Computational Fluid Dynamics

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Lecture – 32
Finite Element Shape Functions and Numerical Integration

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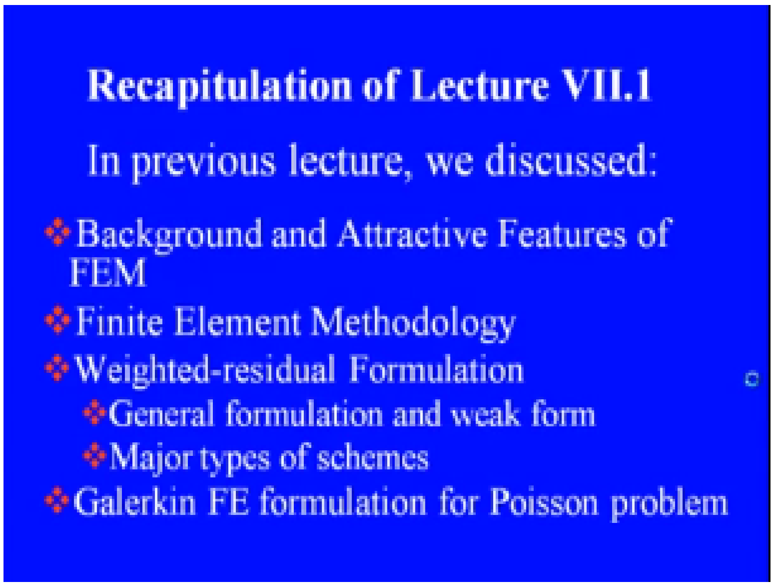

Welcome back to the second lecture in module 2 on finite element method. In the previous lecture, we had introduction to finite element and weighted residual formulation, we will next take a variation formulation, finite element shape functions and numerical integration and then applications to the scalar transfer problem in next lecture. So, let us have a recap of what we did in the previous lecture.

We discussed the background and the attractive features of finite element method, we also discussed the basic finite element methodology and we discussed one particular type of finite element formulation which is most powerful one is called weighted residual formulation. We looked at the general formulation of weighted residual method and the weak form which we can obtain from here.

And what are the major types of schemes which can result from the generic weighted residual formulation, we then applied a particular variety of weighted residual formulation which is using finite element formulation called Galerkin finite element for Poisson problems and todays lecture, we would focus primarily on finite element shape functions and numerical integration but we would also include a variational formulation, a very very brief look at this particular methodology.

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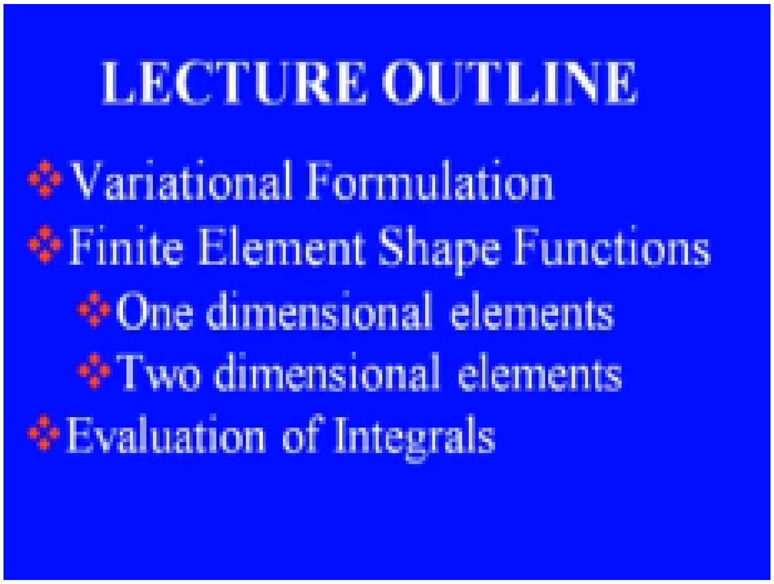

And then we will start off with finite element shape functions specifically for one dimensional problems and 2 dimensional problems in detail and then we will have a look at numerical evaluation of the integrals which occur in finite element analysis. So, let us have a look at variational principle, yesterday while looking at the basics finite element method we said, this starting point was energy minimization for mechanics problems.

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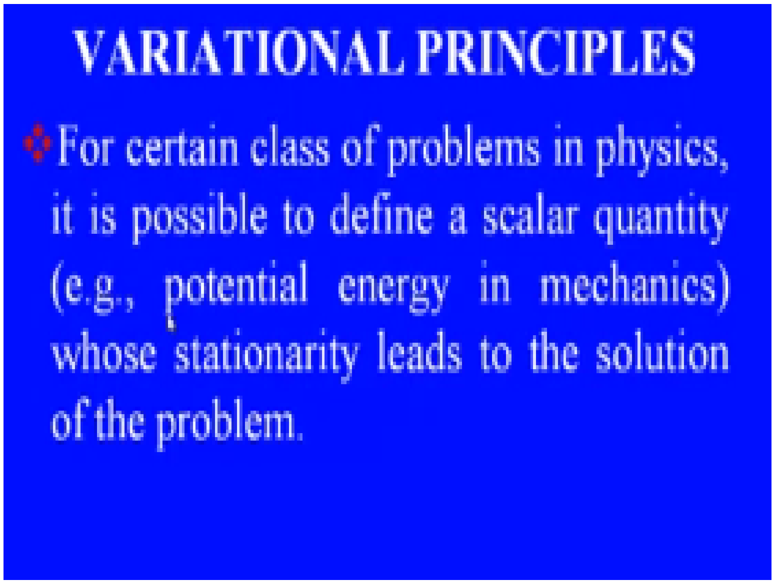

So, if you can minimize potential energy; the potential energy is a function of what is referred to as a functional and if that can be minimized then that would lead us to a state stable; state of the system that is to say our solution, so just, what happens for a certain class of problems in physics it is possible for us to define a scalar quantity, for example potential energy in mechanics, whose stationarity leads to this says; the solution of the problem.

