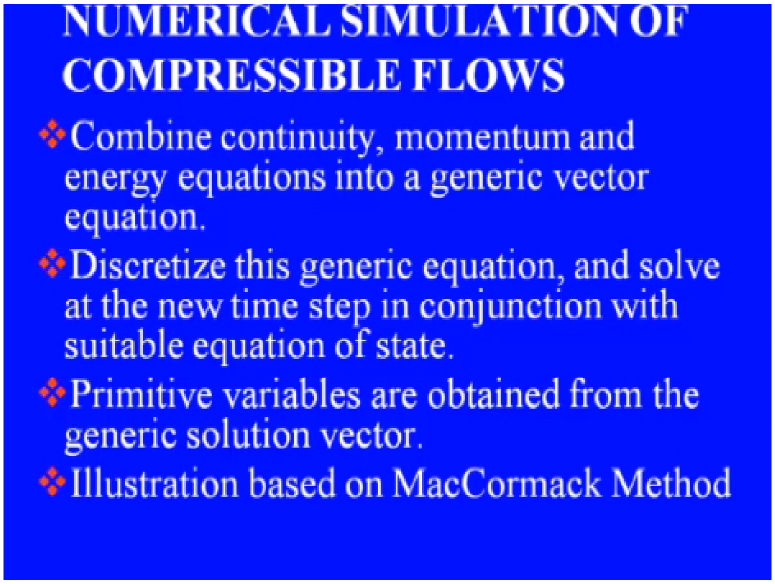
(Refer Slide Time: 04:19)



So outline, we will have brief look at Numerical Simulation of Compressible Flows. We will have a look at one particular type of explicit integration scheme for compressible flows. And since it is a introductory course we would not going to any further detail about the compressible flows and we would refer to appropriate references wherein you can find detailed algorithms for dealing with low speed as well as high speed compressible flows.

We will focus mainly on the numerical simulation of incompressible flows and we will have a look at, Explicit Time Integration procedure then an Implicit Time Integration procedure and Fractional Step methods. So well, let us first have a look at numerical simulation of compressible flows.

(Refer Slide Time: 05:14)

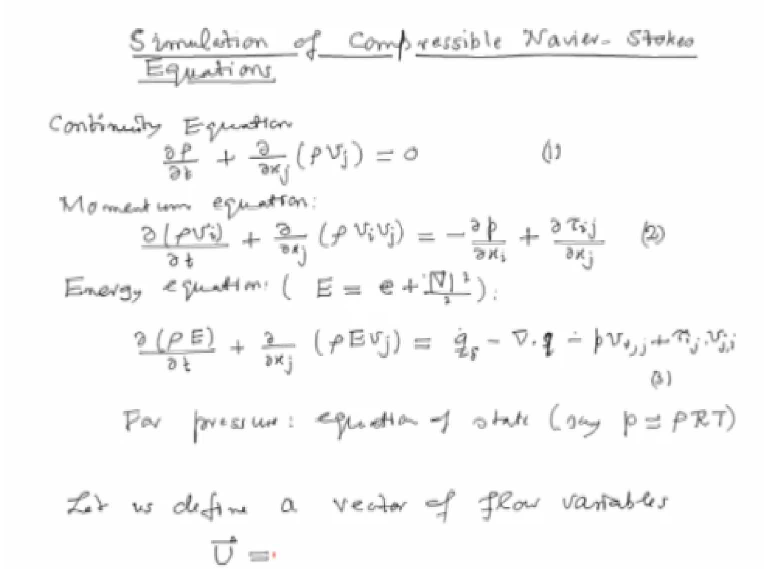


We had a look at the conceptual procedure in the last lecture, that is to say what we have to (()) (05:11) continuity momentum the energy equation they represent, transport equations of density, free velocity components and the total energy. So can we combine this into a generic vector transport equation, so this what we will have a brief look at. So we will combine this and next this generic vector transport equation can be discretized using our choice of special discretized scheme.

And then we can also apply a suitable discretized scheme in time to solve the flow variables making use of the suitable equation of the state to take you to the pressure in momentum equation. And we would see that primitive variables which are velocity components density and temperature they are obtained from our generic solution vector in this equation. And we will have one demonstration of this solution process based on an explicit scheme which is popularly refer to as MacCormack method which is essentially a predictor corrected method.

