Computational Fluid Dynamics
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Lecture - 33
Finite Element Shape Functions and Numerical Integration-2

Welcome to the 3rd lecture module 7 on Finite Element Method. We have finished with the introduction and weighted residual formulation. In the second lecture, we also had a look at variation formulation and we looked at some shape functions. In this lecture, we are going to continue with shape functions and numerical integration and then we would follow up with same application to scalar transport. This is a brief recap of what we did in the previous lecture.

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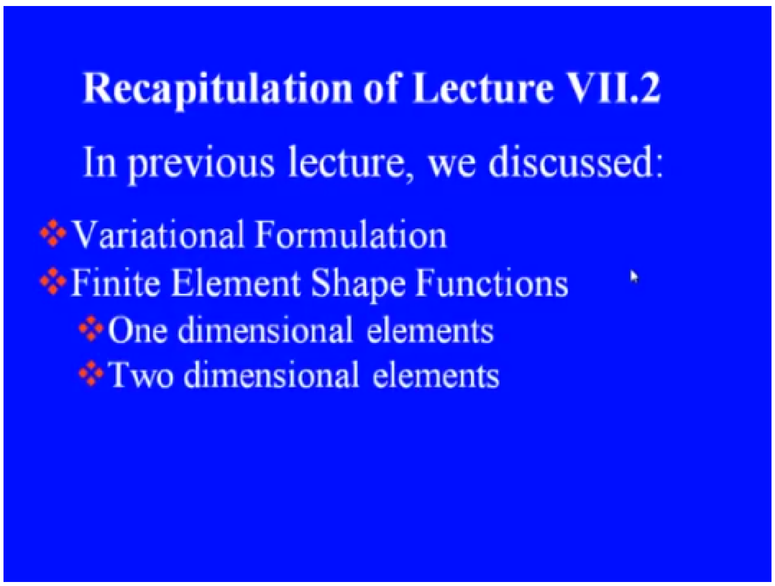

We had briefly discussed variational formulation and then we have started discussing different types of shape functions using Finite Element Analysis. We discussed one-dimensional elements. We also discussed few 2-dimensional elements with rectangular elements. Now, in this lecture which is third lecture in the series, we are going to discuss further few more shape functions and the numerical integration which we require in Finite Element Analysis.

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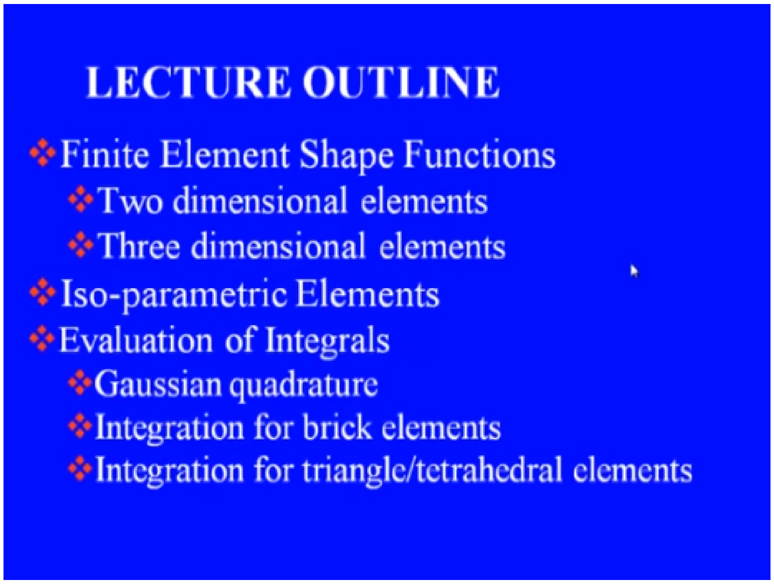

So, we will continue with shape shift functions. In 2-dimensional elements, we will take a look at triangular elements and then we will also have a look at few 3-dimensional elements; and then, we will define what we mean by iso-parametric elements which are very commonly used in curvilinear domains; and then, we will have a look at how do we evaluate the integrals which cannot be evaluated analytically.

So, we will look at 2 approaches; one is Gaussian quadrature which is used for integration of large elements or brick elements and then we will have a look at the integration procedure for triangle or tetrahedral elements. So, we were discussing the standard shape functions for 2-dimensional elements (()) (02:14) would rectangular elements.

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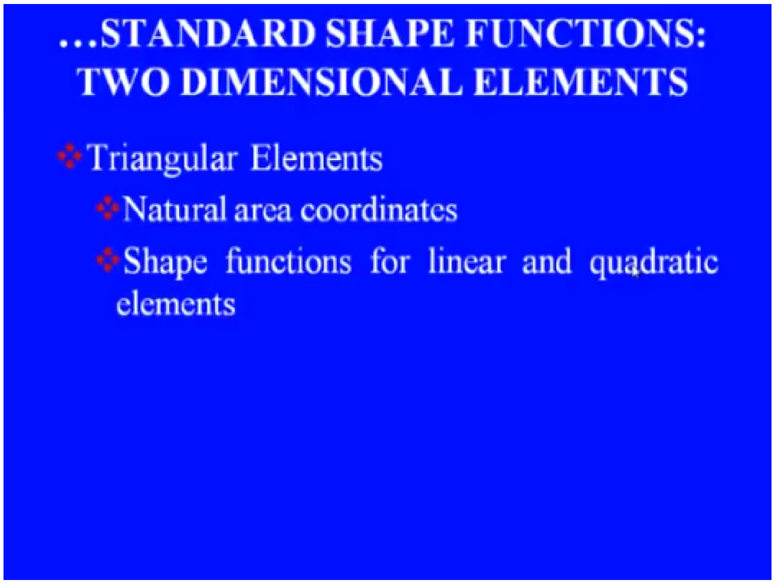

Now, I will start off with triangular elements. We will first define what we mean by natural area coordinates for a triangular element and in terms of this natural area coordinates, we will define the shape functions for linear and quadratic elements. So, let us first have a look at what we mean by area coordinates for a triangular element.

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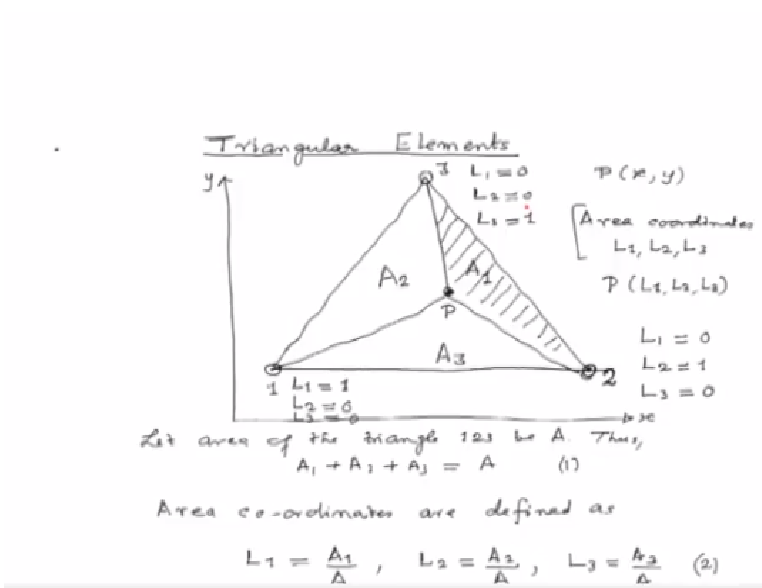

So, let us draw a triangle in XY plane. So, why we need this area coordinates, for a simple reason. Though, we can draw our shape functions in terms of X and Y, the expression is a bit more complicated. The integration is also bit more involved but if we convert it into what we call a natural coordinate for triangular elements similar to what we had had in the case of rectangular elements, the life becomes a lot easier.