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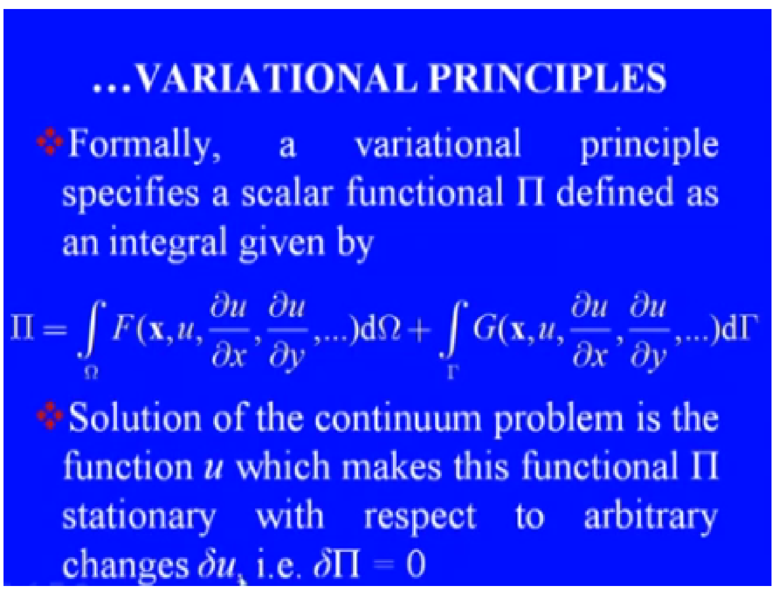

That is to say, we have to reach either an extrema; either extrema it could be minima or maxima this is what we mean by a stationarity here. So, formally we can define this variational principle as the specification of a scalar functional pi, which is defined by an integral, so pi is integral over the domain Fx, u, del u/del x + del u/ del y, so on and so forth; d Omega so, here F is an operator, x is a spatial coordinate.

And u is a dependent variable which you want to solve for plus we might have a surface integral; it is integral over gamma which is the boundary of the domain omega, yet another functional G which depends on x, u and the derivatives of u, so this integral leads to a scalar quantity which we call popularly called a functional and popular symbol, we use capital PI. Now, we seek solution of the continuum problem u by finding out a function u, which will make this functional PI stationary with respect to arbitrary changes in delta u.

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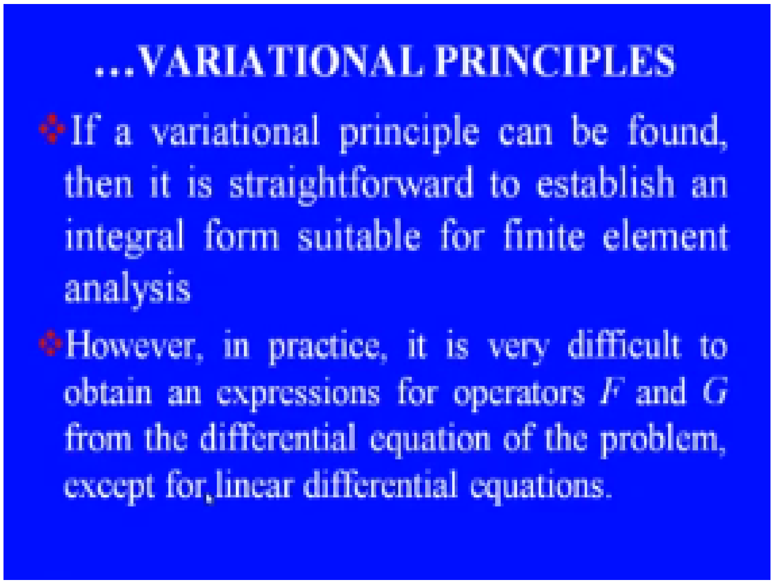

So, what we can say that arbitrary variations in delta u, which leads to delta pi = 0, if you can find such a u that u would represent this solution of our problem. Now, if we can find a variational principle then it is straightforward to establish an integral form suitable for finite element analysis we have already got that integral definition of pi and that is what we can use in our finite element methodology.

However, please note that in practice it is very difficult to obtain an expression for operators F and G from the differential equation of the problem, so normally the way we had seen when we derived our conservation equations we derived it either an integral form but that did not correspond to a variational principle and there were after we obtained a differential form. So, what we would have most of the time is a differential equation.

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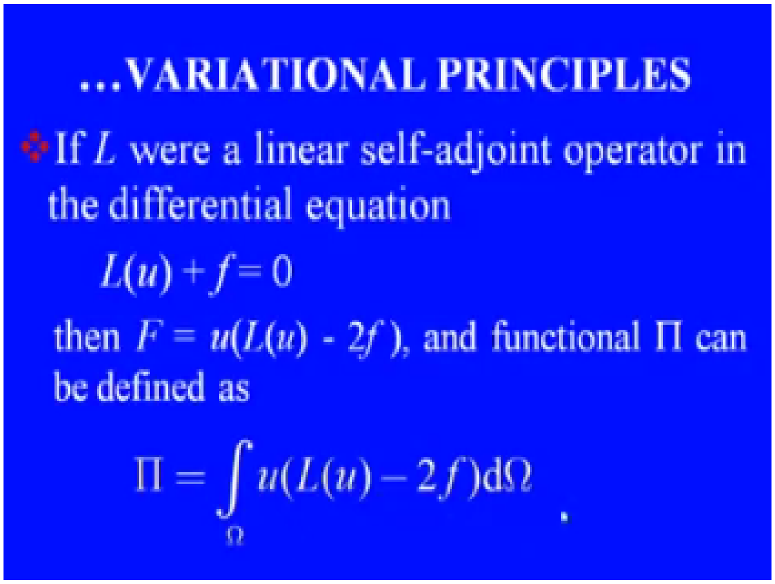

And looking at that differential equation is very difficult for us to formulate or to come up with the forms for the operators F and G except when our differential equation involves what we call a linear operator that is what we say linear differential equations. So, in the case of linear differential equation suppose, we had L was your differential operator and if it were self with adjoint operator in the differential equation Lu + f = 0.

So, in that case you can look at your maths textbook and will tell you that in such situations, we can obtain this function or operator F give which is given by u times Lu – 2f and functional capital pi can be refined as pi = integral over omega of fd omega, we substitute for F that is u multiplied by Lu – 2f, so now this is our scalar functional, we minimize it and thereby we would have our finite element formulation.