So now let us switch over to about and have a look at the Simulation of Compressible Flow.

(Refer Slide Time: 06:37)



We would focus only on Navier-Stokes equations, they are very simplified cases in compressible flow, for instance for very high speed flows, the affect of its (()) (07:13) negligible except close to the boundaries in that case we might solve for Euler's equation or if you are too far off and flow is ir-rotational we can as well solve for the potential equation. So we are not going to deal with those equations.

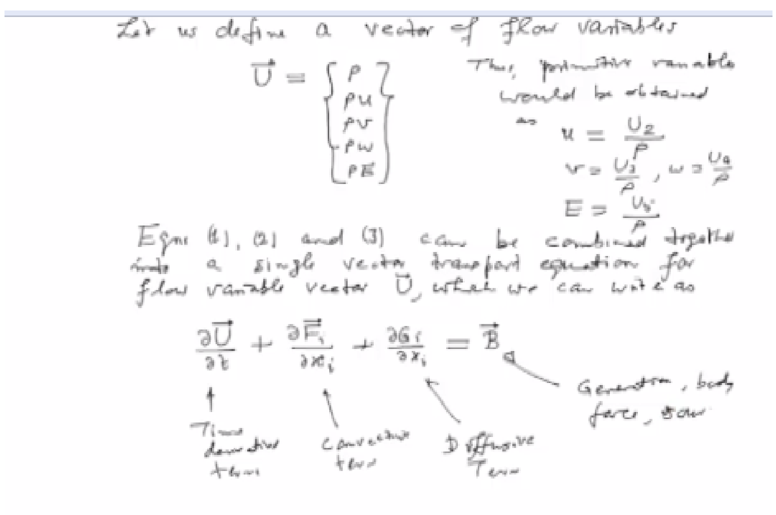
We will focus primarily on the most general case wherein the viscous affect already involved because of the present of boundaries and this is what we would focus on our Navier-Stokes equations. So now let us first write down equations the way we have learnt them that we have derived them earlier. So first is Continuity Equation, let us use Cartesian Tensor notation. So this is del rho/del b+del over xj of rho vj = 0.

A momentum equation, del of rho Vi over del t+del over del xj of rho Vi Vj-del p over del xi+del tau i,j over del xj, and suppose we ignore this source term or body force term which could be combined with the pressure if you want to or let us neglect for the time being. Similarly, we can try to Energy equation, wherein we would write the equation for total energy that is E= a internal energy + the kinetic energy = our heat generation term, moisture diffusion term pVi, j, tau ij vj, the last 2 terms they come because of what we call flow work Q is our heat (()) (10:19). Now these 3 equations are in terms of what we call one variable is our density the next variable is the 3 components velocity so 4+a total energy 5, and we have got pressure 6.

So we have to supply a separate equation of pressure, so for pressure we would have equation of a state say B=Euler equation of state in some situation so p = rho RT and T could be obtained from our total energy E, if you know capital E velocity and density which should be able to calculate p and give that p in the solution of momentum equation and so on. Now let us define a new solution vector. So let us define a vector of flow variables.

Now this would be—we are going let us use symbol capital U for this. This will contain the primary unknown and each of these equations.

(Refer Slide Time: 12:14)



So primary unknowns and are let us say density for continuity equation, momentum equation we have got less use Cartesian component so rho U, rho V and rho W as similarly the last equation we have got rho capital E. Please be careful here, we have not used U, V, W and E we have instead used the combination of rho U, rho V, rho W and rho E.

And getting the primitives variables U, V and E would be easier once we get the components, so the solutions at primitive variables would be obtained as like U that would be the second component vector U so let us called as U2/rho, the rho is U1 V=YU3/rho and W=U4/rho and capital E would be U5/Rho. So its capital U is a vector, it is got 5 components. And in terms of E we can collect all these equations together.

So equations 1, 2 and 3 can be combined together into a single vector transport equation for the flow variable vector capital U. We can write as del of U over del t+del of Fi over del xi+del of Gi over del xi= a vector b. One of this f and j they would the del u/del t that contains the collection of our time, derivatives terms f will contain the convective terms, so this is time derivatives terms, convective terms and diffusive terms.