So, let us draw a triangle. So, a typical triangular element, this is vertices of triangle. Let us number them as 1, 2 and 3. Let us consider an arbitrary point P which is inside the strength elements. Now, we can have 2 sets of coordinates, so in our XY coordinate, P would be given by let us say P x, y. Now, we will define area coordinates which you are going to call coordinates. You would use symbol L1, L2 and L3.

We will define what do you mean by these 3. So, in terms of these area coordinates point, P can also be represented as L1, L2 and L3. Now, you might be surprised here that in the Cartesian reference frame, P is represented only by 2 numbers. Here, we need 3 numbers and we would see these 3 numbers are not independent. There only 2 independent numbers here. Now, let us join this point with 3 vertices.

So, thereby we effectively divide our triangle into 3 sub-triangles. The sub-triangle which is straight in front of the node 1, this area will be called a A subscript 1. So, this is which lies in front of node 1, rather we can that the base of this triangle is the side which is opposite to our vertex 1. Similarly, the triangle whose base is opposite to vertex 2, area of that we will denote by A2 and the area of the triangle which is opposite to vertex 3, we will denote it by A3.

Now, let area of the triangle 1, 2, 3 be A. So, we can clearly see. Thus, A1+A2+A3=A. Let us call this equation 1. Now, how do we define our area coordinates. So, this area coordinates are defined as the coordinate L1=A1/A, coordinate L2=A2/A, coordinate L3=A3/A. Let us call this as equation 2. So, now from 1 and 2 it is obvious thus, what we have at L1+L2+L3 = A1+A2+A3/A, that is = 1.

So, what we say that these 3 coordinates are inter-related. So, we have got only 2 independent coordinates. If we know L1 and L2, we can find out what the value of the third coordinate is. So, in a sense again we have got 2 independent numbers to represent the coordinates of given point P in our triangular element.

Now, using these definitions we can clearly put the coordinates of 3 vertices which were defined for vertex 1, this L1 coordinate would be 1 because when P lies at 1 even is whole of the triangle, L2 coordinate of 1 is 0 and similarly its L3 coordinate is also equal to 0. When we come to point 2, the P is constant with the vertex 2 A to whole of the triangle area. So, we will have L2=1 at vertex 2, L1=0 and L2=0.

Similarly, at vertex 3, we have got L1=0, L2=0 and L3=1. Some of things which you can observe that along line 2, 3, if P lies at the side 2, 3 what will happen, that A1 would be 0. So, that L1 coordinate along this line would be 0 and L2 and L3 they will range in the range 0 to 1. Same holds good between 1 and 3, that is along the side 1 and 3, L2 coordinate would be 0 for all the points.

So, this line 1-3 represents basically L2=0 line. Line 1-2, it represents L3=0 line and the side 3-2 represents L1=0 lines. So, this is our definition of the area coordinates. Next, we would be interested in finding out what is the relation between the area coordinates and X, Y coordinates. S, to find out relationship between XY and this L1, L2 and L3. The XY coordinate of any point that can be represented in terms of these natural coordinates, so what we will have is at X coordinate of a point P would be given by L1X1+L2X2+ L3X3.

Similarly, Y coordinate is given by L1Y1+L2Y2+L3Y3. Okay, now we can collect these equations together. The equations 3, 4 and 5, they can be written together in a matrix form. If you write them in matrix form, equation 3 can be written as the product of matrix with these coordinates L1, L2, L3 and then on the right-hand we are going put for equation 3, the RHS was 1. From equations 4 and 5, we would take the left hand side things which is our X and Y because the terms involving natural coordinates, we are going to put on the left hand side.

So, first (()) (12:40) basically 1, 1, 1, i.e., it is multiplied by this vector L1, L2, L3 that would give us 1. So, L1+L2+L3=1, so this is our first line represents equation 3. The second one, we will put X coordinates X1, X2, X3 so that X1L1+X2L3+L2+X3L3 that would become our X coordinate and then Y1Y2Y3 in the third row so that Y1L1+Y2L2+Y3L3 that gives us our coordinate.

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... Triangular Element

Solve the system of eqns to L_1, L_2 and L_3 in terms of the Cartesian coordinates (x,y):

$$L_\alpha = \frac{a_\alpha + b_\alpha x + c_\alpha y}{2A} \qquad A = \frac{1}{2}\begin{vmatrix} 1 & 1 & 1 \\ x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{vmatrix}$$

$(\alpha = 1, 2, 3)$

Here, $a_1 = x_2 y_3 - x_3 y_2$, $b_1 = y_2 - y_3$, $c_1 = x_3 - x_2$

$\alpha=2$: $a_2 = x_3 y_1 - x_1 y_3$, $b_2 = y_3 - y_1$, $c_2 = x_1 - x_3$

$a_3 = x_1 y_2 - x_2 y_1$, $b_3 = y_1 - y_2$, $c_3 = y_2 - y_1$

I 3-Node (Linear) Element

So, now we have a got a system equations which relate L1, L2, L3 and XY coordinates. So, now we can solve the system of equations to get L1, L2 and L3 in terms of the Cartesian coordinates X, Y and if you do that what we will see that we can write in a short-hand notation. Let us call it L alpha, this can be written as A alpha+B alpha X+C alpha Y/2 times area of the triangle, okay. We can clearly see that how we define the area of the triangle.

Area of the triangle is basically half of this determinant 1, 1, 1, X1, X2, X3 Y1, Y2, Y3, okay. In expanded form, we can substitute it. Now, this alpha will take the values 1, 2 and 3. In fact, we need to compete only 2 of them. The third one is obtained by the relationship L1+L2+L3=1. So, here using Cramer's rule you can easily see that if alpha=1, A1 is X2Y3-X3Y2, B1 is Y2-Y3 and C1 is X3-X2.

Similarly, for the second coordinate when alpha=2, we get A2=X3Y1-X1Y3, B1=Y3-Y1 and C1=X1-X3. A3=X1Y2-X2Y1, B3=Y1-Y2 and C3=Y2-Y1. So, we can use these values to find out what are the natural coordinates L1, L2 and L3 of a point if we know its Cartesian coordinates. Now, in terms of these, it is very easy for us to define now the shape functions of the triangular elements.

So, now let us take the first scenario as what we call 3 node or we also call it linear element, okay. So, this is the most basic 2D element.