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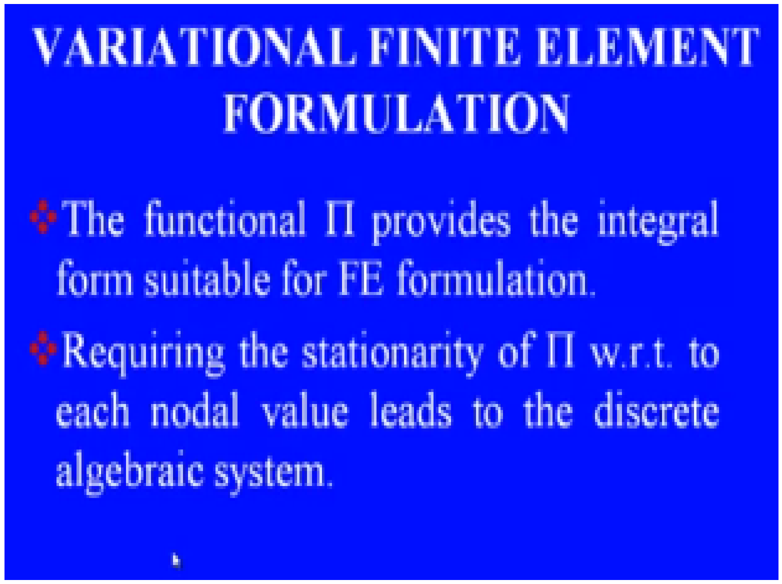

So, now let us have a look at the variational finite element formulation. We have got the functional pi, so that is the one which provides the integral form, this is similar to the weak form which you derived earlier using weighted residual formulation, so we have already got something similar to a weak form for finite element formulation and all the people require is now we would require stationarity of pi with respect to each nodal value which should lead us to the discrete algebraic system.

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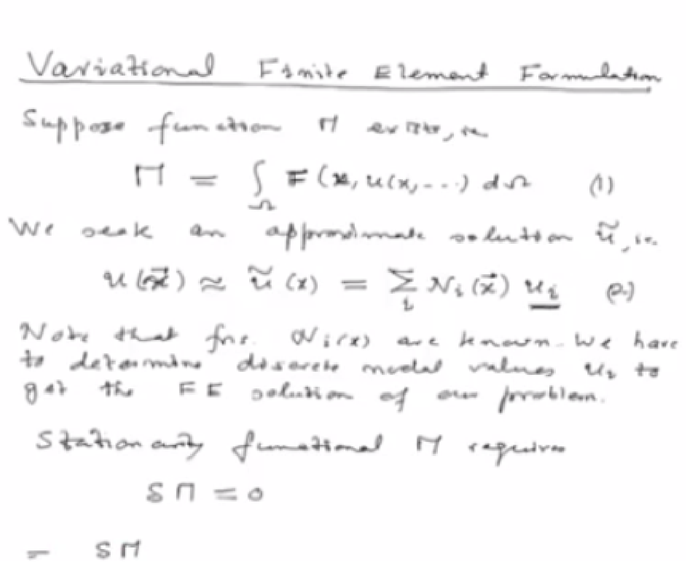

And if pi is quadratic in the proceeding step results in a linear system k = b, which will have some nice properties but before that, let us have a look at how we reach at this stage. So, variational finite element formulation, suppose we have got our functional pi defined, so our functional is pi and then we will use your finite element approximation, so suppose functional pi exists.

That is; this we have got a scalar functional pi defined as a function operator F times x, u of x and so on d omega. Now, what we want in finite element analysis; we seek an approximate solution; we seek an approximate solution; solution u ~, that is we will approximate our u of x/ u ~ x and this u ~ x defined in terms of our interpolation functions that is Ni which depend on the spatial coordinates and the discrete nodal values.

So, u ~ x is summation over i of Nix into ui and our task is it now to determine ui, so functions note that the functions Nix are known and we have to determine discrete nodal values ui to get the approximate solution finite element solution of our problem okay. Now, we want to find out of a solution or we seek a solution which would make pi stationery that is what we seek is at stationarity of pi; functional PI requires that delta of pi this is equal to zero.

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Now, any variation in delta that can be expressed now in terms of the delta pi, this is given by summation over i del pi over del ui delta ui because that is the way we have expressed our approximate solution; my approximate solution depends on these unknown discrete parameter cui, so this how; the small change in pi can be represented as this. Now, this is what we want to set as 0 okay.

So, this; let us call this equation as 3 and 3 must hold for arbitrary variation variation delta ui for each ui. So, that is possible only if the multiplier which we had in this summation is identically 0; delta ui; there could be arbitrary for each i and if we assume that looks for some I, this del pi/ del ui is positive, for some it is negative, so that itself does not hold good okay, though that it is possible for us to argue that there could be certain variation for certain values of i which are positive, for other set is negative.

So, we get a summation is equal to 0, but that will not hold good for an arbitrary variation in each of ui, so that is why what we would require that for stationarity of the functional for arbitrary variations in ui, these derivatives delta pi over delta ui, this would be 0 for each i and this is what gives us our discrete algebraic equations for discrete algebraic equations which we seek as a part of a finite element formulation.

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And of course, those discrete equations would be in terms of the element and integrals and we need to evaluate those integrals to get the final discrete system, so the intermediate steps would be similar to what we earlier looked at various residual analysis, on final discrete system would be Ku = b or K is system matrix; u is a vector of unknown quantity, b is our right hand side, so this is Nut shell, the basic variational finite element formulation.

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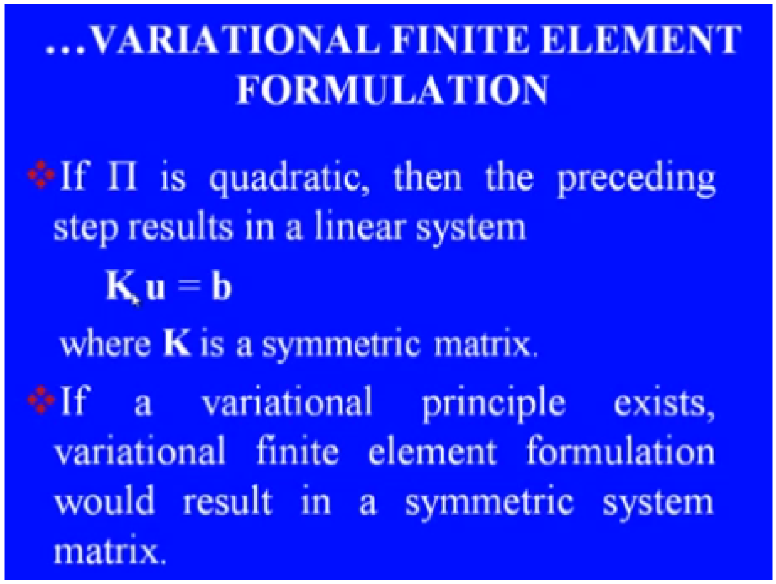

Now, there is some beautiful features of this variational finite element formulation and one of these beautiful features is that; if pi is quadratic, then the system of equations which we get from finite element formulation Ku =b, this K is symmetric and positive definite, so that gives us specific advantages in terms of solving the system. In fact, for many of the problems what has been observed in mechanics is that if there is a variational principle in existence variational finite element formulation invariably results in a symmetric system matrix.