This will contain all generation terms, generation or body force, source terms. For actual numerical implementation what we can do is instead of writing in generic form let us use Cartesian coordinate system and in Cartesian coordinate system we can correct or rearrange the terms that will definitely--

(Refer Slide Time: 16:09)

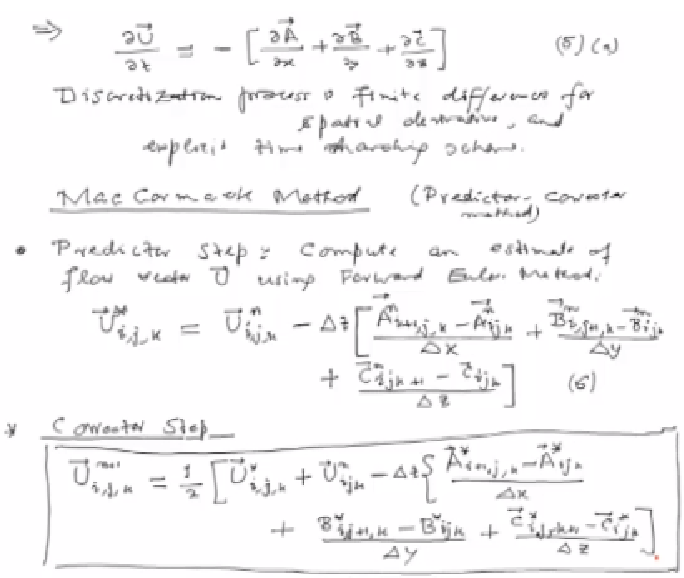
So for numerical implementation in Cartesian Coordinates, let us rearrange. We will keep the derivatives of x, y and z separately, the above equation in terms of derivatives of x, y, z and t and rewrite it has del of U/del t+del A over del x+del B over del y+del C over del Z=0. So let us call this equation as a equation number 5.

Now here we can again rewrite in terms of ODE to enable us to apply the time integration scheme, so we can put it as del u/del t-del A over del x+del B over del y. Now we can use a suitable time integration scheme and a discretization procedure to solve this equation, okay. So in the context of finite differences, so let us choose our discretization procedure. Let us call it a finite difference for spatial derivatives and explicit time marching scheme.

So you can use one such thing and one such a scheme is what we call MacCormack method. Now this MacCormack method it is a predictor corrected method. So what do you do-- in first step of this method the predictor step we would use this solution which is available at time T=t and that is the known time to obtain a predictor value for the variable, or our solution vector capital U.

Use this predictor value again to update, let us say our material properties as a terms and thereafter thereby get a final solution which as an average of the predictor solution as well the initial guess or initial solution at time t=t. So here, the first step is our predictor step. Okay.

(Refer Slide Time: 21:24)

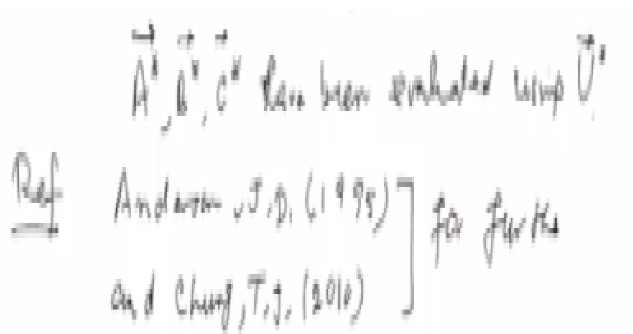


To compute an estimate of flow vector U using forward Euler method, so once we apply forward Euler method this straightway get our capital U at—U * this at the grid point i, j, k basically U at i, j, k at time level n-delta t times. Now we can—we have to use the spatial derivatives for A, B and C. So let us substitute let use differences for A, B, C.

So A become Ai+1, j, k-Ai, jk, this H vector-- H would be evaluated using this solution at time level n/delta x. Similarly, Bi, j, k+1k-Bi,j,k evaluated at time level n/delta y+Ci, j, k at an k+1-Ci,j,k/delta Z. So these are our predictor value. Now with this predictor value we would use in the corrected step. So all these A, Bs and Cs there component would be computed using this value U* and solution would be obtained as a simple average of an Euler step value plus this predicted value.