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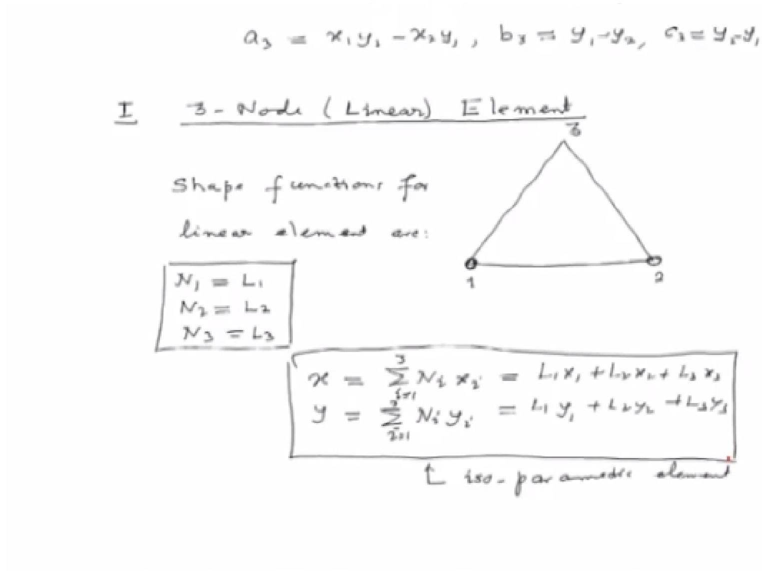

So, in this case our nodes of the 3 vertices of the triangle 1, 2 and 3 and the shape functions for the linear elements N1=L1, N2=L2 and N3=L3. So, now you can easily see the simplicity which has been introduced by the use of area coordinates that our shape functions are expressed in a very simple form. You can also see this is an iso-parametric element which will introduce here just for the sake of revision.

Please note down that this X coordinate is again given by this sigma NiXi, I=1, 2, 3. L1X1+L2X2+L3X3 and similarly Y coordinate, I=1, 2, 3, NiYi. So, since Ni, they get values of Li, so we have L1Y1+L2Y2+L3Y3. So, in effect our basic linear triangular element is an iso-parametric element.

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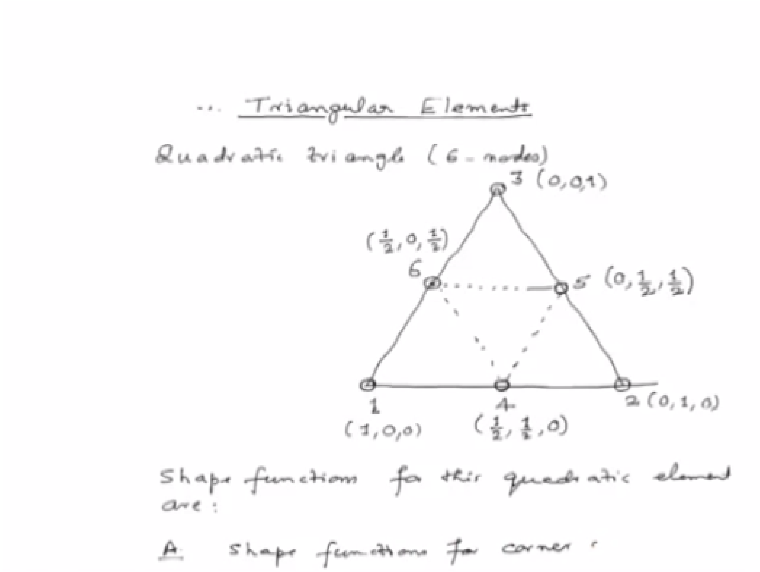

Let us have a look at one more triangular element. Let us take a triangular element for which the shape functions are of second order, i.e., we will call it as quadratic and in this case we will have 6 nodes. So, let us have a geometrical presentation of this quadratic element and the numbering convention which is commonly employed 1, 2 and 3. These are our 3 primary nodes. Mid-sized nodes are numbered again in similar order, i.e., the mid-point of 1 and 2, that could be numbered as node 4 in same cyclic order.

The mid-point of 2-3 that would be called node 5 and midpoint of 1 and 3 that would be called node 6. So, remember these 4, 5 and 6, they are mid-points of their respective sides. So, in terms of area coordinates, let us note down coordinate of each of these nodes. So, node 1 we will have the coordinates 1, 0, 0. Node 2 has got coordinates 0, 1, 0. Node 3 has got coordinates 0, 0, 1. Node 4, it is midway 1 and 2, so L1 is half, L2 will also be half and L3 coordinate is 0 along this line.

Node 5, L1 coordinate is 0. L2, L3 they both have value equal to half. Node 6, this is half 1/2, 0, 1/2. So, now how do you define the shape functions. So, shape functions for this quadratic element, we will first define the shape functions for corner nodes. So, that is case A, shape functions for corner nodes that is nodes 1, 2 and 3. We will write in a fairly compact form.

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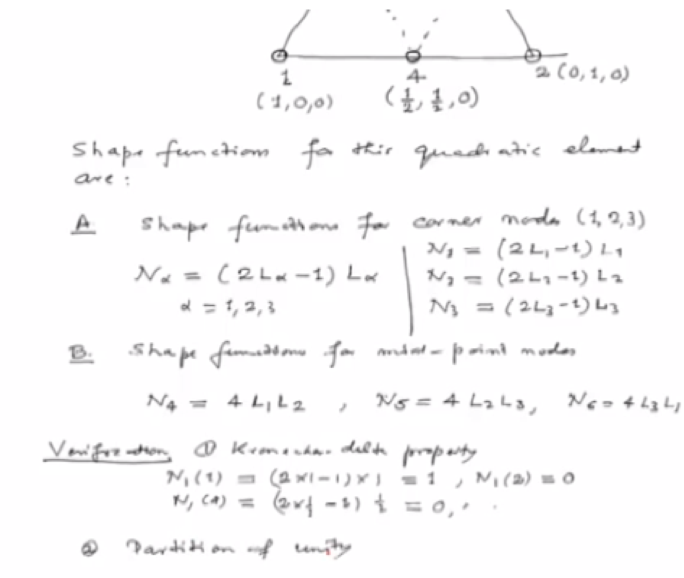

N alpha=2L alpha-1 times L alpha. So, alpha takes the values 1, 2 or 3. Let us demystify and write in expanded form. Let us write for each node separately. So, your shape functions N1=Twice of L1-1 times L1. N2=2L2-1 time L2 and N3=2L3-1 times L3. Now, can verify whether these satisfy the requirements which we had mandated for each shape function. Before we do that, let us write the shape functions for the remaining 3 nodes, i.e., midpoint nodes.

Shape functions for midpoint notes. N4=4 of L1, L2, N5=4 times L2, L3 and N6=4 times L3L1. It is very easy to verify these things because at node 4 for instance, N4 must be equal to 1 and since coordinates are 1/2, 1/2 there product of L1, L2 is 1/4. So, this must be multiplied by 4 and so on. So, now you can verify the 2 properties which each shape functions should satisfy, so verification.