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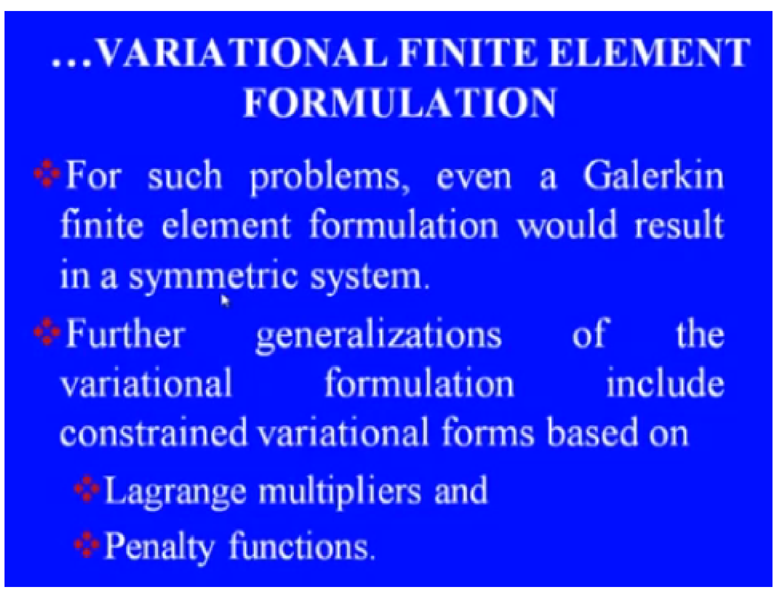

This also yet another feature in such situations that is to say that if we have a problem for which a variational formulation is possible that is to say we can find a functional in such cases, even if we imply a Galerkin finite element formulation that would also result in a symmetric system, so that is the reason why nowadays instead of trying to search and formulate the variational functional of a problem.

Most of time we prefer to go with Galerkin finite element formulation (()) (14:48) that if there were a variational formulation which you was seeking to obtain a symmetric system that we would anywhere get by using our Galerkin finite element formulation, so we just had a look at one form there are many generalization of these variation formulations available in the literature which are called constrained variational principles.

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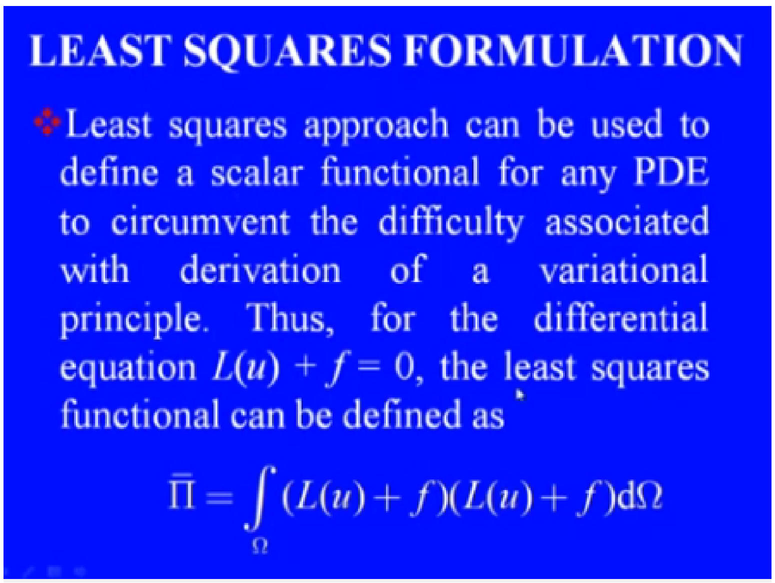

Some of them are based on Lagrange multipliers and penalty functions, for details please have a look at any standard finite element book for instance, you can have a look at the book by Zienkiewicz or Reddy we will give the full reference of these books towards the end of the lecture. There is a yet another category of somewhat similar to variational formulation wherein we can construct a functional based on least squares approach.

Now, I have just given the sketch of this approach here because this is the one which is used occasionally in finite element analysis of fluid problems, so what we can do is; instead of trying to search for a variational principle for a given partial differential equation Lu = F, if you substitute for u, our approximate solution we will get the residual or okay and now let us define a functional pi bar, which is given by R square d omega.

So, R is our residual, what we have here is square of that residual and to get a solution, we want to minimize that region, why this is called least squares formulation. So, for details once again I would encourage you to have a look at the book by Zienkiewicz. Now, let us come back to the finite element shape functions in todays lecture we are going to focus on some most commonly used polynomial shape functions.

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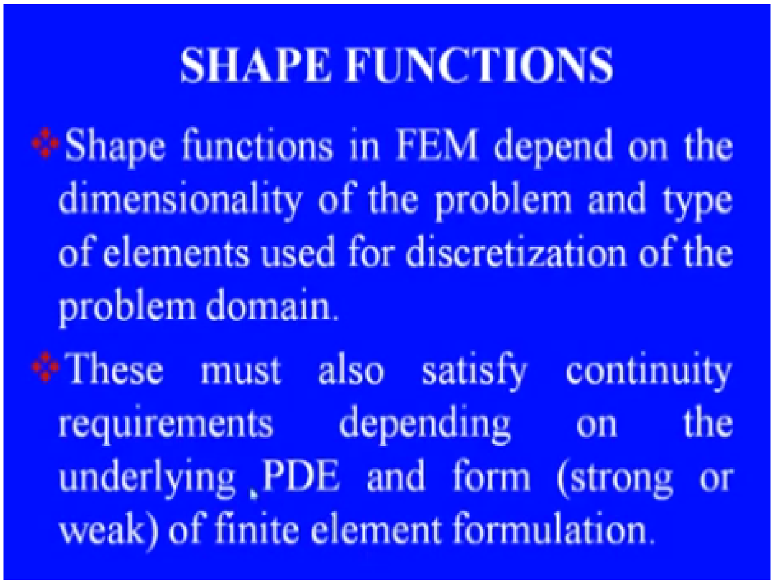

Though, if you look into the finite element books and literatures, there are a variety of finite element shape functions which have been derived for different physical problems for different scenarios, for instance; in solid mechanics we have got a set of functions which are used for plates and shells and so on but we would focus primarily on the most popular shape functions of polynomial type.

And please remember, the shape functions in finite element analysis would depend on dimensionality of the problem whether we are dealing with one dimensional problem or two dimensional problems or three dimensional problems, in each case we will have a different shape function. They will also depend on type of elements which have used for discretization whether our elements permit quadratic variation, linear variation and so on.