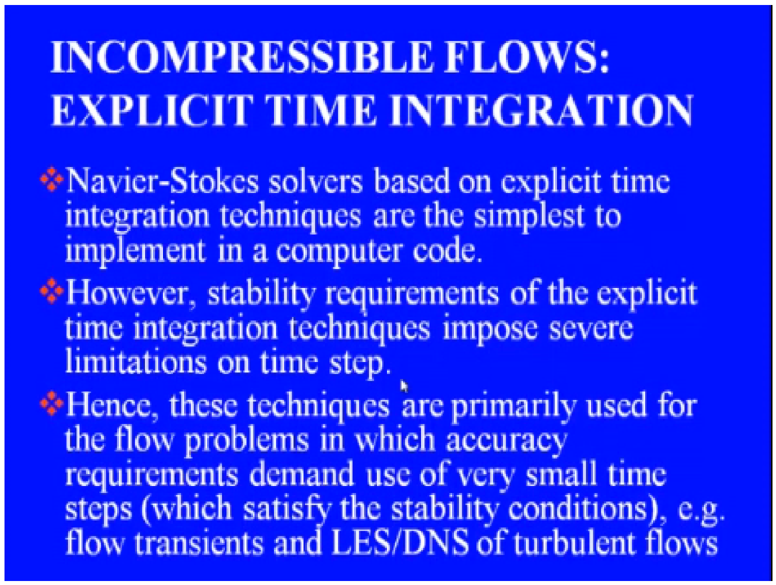
So U at n+1i, j, k this would be given as an average of this 2 so 1/2 of U* i, j, k which we computed at the previous step. And if we use this U* i,j,k for the evaluation of A, B, Cs and start off within initial condition at t=tn that is our Un so we get U at n i,j,k-delta t times A* i+1j,k, A* i,j,k/delta x B* i,j+1k-B i,j,k*/delta y+C* i,j,k+1-C i,j,k*/delta Z. So now we have obtained our solution vector at the new time step and grid point i, j, k using an explicit scheme. And remember that this A*, B* and C* they are the ones.

(Refer Slide Time: 26:11)



So this start values A* B* C* have been evaluated using U*. Now remember this MacCormack scheme is an explicit scheme so (()) (26:35) stability constant and I would refer you to the books by Anderson, J.D (1995) and Chung, T.J (2010), these 2 books the complete reference are provided at the end of the lecture for further details. So we are not going to discuss any further about a compressible flows. We will not focus on the incompressible flow time integration scheme.

(Refer Slide Time: 27:24)



Now let us have a look at numerical simulation of incompressible flows. We have already seen that is in this case we need to derive a Pressure Poisson Equation for the continuity. We can make use of an explicit time integration scheme; we can go for implicit time integration or we can use category of method feature called fractional steps methods. So these are the 3 which we are going to discuss in today's lecture.

And this (()) (27:52) category methods which is called the Implicit Correction methods which we would discuss in the next lecture. Now let us have a look at the characteristic of explicit time integration. Now if you try and implement them this Navier-Stoke solvers based on explicit time integration techniques they are simplest to implement because we can straightway obtain the values of the flow variables in terms of the values of the preceding (()) (28:21) time; that is what we saw in the case of our MacCormack method for compressible flow.

Same way here is the velocity components could be directly obtained from or momentum equations in terms of the values at the previous time step. Okay. But there is one problem here that explicit time integration techniques, they have got severe stability constraints and that puts that imposes limitations on the time step which we can choose.

So explicit time integration schemes are recommended or they are primarily used for the flow problems in which the accuracy requirements demand use of a very small time steps. So such would be the situations wherein in for instance we want the flow transient something is happened suddenly all of the sudden there is some disturbance flow field and you want to catch those flow transients.

Or if you want to perform a large simulation or direct numerical simulation turbulent flows in each of these cases we require a very accurate time history and for this accurate time history we should choose very small time steps. And this time step maybe small enough to satisfy the stability conditions, so these are the 3 prime candidates in which we should prefer explicit time integration scheme that is the problems involved in flow transients and Large Eddy Simulation and Direct Numerical Simulation for turbulent flows.