Let us first verify the reverse order. First let us say Kronecker delta property for the nodes 4 and 5, this N4, N5 and N6 straightforward, the values of 2 coordinates which are involved in definition are 1/2 each, so multiply by 4, that easily gives us a value of 1. Definitely, their value elsewhere is 0 at all other nodes. The value of N4 for instance, if we find out its value at the node 3 or node 1, it will become 0.

So, this Kronecker delta property is very easy to verify. We can also verify this property for let us say for N1. So, N1 at node 1=2*1-1*1, this is definitely 1 and how about N1 at node 2. At node 2 the coordinate L1 is 0, so it becomes identically 0. At node 4, N1 at 4=2*1/2-1*1/2, so that is again 0. So, we can verify it for each node. So, all the shape functions do satisfy Kronecker delta property.

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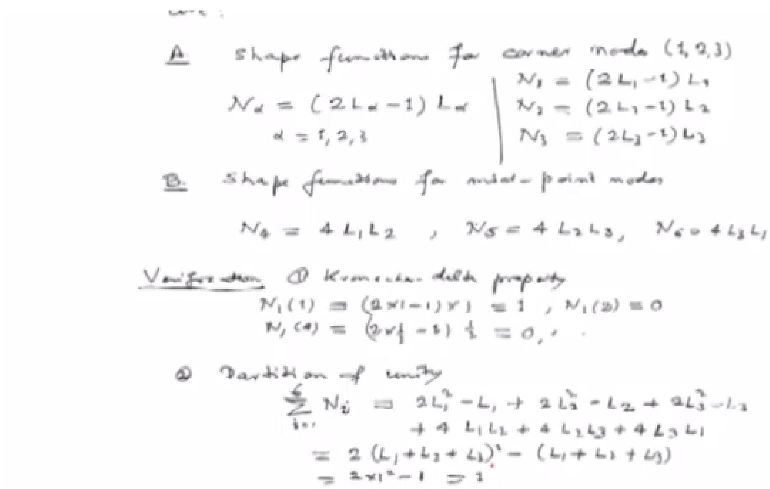

Next, the partition of unity. For that, we need to find out whether the summation of all these shape functions NI, I is equal to 1 to 6, that is what we need to find out. So, now let us do that. Let us write each one in expanded forms. N1 becomes 2L1 square-L1. L2 would become 2L2 square- L2. N3 is 2L3 square- L3+4 L1L2+4 L2L3+4 L3L1. So, this we can write as twice of L1+L2+L3 whole square-L1+L2+L3 and we have L1+L2+L3=1.

So, we have got 2*1 square-1 which is equal to 1. So, these shape functions do satisfy the partition of unity. So, there is absolutely no problem with satisfaction 2 primary conditions which these shape functions must satisfy, okay.

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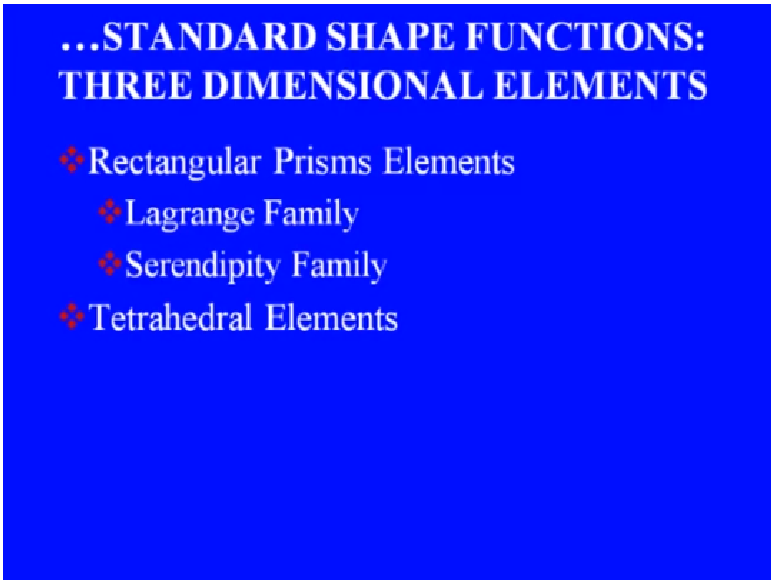

Next, let us move on to 3-dimensional. So, in 3-dimensional, we will have a look at what we call rectangular prism elements. We have got variety of options. We can have rectangular prism elements and we can have triangular prism elements. We can have tetrahedral elements. We can have what we call wedge elements. There are variety of options available. We will have a look at 2 of the basic ones.

Basically, we will have a look at rectangular prism elements and we will have both the types; Lagrange family and Serendipity family. Tetrahedral elements are very similar to our triangular elements and we have to introduce what we call volume coordinates in place of area coordinates to define this tetrahedral elements. Now, please remember that both of these types, i.e., rectangular prism elements and tetrahedral elements are the ones which are most widely used in finite element analysis.

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3-D Rectangular Elements

A. Lagrange Family

Shape functions for element of any order can be expressed as product of 1-D Lagrange polynomials, i.e

$$N_\alpha \equiv N_I N_J N_K = L_I^n(\xi)\, L_J^m(\eta)\, L_K^p(\zeta)$$

where n, m and p denote no. of sub-divisions along each side.

Definition of coordinate ξ, η, and ζ:

$$\xi = \frac{x - x_c}{l_x}, \quad y = \frac{y - y_c}{l_y}, \quad z = \frac{z - z_c}{l_z}$$

B. Serendipity Family

So, now let us have a bit detailed look at our 3D rectangular elements. As usual, we can have 2 sets here. The first one we are going to look at is what we call Lagrange family and for the Lagrange family elements, the shape functions can be written as, the shape functions for element of any order, i.e., whether we are dealing with linear elements, quadratic elements, cubic elements and so on can be expressed as 1D Lagrange polynomial.

That is if you want to write a generic shape function N alpha for a node alpha, this can be expressed in terms of what we call NI NJ NK. Now, I, J, N, K, they are being used to represent one-dimensional Lagrange shape functions in X, Y and Z directions respectively. So, if we use the symbol for Lagrange Polynomial, we can write this as L of I XI n, LJ eta m and LK zeta p where n, m and p denote the number of subdivision along each side.

How many subdivisions for this element we have got along X direction, Y direction, and Z direction for such an element and definitions of natural coordinates xi, eta and zeta. This is very similar to what we did for the 2-dimensional case, i.e., over xi is defined as X-XC/L of X, okay. Now, Lagrangian 3D rectangular elements, they will have interior nodes if the order is more than linear.

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$N_{\alpha} = \ell_I \ell_J \ell_K = \ell_I(\xi)\,\ell_J(\eta)\,\ell_K(\zeta)$

where n, m and p denote no. of sub-divisions along each side.