Or say, for instance we have used in solid mechanics we want to use a plate element or a shell element and so on. So, what type of element we have used that also dictates the form for shape function. Now, the shape functions must satisfy certain continuity requirements, now this quantity requirement will depend on the type of formulation which we have used which will specify that order to which our shape functions must be differentiable.

So, that is; what we saw in previous lecture that weak form results in a lower continuity requirements or a shape functions, so depending on let us say, if you want to use strong form or weak form finite element formulation and the order of differential operators in PDE, these two factors will govern the continuity requirements on the shape functions but there are some basic properties which our shape functions must satisfy.

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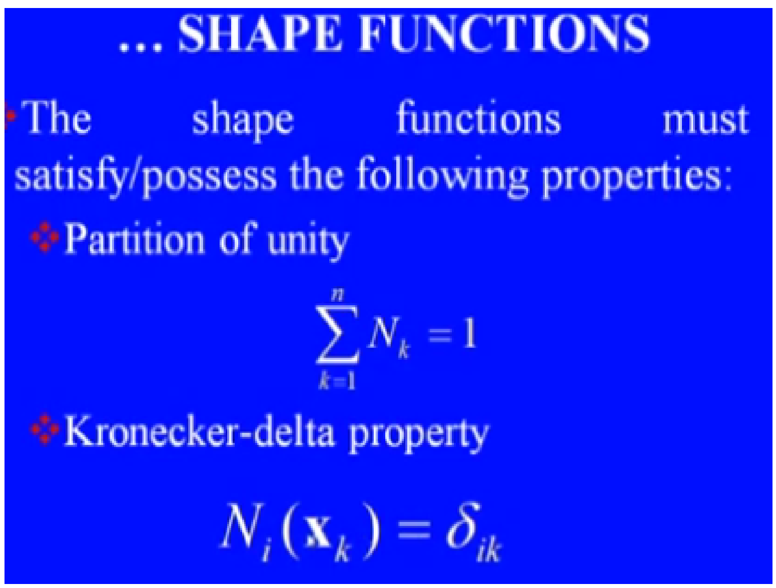

So, 2 basic fundamental property; the first one is called partition of unity that is at any point in our domain, so if we sum up all the shape functions for that element so, sigma K = 1 to N, NK at any point in that element, it should sum up to 1. So, that is this property is called partition of unity. The next is Kronecker delta property that is simply say that our shape functions, say Ni at node Xk should be equal to delta ik, so if i pertains that particular node, if i=K the shape function should take a value unity there.

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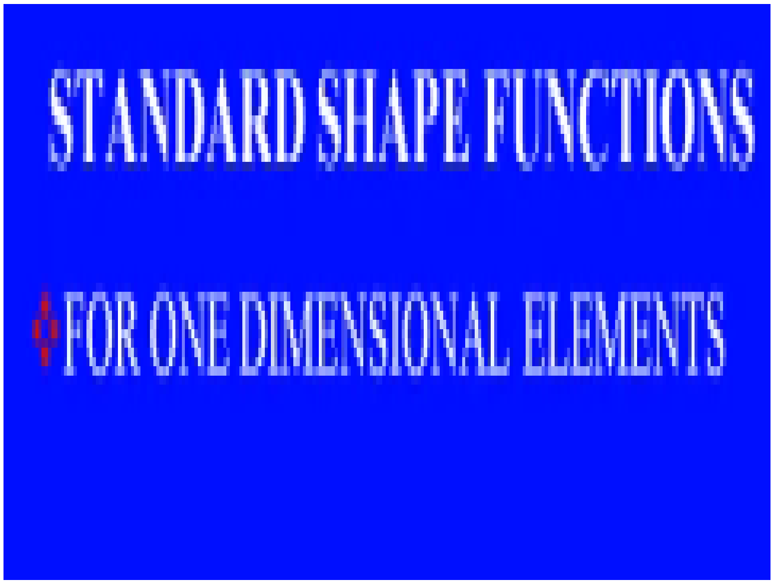

Otherwise, it should evaluate to 0 at that elemental node. Now, let us have a look at some standard shape functions, we will first discuss the shape functions for one dimensional element. We will concentrate on 2 types; 2 node linear elements and 3 node quadratic elements, for description let us switch over to our board, one dimensional shape functions.

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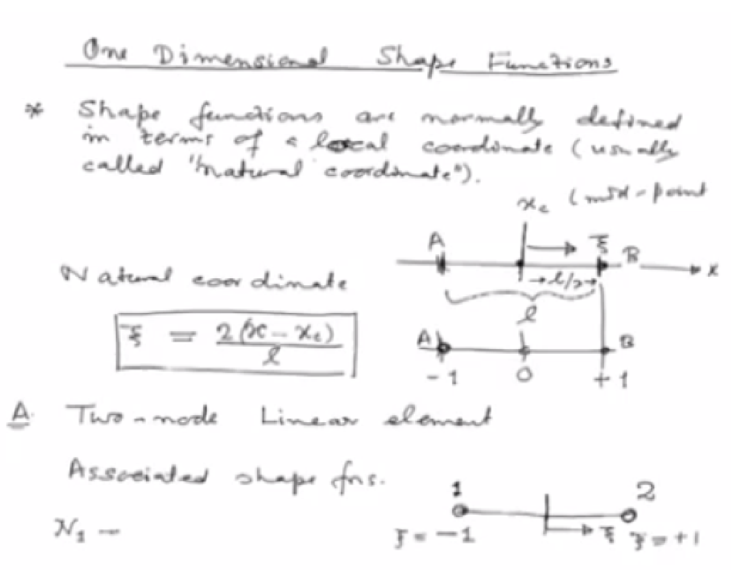

Now, it is customary to define these shape functions in terms of what we call local variables or natural coordinates, so shape functions are normally defined in terms of a local coordinate which is usually called natural coordinate. So, how would you find the natural coordinate for a one dimensional element, so let us say we have got an one dimensional element, the extent of this element in x direction, the length of this element is l.

Now, the origin of this x coordinate could be anywhere, so for each finite element if we define our shape function in terms of x, the definitions should change but if you introduce this natural coordinate we can have identical definition for shape function in each of those elements. So, how do you find the natural coordinate? We will define the origin of natural coordinate at the centroid that is the middle point of the element xc, so xc is the midpoint of our finite element.

And we would define just use the symbol xi to denote our natural coordinate, so this natural coordinate x is defined as x - xc twice of it divided by l. Now, please remember this now xc is the middle point, so at midpoint the way we have defined it should evaluate to 0, what happens to the extreme ends? So, in terms of xi, if you draw the same element; the straight again, so our xy coordinates are 0-1+1.