Now let us see how do we obtain or how does this explicit time integration scheme works, let us have schematic, the algorithm and then we would summarize it. So let us back to a board.

(Refer Slide Time: 30:17)

Explicit Navier-Stokes Solvers (Incompressible Flow)

Momentum eqns.

$$\frac{\partial(\rho v_i)}{\partial t} + \frac{\partial}{\partial x_j}(\rho v_i v_j) = -\frac{\partial p}{\partial x_i} + \frac{\partial \tau_{ij}}{\partial x_j} \quad (1)$$

Application of a spatial discretization technique leads to semi-discrete system given by

$$\frac{\partial(\rho v_i)}{\partial t} = \underbrace{-\frac{\delta}{\delta x_j}(\rho v_i v_j)}_{\text{convective}} - \underbrace{\frac{\delta p}{\delta x_i}}_{} + \underbrace{\frac{\delta \tau_{ij}}{\delta x_j}}_{\text{Diffusive term}} \quad (2)$$

Operator $\frac{\delta}{\delta x_i}$ represents spatial discretization operator, and it could be different for convective, diffusive and pressure term.

Let $H_i = -\frac{\delta}{\delta x_j}(\rho v_i v_j) +$

So Explicit Naiver-Stokes Solvers. And remember we are dealing with only incompressible flows. So, here most of the time you would be solving only the momentum equation not the energy equation; continuity equation would be used only (()) (30:46) constant, so our primary focus would be on the solution of momentum equations.

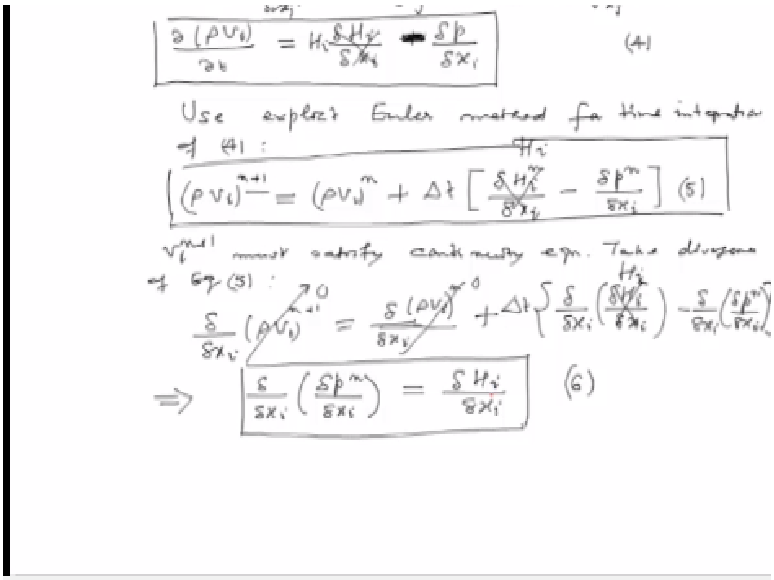
So let us write a momentum equations del of rho Vi over del t+del over del xj of rho Vi Vj–del p over del xi+del tau i,j over del xj. For the time being let us ignore the source terms if they present they could always be combine together with the pressure. Now we would like to apply spatial discretization techniques and let us write the discretized equation.

So application of spatial discretization techniques which could be finite difference, finite volume or finite element that does not really a matter that will lead to a semi-discrete that would be discrete in the space but still contains in time, semi-discrete system given by rho Vi over del t=-delta over delta xj rho Vi Vj-delta p over delta xi+delta over delta xj of tau i,j.

Now here we have introduced this operator delta so operator del/del xi represents spatial discretization operator and it could be different for each term that is for convective term, convective, diffusive and precautive. So what we would normally do is we would use central differencing if you are dealing with finite difference of finite volume taken x central differencing would be used for the pressure term and diffusive terms that is our system term.

And we would use some sort of an up winding scheme for the convective term. Okay, so this was a convective term, pressure term and this is a diffusive term. Now further for sake of notational convenience we are going to introduce attain the short term notation. So let represent by Hi, Hi contains both our convective as well diffusive terms.