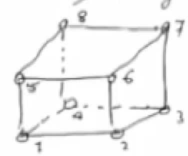
Definition of coordinate ξ, η, and ζ:

$$\xi = \frac{x-x_c}{l_x}, \quad y = \frac{y-y_c}{l_y}, \quad z = \frac{z-z_c}{l_z}$$

B. Serendipity Family (nodes are only along the boundaries of the element).

① 8-node Linear element

$$N_a = \frac{1}{8}(1+\xi_a\xi)(1+\eta_a\eta)(1+\zeta_a\eta)$$



where $(\xi_a, \eta_a, \zeta_a) \Rightarrow$ coordinates of node 'a'

So, we can have another family which is more widely used, which is called Serendipity family in which nodes are only along the boundaries of the element, and let us try to define 1 or 2, we will restrict to linear and quadratic. So, linear one is very simple. This is 8-node element which is also referred to as simple brick 1, 2, 3, 4, 5, 6, 7, 8. So, this is our 8-node linear element, the shape functions are identical to what will get for linear Lagrangian element and we have got all corner nodes.

So, the shape function is given as NA=1/8*1+xi A xi*1+eta A*eta times 1+zeta A*zeta where xi A, eta A and zeta A, these are coordinates of node number A. So, by substituting the appropriate values, we can find out the shape function corresponding to each node for this 8-node linear element.

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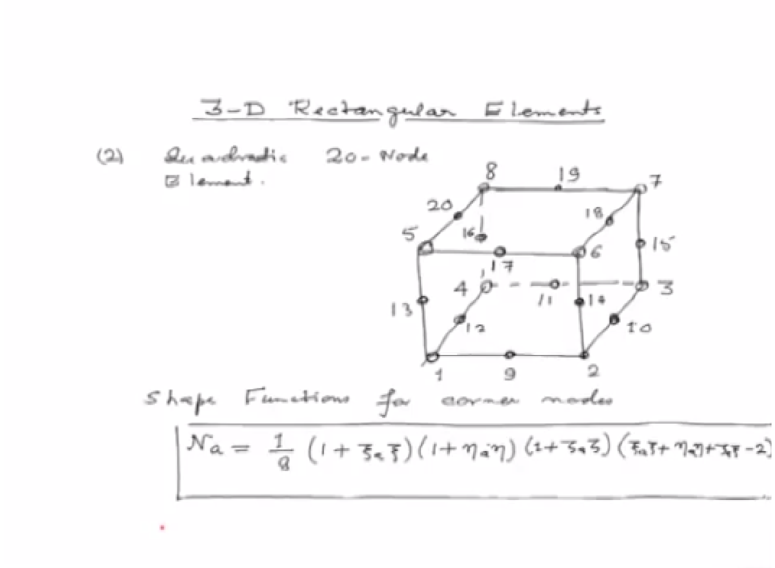

Next, let us have a look at quadratic serendipity element. So, let us first draw a small diagram. As usual, we have to number these nodes starting in a specific order, so 1, 2, 3 and 4. There are 4 corner nodes in the bottom line and 5, 6, 7, 8 these are corner nodes on the top line. This is quadratic 20-node element. Okay, now how do we define the big nodes. Their numbering now will start in order.

Let us say 1 midpoint of 1 and 2, this will be numbered as 9. So, go in the same fashion 2-3, that gives us the node number 10. 3-4 midpoint that gives us 11. 4 and 1, 12. Next, we will have a look at the vertical things 13, 14, 15 and 16 and after that we will have these midpoints on the top sides 17, 18, 19 and 20. So, in 3-dimensions you quickly say that if you move from linear to quadratic, the number of nodes has increased considerably.

In fact, if we had a Lagrangian family, we will have much larger element of nodes. There will be many more nodes in the interior. There are 7 more nodes to be precise. Now, how do we represent our shape functions in this case. Shape functions for corner notes. You can just remember the way we did it for 2D case. So, NA=1/8 1+xi*xi 1+ eta A*eta 1+zeta A*zeta*xi A xi+eta A eta+zeta A zeta-2.

So, this very compact form expressing for our corner nodes, substitute their natural coordinates xi A, eta A and zeta A to get the corresponding shape functions. Now, let us have a look at what would be the form for midpoint nodes. So, let us take one typical case, remaining ones you can workout yourself.

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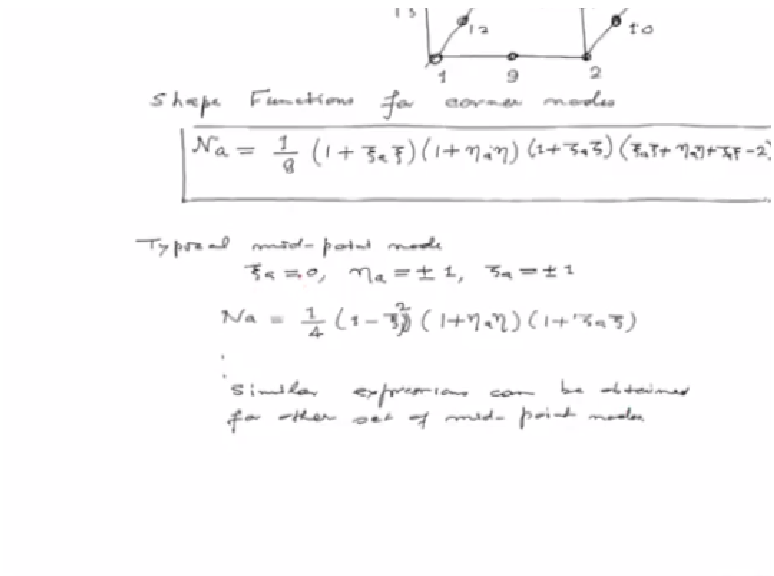

So, typical midpoint node for which you xi A=0 and eta A takes the value plus or minus 1. Similarly, zeta A takes the value plus or minus 1. Your NA is expressed as 1/4 1-xi square*1+eta A eta*1+zeta A zeta and so on. So, similar expressions can be obtained for other set of midpoint nodes. So, what we have done is identify one set for which, let us say this xi head to 0. So, the first term was 1-xi square.

Remaining terms very similar to what we had that with product of 2 linear terms in eta and zeta. So, if you move on to the case of eta A=0, then the shape functions would involve the linear combination in terms of xi and zeta multiplied by 1-etas square and so on. So, that is how you can easily write down the expressions for the remaining shape functions and that I would leave as an exercise for you to complete.

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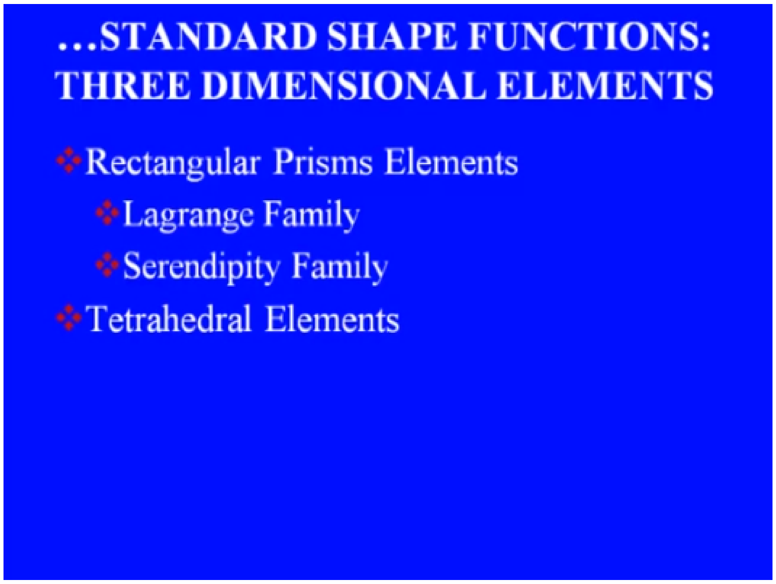

Tetrahedral elements as I said that is reading assignment for you.