So, at the extreme right end is very easy to see this part as l/2, so x- xcl/2 twice of that that is l divided by l that will give us a value +1 and same holds. Similarly, we can see why the natural coordinate of the node a, should be -1, so this is our definition of natural coordinate for one dimensional elements. Now, depending on the number of nodes, we can specify what I have polynomial variations or shape functions we would like to have.

So, now let us have a look at two cases; first is what we call two node linear elements. So, we will start numbering from left, the left most node is numbered as 1 and the rightmost node is numbered as 2 and the way we have defined our coordinate system, the middle of the element that is where the origin of xi lies, node 1 has got coordinate -1 and node 2 has got coordinate as xi=+1, this is xi=-1.

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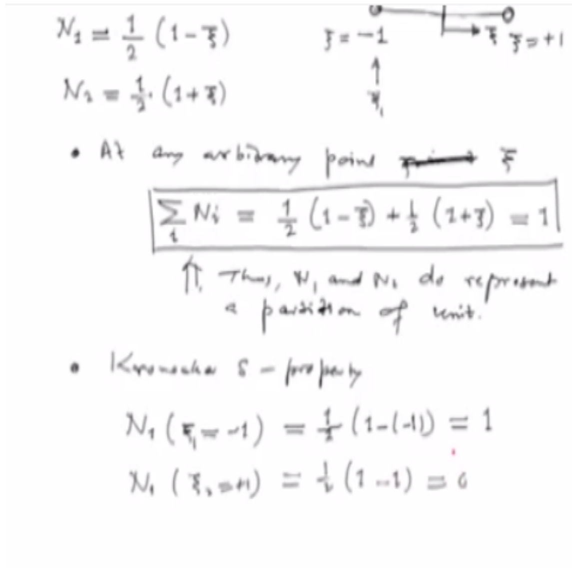

And associated shape functions in this case we will have 2 shape functions; shape functions N1 subscript 1 say it now, this is associated with the node 1, this would be defined as 1/2 into 1- xi and N2 is given by 1/2* 1+ xi. Now, you can easily verify the 2 properties, which we have mandated the shape functions should have okay at any; at any arbitrary point xi in the element, what happens to this summation of this shape functions?

We have got 1/2 1-xi + 1/2 1+xi, which gives us 1, so these shape functions they do satisfy, so thus N1 and N2 do represent a partition of unity. Similarly, can you do verify the Kronecker delta property? Let us see, what happens to Kronecker delta property. N1 at; let us call the coordinates of node 1 as xi1; N1 x i1, where xi is -1. What is this? This 1/2 * 1-, -1, which is equal to 1 and what will happen if we evaluate N1 at xi2 = +1?

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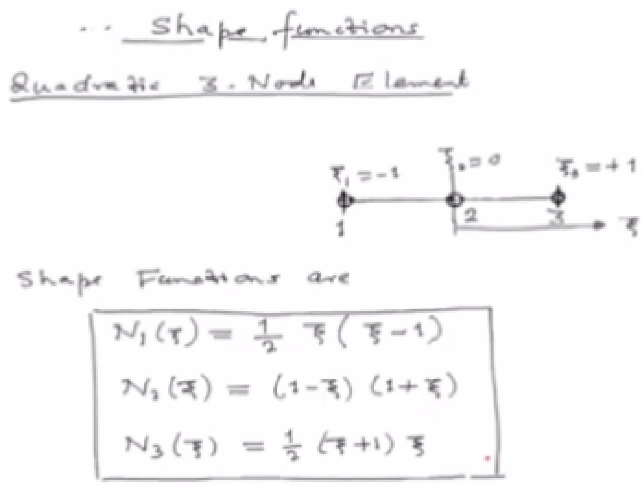

This is 1/2 1-1=0. Now, let us have a look at one more one dimensional element. Let us call it quadratic 3 node element. Now, if you have got this 3 node element, let us draw our element diagram; this is centre point which is the original of our local coordinate system xi, the left most node is numbered as 1, the centre node we will number it as 2, the centre node is at xi2 = 0, xi1 = -1 and the rightmost node we will number as 3; xi3 = +1.

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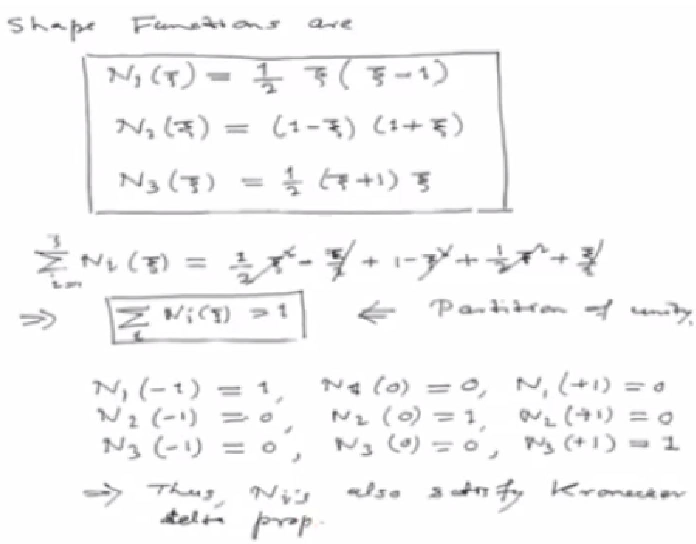

Now, the shape functions are N1xi, this is given as 1/2 xi * xi -1, the shape function a link to the middle node this is given by 1-xi 1+xi and the last one N3xi that is given as 1/2 xi+1 * xi, so these are shape functions for quadratic one dimensional elements. Now, do this satisfy the two properties which we require that is our partition of unity, so let us sum them up; sigma Nixi; 1=1, 2, 3 and what do we get? 1/2 xi square – 1/2 + 1 - xi square + 1/2 xi square + xi/2; xi/2 gets cancelled; now these 2 xi square, these once again will cancel.

So, what we are left with this now is; sigma i Nixi =1, so thus these shape function they do satisfy this partition of unity property, they also satisfy Kronecker d elta property which we can easily see, for instance; our N1 at -1 this is equal to 1 but if you evaluate N1 at 0, it is 0. Similarly, N1 at +1 is also equal to 0, the same holds good with N2, N2 at -1 is 0, N2 at the node point 2 is equal to 1 and N2 at the node point 3 is equal to 0.