(Refer Slide Time: 34:54)



So then we can rewrite to as del of rho Vi over del t= sorry this is del tau xi. This can be written as del Hi over del xi, so this is same as discrete equation in terms of time. Suppose we want to use an explicit scheme the simplest one would be our Explicit Euler method. So use Explicit Euler method for time integration of 4. Then what we will get, we will get rho Vi at time instance n+1=rho Vi at time n+delta t times del Hi at n/del xi-delta pn over del xi.

Now in this equation we have only one problem, Hi which contains our convective term and diffusive term taken together that is explicitly known in terms of the flow variables at time level t=tn. But same thing we cannot say about pressure this pn would usually be unknown. Now the velocity field which we have calculated that must satisfy continuity equation, so this Vi+n+1 must satisfy continuity equation, so let us take divergences of equation 5.

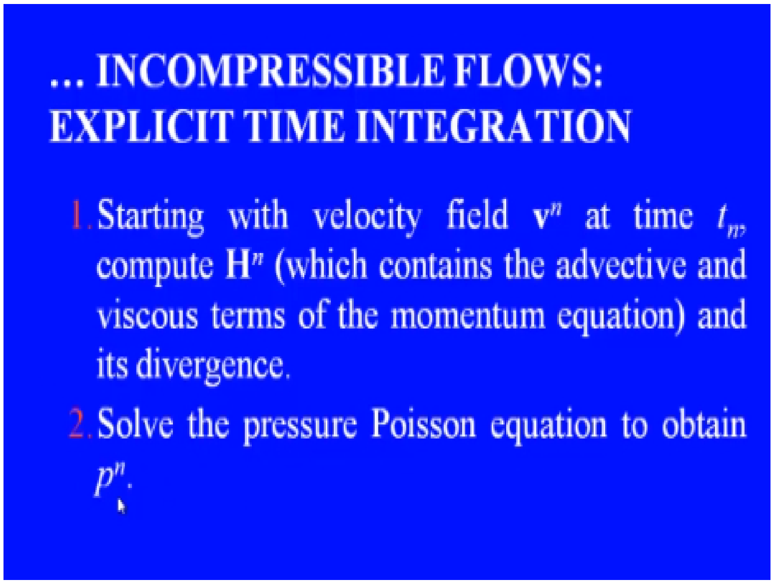
So what do we get, del over del xi of rho Vi n+1 del over del xi of rho Vi at n delta t times del over del xi of del Hi n over del xi, so Hi we will have here, let Hi-del over del xi of del pn over del xi. Now remember we have to satisfy the continuity equation. The velocity field must satisfy the time tn+1 so that the first term, the term one LHS must vanished.

The velocity field which was available there at time t=tn that would also satisfy our continuity equation so this term also vanished, so these 2 terms vanished for continuity. So thereafter, what we are left with is del over del of xi delta pn over delta xi=del Hi over del xi. So this is what is our pressure Poisson equation. So now we have got all the tools which we require for explicit time integration. If you wanted to use the formula given by this equation 5.

In equation 5, to obtain the value at time level n+1 we needed the evaluation of Hi and this del pn over del xi. Hi can be computed in terms of the values at time level n and pn can be computed by solution of a pressure Poisson equation. So steps are clear now. First let us see evaluate this Hi then obtain numerical divergence of Hi use that as a source term for the pressure Poisson equation, solve for the pressure Poisson equation, obtain the pressure field, take its gradient and use it in our explicit formula for the velocity components to get our flow field.

Now let us summarize these steps.

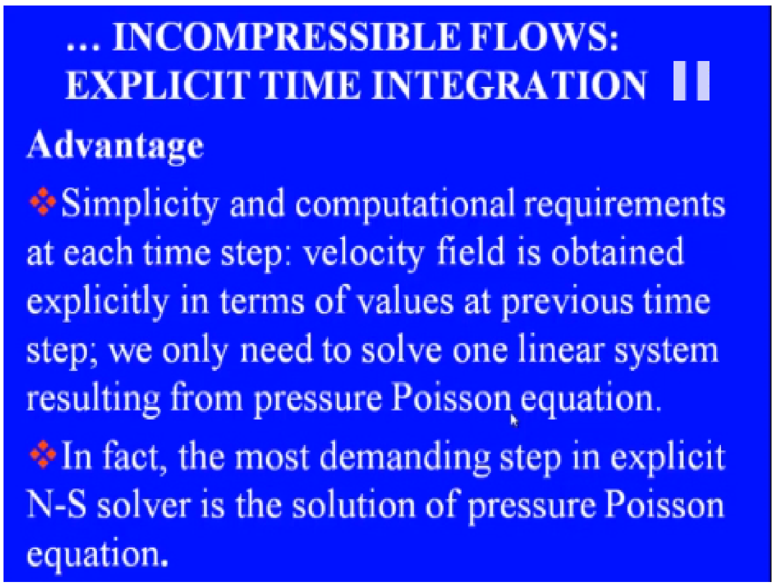
(Refer Slide Time: 41:37)