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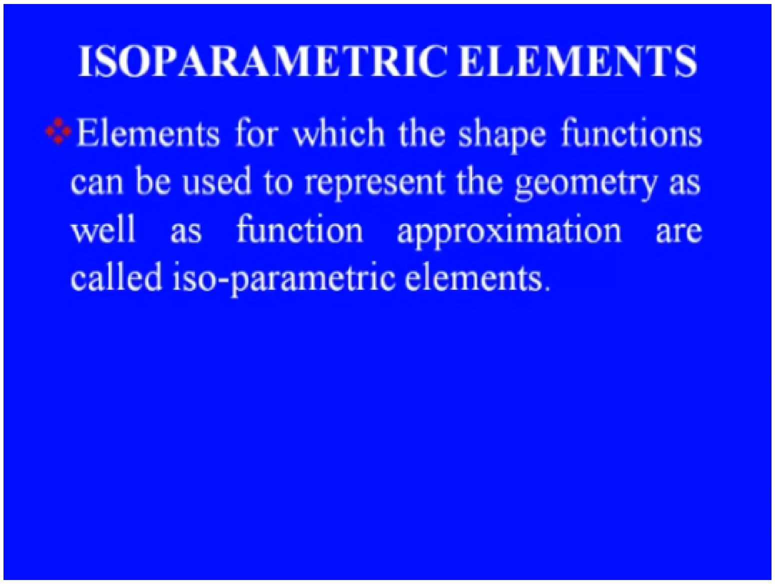

Then, we used the word iso-parametric element, specifically when we discussed our triangular element, we said look it also works out to be linear triangular iso-parametric element. So, what do you mean by an iso-parametric element in general. The definition is the elements for which shape functions can be used to represent the geometry as well as the function approximation, they are called iso-parametric elements.

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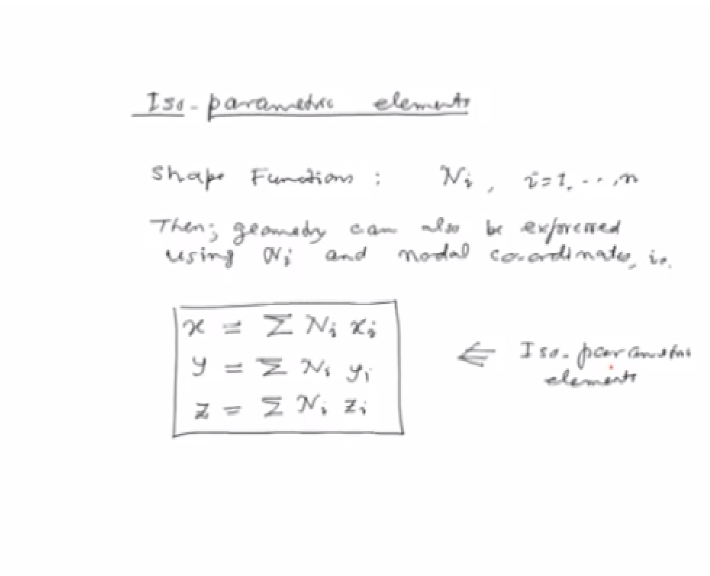

So, that is to say if you got any element for which shape functions are represented as an iso-parametric elements. So, shape functions they are given by NI where I is equal to 1 to number of nodes in this element. Then, the geometry can also be expressed using NI and nodal coordinates, i.e., we can write X as sigma NI XI, Y would be given as sigma NI YI and Z would be represented in terms of sigma NI ZI.

So, that were the case such elements are called iso-parametric element. Iso means identical of sign. So, here the variation of the variable in our element and variation of the coordinates they follow the similar pattern, they are related by these shape functions. So, this is a reason why we call such elements as iso-parametric elements.

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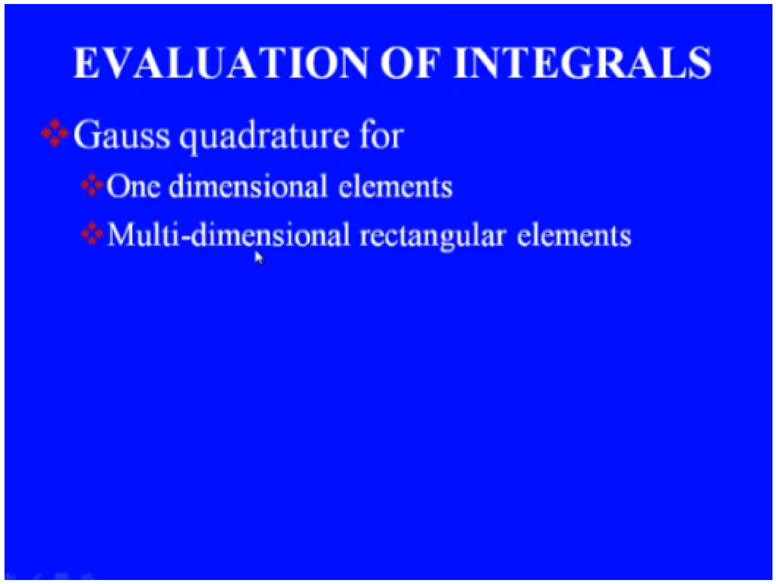

Now, let us come to the evaluation of integrals. For most of the finite elements, analytical integration would be very difficult if not impossible. So, we have got to go for what we call numerical integration. There are separate set of formulae which are available for rectangular elements and triangular elements. So, if you are dealing with one-dimensions or in 2D rectangular or 3D brick elements, so we will use Gauss quadrature formula.

So, Gauss quadrature would be used for one-dimensional elements as well as multi-dimensional rectangular elements. So, let us have a look at what we mean by the Gauss quadrature here.

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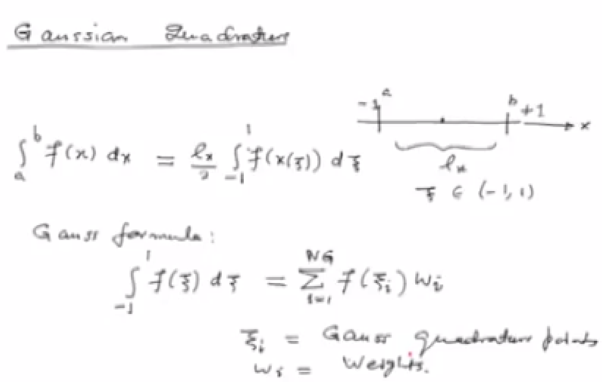

This is to remind you that if you want integrate a function FX, we have got a function which has be to be integrated along a line and suppose the extend of this line, let us call it as LX is A to B. So, we want to find out this integral A to B FX DX. Now, you can verify that this is given by LX/2. If we map it to an iso-parametric element with (()) (48:45) -1 to 1, the natural coordinate xi which ranges from -1 to 1.