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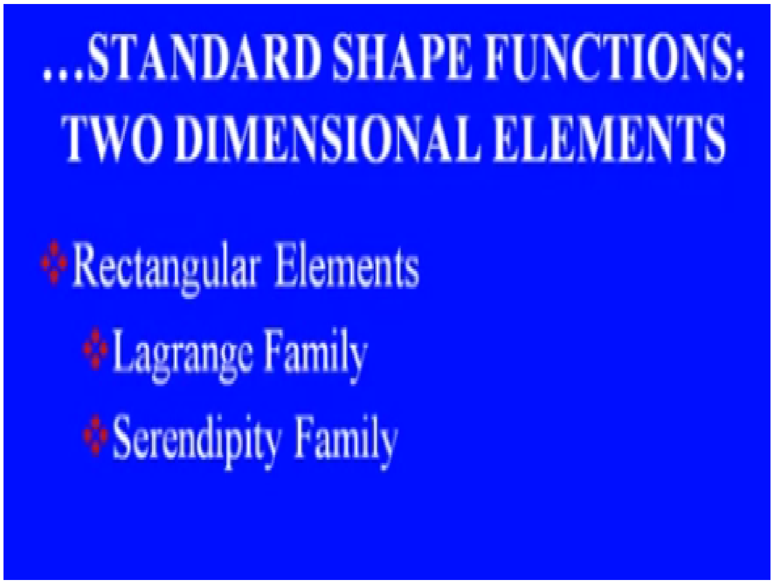

Similarly, our N3-1= 0 N30 is also 0 but N3 at the node 3 that is N3 xi =1= 1, so thus these shape functions our Ni is also satisfy Kronecker delta property. Next, let us have a look at some of the standard shape functions for two dimensional elements, in two dimensions we can have either rectangular elements, in the polynomial class we have got two families; Lagrange family and Serendipity family.

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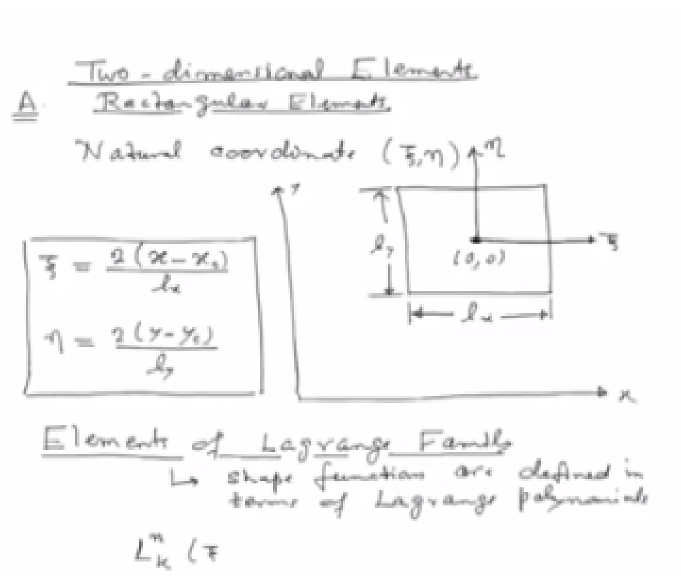

We will have a look at what are these two families separately and then triangular elements, so two dimensional elements; you will first look at rectangular elements, they will be using the word rectangular, the shape functions can also be used for the curvilinear elements which have got 4 sides by using what we call isoparametric mapping or these can also can be used for the quadrilaterals of arbitrary type.

But for the basic definitions, let us stick to that rectangle shape and here again we need to define our natural coordinates, we need to introduce now our 2 natural coordinates; xi and eta, so if you are looking at our xy coordinate system, let us say we have got a rectangular element, the centroid of this element that is what would be chosen as the origin of our natural coordinate system. So the one local coordinate parallel to x that is what we will call as xi, the one parallel to y direction that would be called eta zero zero okay.

And their definition is very similar to what we had defined earlier for the one dimensional case that if the extents of this element; length of this rectangular element in x Direction is lx and its length in y direction is ly, we can easily see the definition of this xi would be defined as twice of x - xc divided by lx and eta would be defined as twice of y - yc divided by ly. So, this is the basic definition of our natural coordinates for a rectangular element.

And now in terms for these coordinates, we will define the shape functions for rectangular elements. First, we will have a look at the elements of Lagranges family, why we call them elements of Lagrange family is because of simple reason that the shape functions are defined; shape functions are defined in terms of Lagrange polynomials okay, that is say our typical Lagrange polynomials is called as L.

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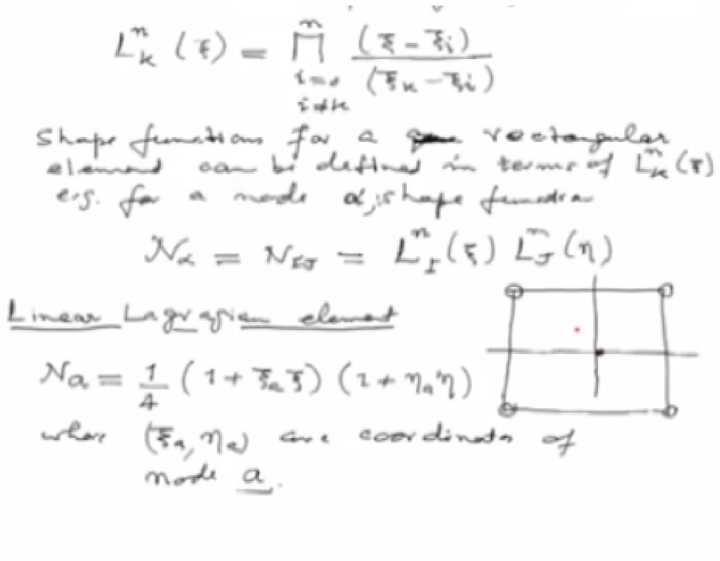

L kn for a one dimensional variable xi, this is defined as a product of i=0 to n, i0 = k x-xi divided by xik – xi, so this is a definition of a Lagrange polynomial. Now, in terms of Lagrange polynomial, the shape functions are defined, so shape functions for a quadratic element; sorry a quadrilateral or rectangular element can be defined in terms of Lk xi, that is for example, for a node a.

Or let us use this index alpha, the shape function N of alpha that can be represented as N capital I, capital J, where this capital I capital J can be used to represent that which side; left side or right side we can assign the integer indices for these ones. similarly we can also assign the integer indices for the bottom side and the top side and now this Ni is in fact it represents in or it is expressed in terms of these Lagrange polynomials; Ni n xi LJ m eta, where this n and m will tell us that how many number of divisions we have got in that particular direction.

To make it clear, put it more clearly let us define linear elements, so linear Lagrangian element, now this linear Lagrangian element would have basically 4 nodes at each corner of this rectangle and this shape function Na can be represented as 1/4 of 1+ xia * xi 1 + eta a * eta where xi a, eta a are coordinates of node a, so what you can easily see? that these are basically the products of the shape functions in one dimension.