Starting with the velocity field vn at time tn, compute Hn which contains the advective and viscous terms of the momentum equation and find out its divergences that we need as a source term of the pressure Poisson equation. The next is, solve the pressure Poisson equation to obtain pn and once we know pn then we know this explicit formula in which we substitute for del pn/del xi and H at n and that gives us the velocity field at time level tn+1.

Now we have demonstrated this explicit integration with Forward Euler scheme. Now process would be almost same, if you want to use a higher order schemes such as Adams-Bashforth or Runge Kutta methods.

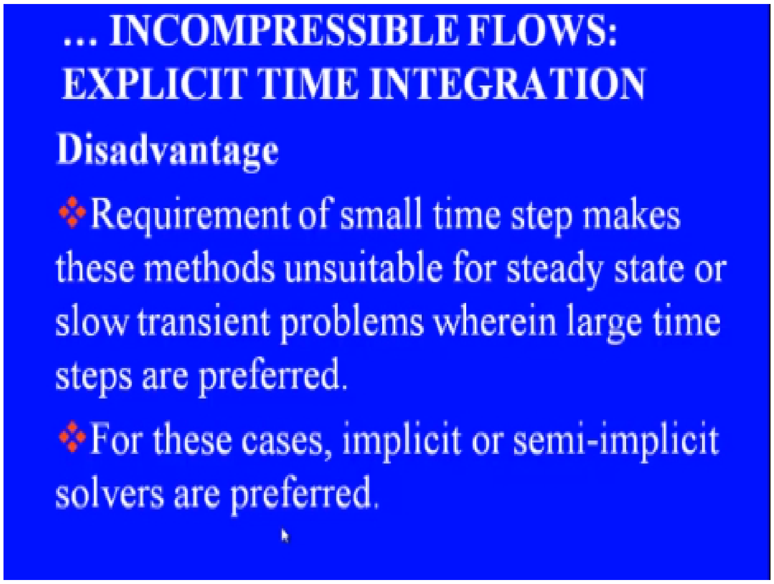
(Refer Slide Time: 42:33)



Now the advantage of the explicit schemes is obvious a very simple implement, the velocity component evaluation requires just straightforward algebraic formula. So velocity field is obtained explicitly in terms of values at previous time steps provided we know the pressure-- we have to solve only one system linear equation which results from pressure Poisson equation. So the implementation of this explicit solver is pretty straightforward.

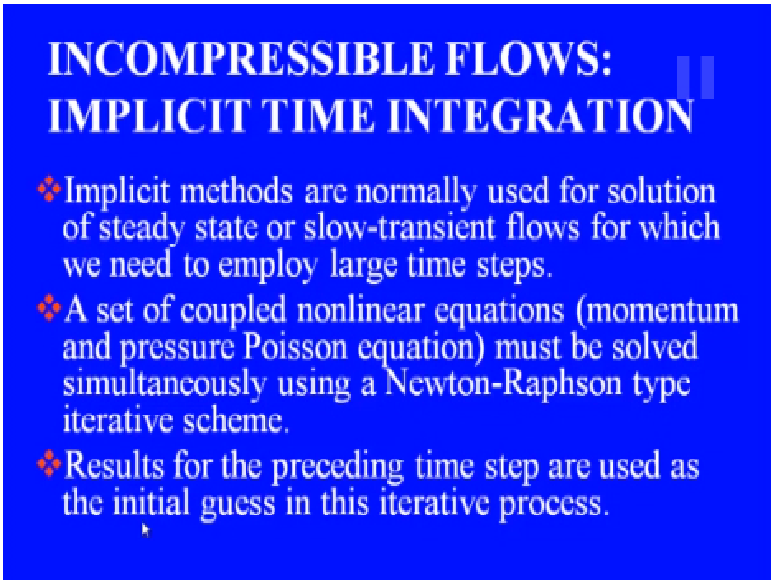
And remember here, the only system of equation which we have to solve that is your pressure Poisson equation so that is that makes this step as the most demanding step in explicit Navier-Stokes solver. And because of this reason the solution of pressure Poisson equation is still one of the most challenging research area at the moment. People trying to develop fast and faster, what typically called fast, Poisson solvers which can be used for large scale flow simulations.