So, in this, we can write with -1 to 1 FX of xi D xi, I the X and Gauss' formula says that now this standard integral in the interval -1 to 1 F xi D xi, it can be obtained by this weighted sum, I is equal to 1 to NG F xi Wi. So, here this xi point, they are called Gauss quadrature points and WI are called weights, okay. So, we can use the Gauss quadrature even for multi-dimensional integrals.

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Gauss formula:

$$\int_{-1}^{1} f(\xi)\,d\xi = \sum_{i=1}^{NG} f(\xi_i)\, w_i$$

ξ_i = Gauss quadrature points

w_i = Weights.

$$I = \int_{\Omega} G(\vec{x})\,d\Omega = \int_{-1}^{1}\int_{-1}^{1}\int_{-1}^{1} G(\vec{x}(\xi,\eta,\zeta))\,|J|\,d\xi\,d\zeta\,d\eta$$

$$\int_{-1}^{1}\int_{-1}^{1}\int_{-1}^{1} G\,|J|\,d\xi\,d\eta\,d\zeta = \sum_{i=1}^{N_1}\sum_{j=1}^{N_2}\sum_{k=1}^{N_3} \bar{G}(\xi_i,\eta_j,\zeta_k) \times w_i w_j w_k$$

$$\bar{G}(\xi,\eta,\zeta) = G(\,)\,|J|$$

For instance, if you want to find out an integral in 3 dimensions, i.e., to say we want to find out the integral given by integral/3-dimensional domain GX D omega. So, this we can easily express as -1 to 1 -1 to 1 -1 to 1 GX, now X would be given in terms of xi eta zeta multiplied by what we call Jacobian, D xi, D eta, D zeta and now this standard integral -1 to 1 -1 to 1 -1 to 1 G times D xi, D eta, D zeta can be of obtained using Gaussian quadrature formula, I=1 to N1, sigma G=1 to N2, sigma K=1 to N3.

Let us call this function as G bar xi I, eta J, zeta K*the weights. So, multiplied by the weights WI, WJ, WK where our G bar xi eta zeta, this is of function G*Jacobian. So, this is how we can use Gaussian quadrature to find out 1D or multi-dimensional integrals numerically.

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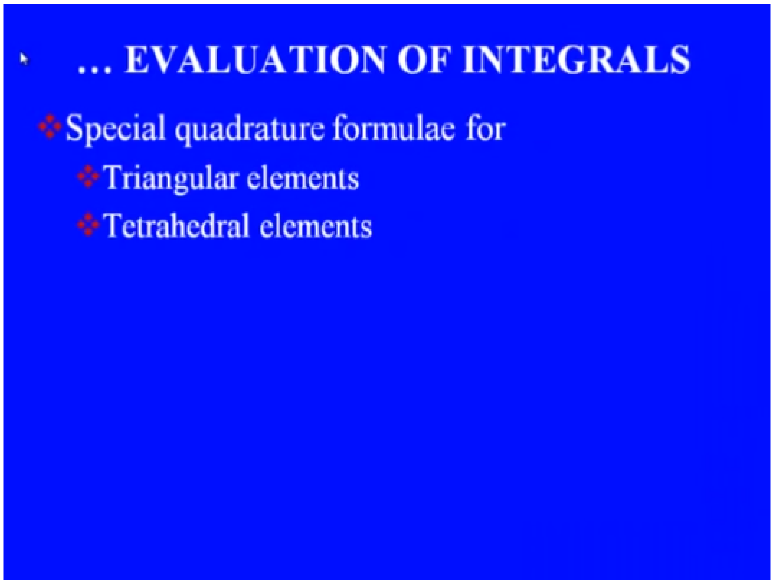

Next, we have got some special quadrature formula for triangular elements. For triangular elements, we already see that there is one possibility of doing some analytical integration and there are special formula which is fairly similar to what we had seen with the Gaussian quadrature, i.e., we will have some Gaussian points and their corresponding weights which can be used to find out the integrals over an area element.

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Integral over a triangular element

$$\boxed{\int_\Delta L_1^a L_2^b L_3^c \, d\Omega = \frac{a!\, b!\, c!}{(a+b+c+2)!}(2A)}$$

Numerical Integration

$$I = \int_A f(\bar{x})\, d\Omega = 2A \underbrace{\int_0^1 \int_0^{1-L_1} f(L_1, L_2, L_3)\, dL_1\, dL_2}_{I_1}$$

$$I_1 = \int_0^1 \int_0^{1-L_1} f(L_1, L_2, L_3)\, dL_1\, dL_2$$

$$= \sum_{i=1}^{N} f(L_1^i, L_2^i, L_3^i)\, w_i$$

So, now let us have a look at the integral over a triangular element. Now, if everything is in terms of the area coordinates, we have one simple relation which we can make use of in the evaluation of these integrals. So, there is one analytical formula, so integration over a triangle we got L1 to the power A, L2 to the power B, L3 to the power C D omega. This is given as factorial A, factorial B, factorial C, divided by A+B+C+2 factorial*2 into the area of the triangle.

So, this is one analytical formula. Now, let us have a look numerical formula similar to the Gaussian quadrature, so numerical integration. If you want to find out the integral which is given by integration over the area of some function FX D omega, this we can write as 2A integral over 0 to 1, integral 0 to 1-L1, F L1, L2, L3 DL1 DL2. Remember that we have not introduced L3 because that is not an independent coordinate that is given in terms of L1 and L2.

So, now this standard integral which we get here for that there are specialised formula. So, this integral, let us call this as I1. So, integral I1=0 to 1 0 to 1-L1 F of L1, L2, L3 DL1 DL2. This can be expressed in a form very similar to our Gaussian quadrature, sigma I=1 to N F at L1I L2I L3I times WI. So, there are standard tables available. Say for instance, if you want I will just put 2 possibilities here.

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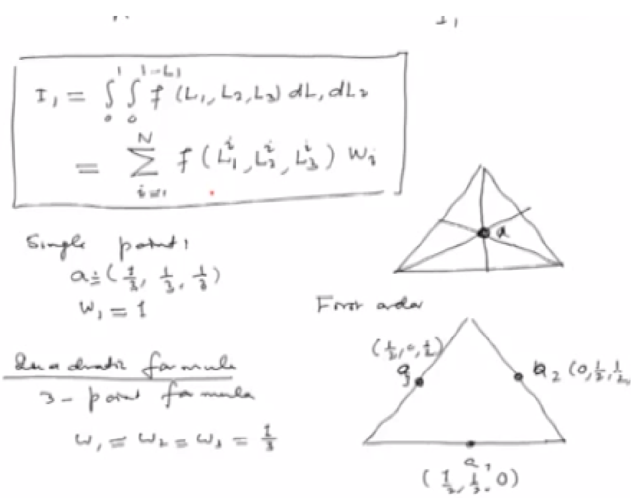

In one case, we have got a single point formula, the coordinates of this point, let us call this point as small A. So, coordinates are 1/3, 1/3, 1/3 and you can easily guess, this would be equal weight, so the weight is 1. So, this gives us first-order accuracy. If you want a quadratic formula, so that will involve use of 3 points which are basically midpoints of the sides and the coordinates you can easily say this is 1 x 2, 1 x 2, 0.