So, what we had shape function for one dimension; 1/2 1+-xi *xi, so this 1+ or –xi, so this is what we will have here that xi a is eta a, they can take the values +1 or -1, okay so the shape functions for the linear Lagrangian element, they are basically product of corresponding linear shape; one dimensional shape functions. The same thing can happen in fact we will have to repeat the thing as same way.

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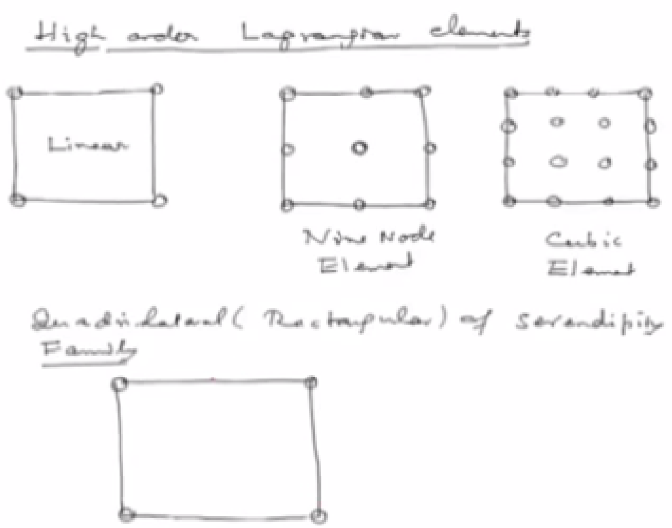

If you take higher order elements quadratic or cubic, we take the corresponding dimensional shape functions in the two directions multiply them and we will get the shape functions for higher order Lagrangian elements. For details, you can see the book by zienkiewicz or just like to illustrate one point that relates to the placement of nodes; in the case of linear elements just we had 4 nodes at the 4 vertices.

But if you take quadratic element, so this is our linear rectangular element, if you take quadratic rectangular element of Lagrangian type, we will have 3 nodes on each side plus we will have one node which one sides with the centroid, so this becomes this is a 9 node element. Similarly, if you want to have cubic one that will have many more interior nodes, so this is cubic element, so if you do not want these interior nodes, we can go for what we call quadrilateral elements quadrilateral or rectangular elements of serendipity family.

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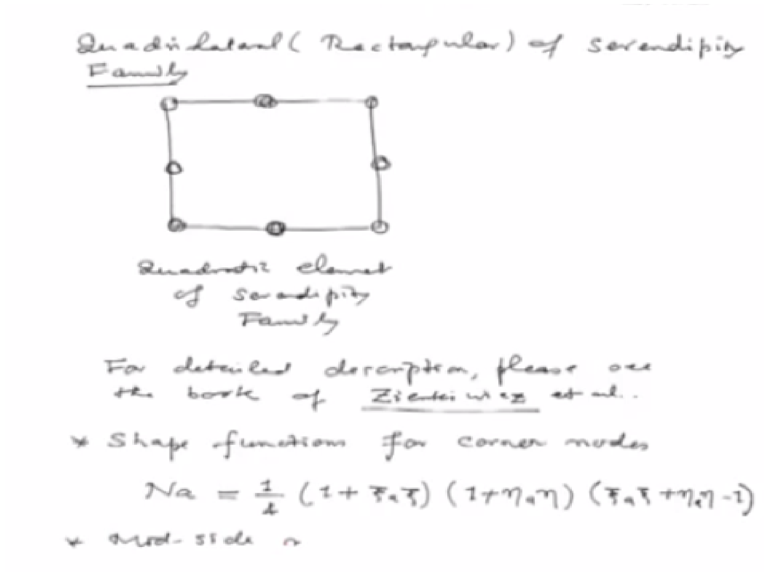

The linear one, of course did not have any node in the middle, so we do not have to worry about the linear node but if you go for the quadratic one say for instance our cubic ones, we can obtain the shape functions or we can differentiate shape functions such a way that we do not need any interior node, so this is our quadratic element of serendipity family. The same as way we can define our cubic element of the serendipity family.

For details; for detailed description please see the book of Zienkiewicz et al. Now, the shape functions or for this element would differ depending on where the node is located. So, suppose we really mean the corner nodes, these are defined the same way as we had earlier had with slight modification, so this is 1/4 1+ xia * xi * 1+ eta a * eta and plus a modifier term, so multiplied a modifier term, so xia xi + eta a eta -1.

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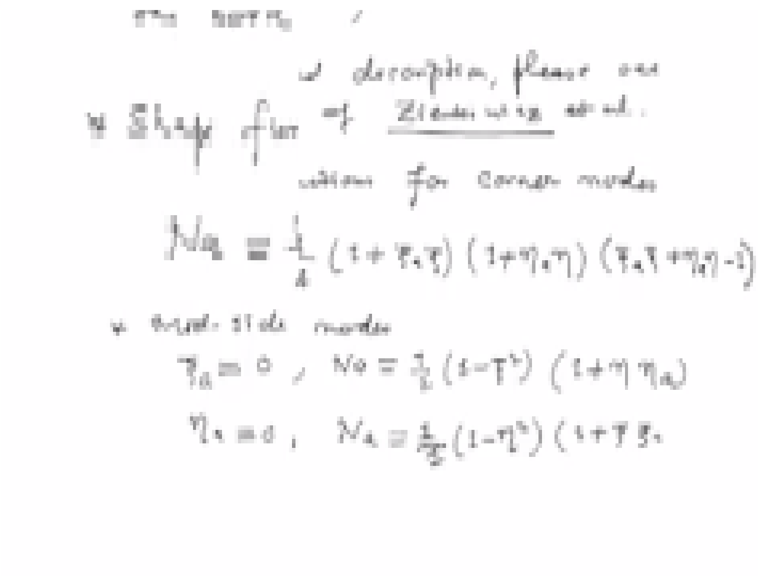

And if we had the mid side nodes or the nodes which are sitting at the centres of each of the sides, so mid side nodes a slightly different version, so if we have let us say the node which sits on the sides defined by unless, whose xi coordinate is 0, so then the shape function is defined as 1/2 1- xi square 1+ eta times eta a and if is eta n = 0, which will correspond to two mates or side nodes.

So, in this case the shape functions Na would be 1/2 1- eta square 1+xi xi a, so in this way you can find the details of the shape functions for other higher order elements and for the further description, we will take up the next familiar of elements, the rectangular elements and numerical integration in our next lecture.