(Refer Slide Time: 43:49)



A disadvantage is that we need to have very small time step that is mandatory by the stability requirements. So these methods are unsuitable for steady state or slow transient problems wherein large time steps would be preferred to reach the steady state very quickly. And for this, steady state problems or the slow transient problems we would prefer implicit or semi-implicit solvers. So let us have a brief look at the implicit schemes.

(Refer Slide Time: 44:35)

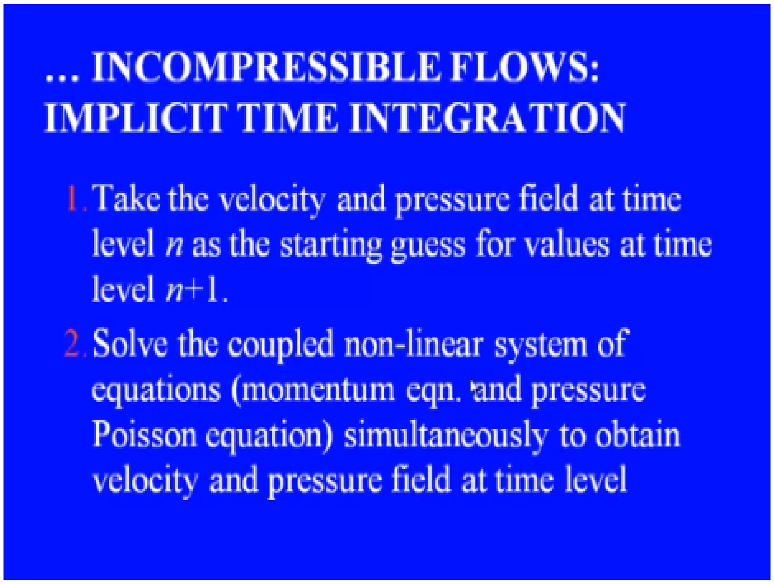


Now these implicit time integration methods are normally used for solution of steady state or slow transient flows for which we need to employee large time steps. But there is one small problem here. We have to solve a set of coupled nonlinear equations; we have got momentum equations which are coupled nonlinear equations and we have the Poisson equation for pressure as again in terms of the products of velocity components which are unknown.

So we have got a coupled system of nonlinear equations which must be solved simultaneously using a Newton-Raphson type iterative scheme. And in our recitative process the results for preceding time step are known to as so they are used as initial guess in the iterative process. Very often what do we do is we linearize these equations; use the value of previous time step in that linearization process, so we get a system of linear equation so solve them, get updated value.

Use that updated in the quasi-linearization process to get a next system of linear equation solve them and continue the iterative process until we are satisfied with the convergence at a given time step. So this quasi-linearization is very often used in conjunction with implicit time integration schemes.

(Refer Slide Time: 46:00)



And solution scheme is very obvious here that is we have to take the velocity of pressure field time level n as the starting guess for values at time tn+1. Solve the coupled nonlinear equations momentum equation and pressure Poisson equations simultaneously to obtain the velocity and pressure field at the new time level. This Pressure Poisson equation we already learnt yesterday in the previous lecture. Okay, so that completes our algorithm part.

And remember that this implicit time integration scheme they are extremely demanding in terms of competition time and memory requirements as we need to solve a very large nonlinear system at each time step. So as I mentioned earlier we would usually go forward we call a quasi-linearization, so that we need to solve only a sequence of linear systems especially for steady state problems.

We will see few specialized schemes which are based on this quasi-linearization approach in a set of schemes which are termed as pressure correction based schemes in the next lecture.