In fact, let me call this as A1, A2, and A3; 0, 1/2, 1/2; 1/2, 0, 1/2. So, this is a 3-point formula. Weights are again equal. So,W1=W2=W3=1/3. So, use these coordinates in this formula which we had used. So, summation would be using the function evaluation at the points A1, A2 and A3 and weight multiplier is 1/3 and this should give us the numerical value of the integral.

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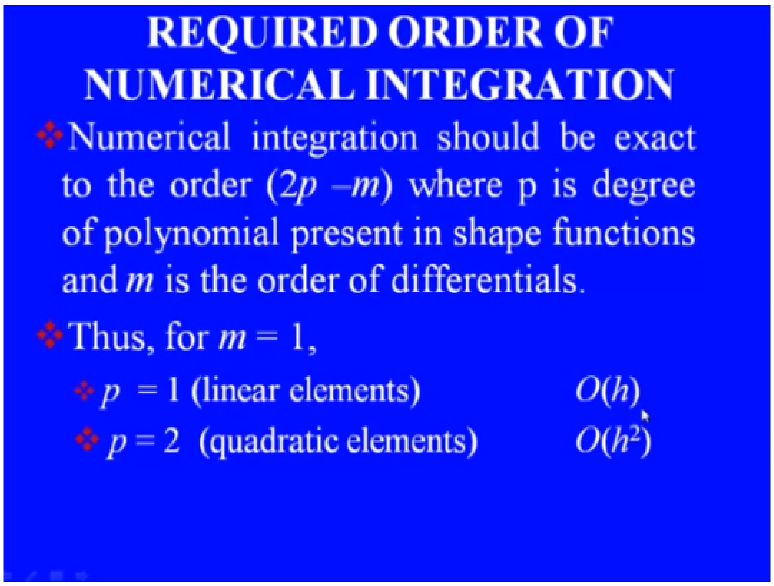

Now, to conclude our numerical integration, there are certain points which I would like to just point out, the required order of numerical integration. Because more number of points we choose in the Gaussian quadrature or for triangular elements or similar formulas are available for tetrahedral elements.

More expensive would be evaluation of the integrals, but do we really need that. There are some simple rules which say no. We can restrict the number of points which we need. So, the total degree to which we need to evaluate our integrals exactly is 2P-1 where P is the degree of polynomial present in our shape function, i.e., whether we have used linear, quadratic and so on shape function and M is the order of differential present in our weak form.

So, if M=1, so if you want to use linear elements we should restrict P2, we should use only first-order formula. For instance, in the case of triangular elements, just taking the value of the centroid that should be good enough, use one point formula. P=2 for quadratic elements, use a quadratic formula or in the Gaussian quadrature use 2/2 formula.

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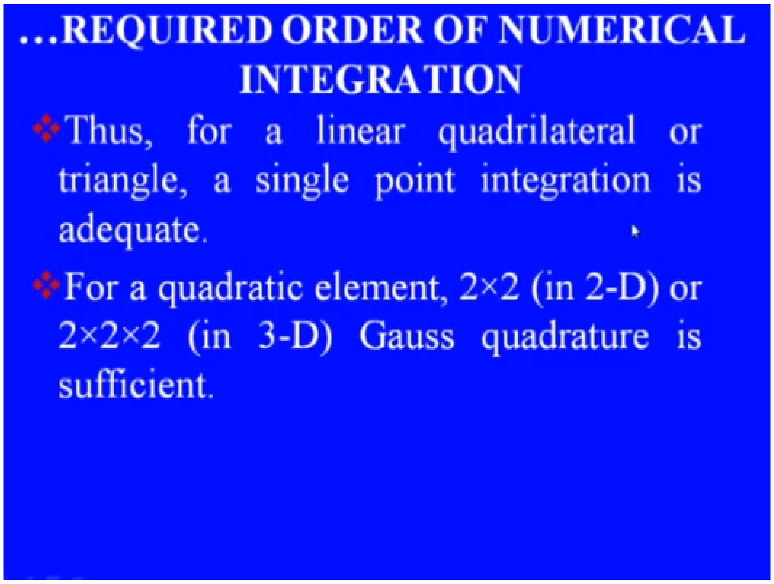

So, that is for a linear quadrilateral or triangle, a single point integration is adequate and for a quadratic element, let say if you are dealing with rectangular one, we can choose 2*2, i.e., 4 Gauss points in 2 dimensions and 2*2*2, i.e., 8 points in 3 dimensions to obtain our value of the elemental integrals. We do not need anything more than that. Same would be the case if you want to use triangular elements, use 3 points in 2D and similar extension in 3D with the tetrahedral elements.

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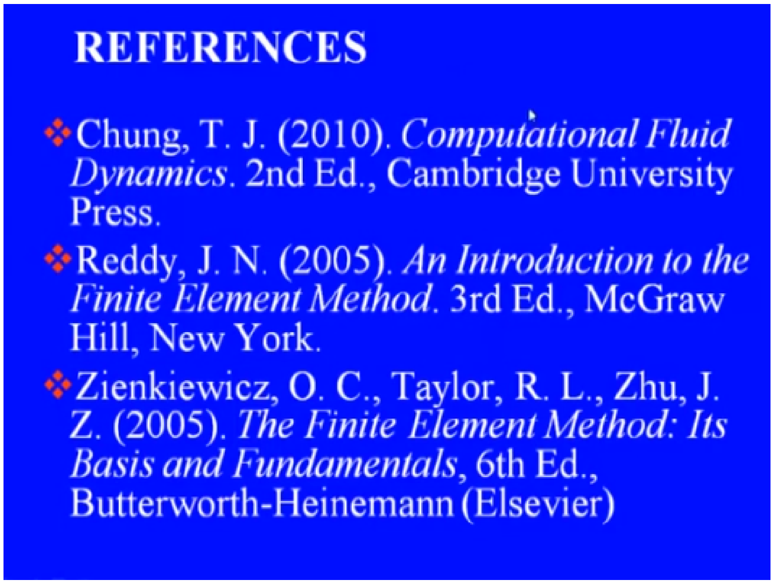

For further details on these procedures, please have a look at the book by Chung on Computational Fluid Dynamics (()) (59:53) or Reddy's book on Introduction to the Finite Element Method or the most definitive book of them all Zienkiewiez, Taylor and Zhu's book on Finite Element Method: Its Basis and Fundamentals. So, we are not going to discuss any further about the shape functions. We are going to put a stop here.

In the next lecture, we will see application of Finite Element Method to heat conduction problem.