Computational Fluid Dynamics
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Lecture - 03
Conservation Laws and Mathematical Preliminaries

Mathematical modeling is the prerequisite for any theoretical or computational analysis of a flow problem. So we would first have a look at the mathematical modeling of flow problems in this module. We would specifically have a look at basic conservation laws of physics.

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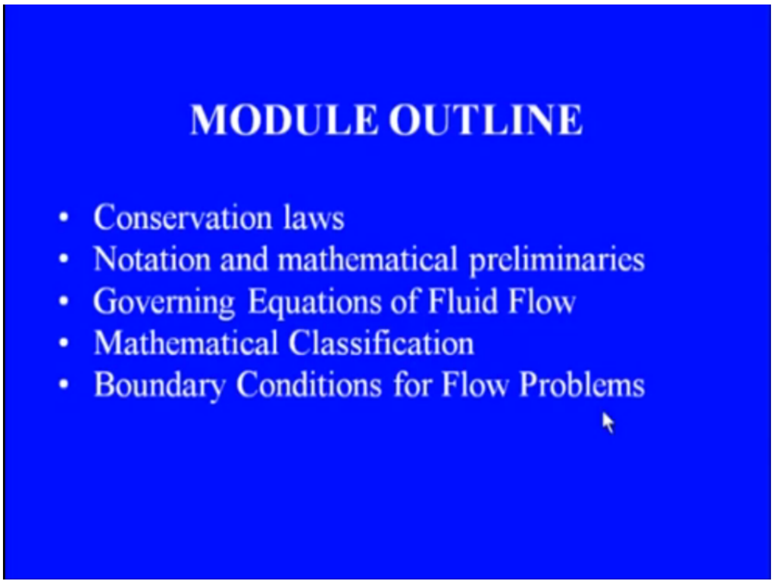

Next, we will have a look at the notations which we employ in mathematical analysis of flow problems. Then we will have a look at derivation of governing equations of the flow problem. The next lecture we will focus on mathematical classification of the governing equations of the fluid flow and in the last lecture in this module, we will have a look at boundary conditions for flow problems.

Let us first start with the 1st lecture in this module, which is conservation laws and mathematical notations and some theorems, which we would need to derive the basic governing equations of the flow problems.

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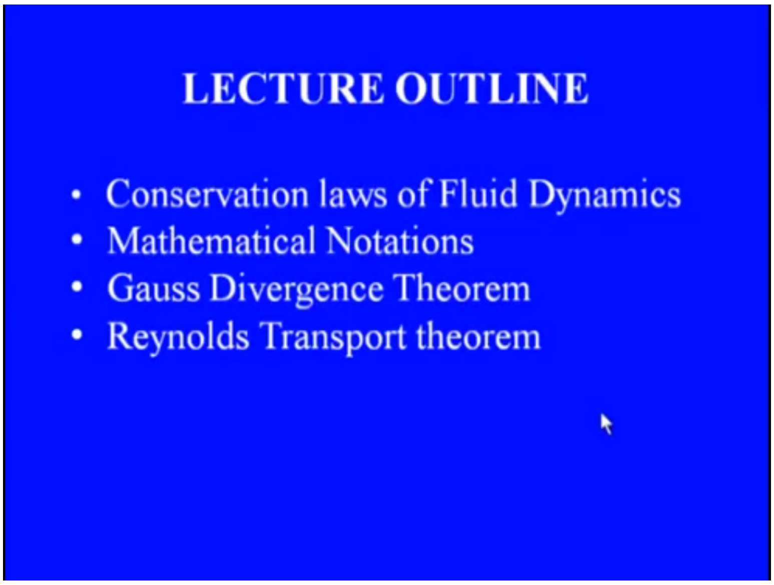

The outline of this lecture would be we will first have a look at the basic conservation laws of fluid mechanics, then mathematical notations, which we adopt in writing the fluid mechanic equations then we would look at few theorems, which we would use in the derivation of the governing equations specifically changing from one form to another namely Gauss divergence theorem and Reynolds transport theorem.

First what are conservation laws of physics?

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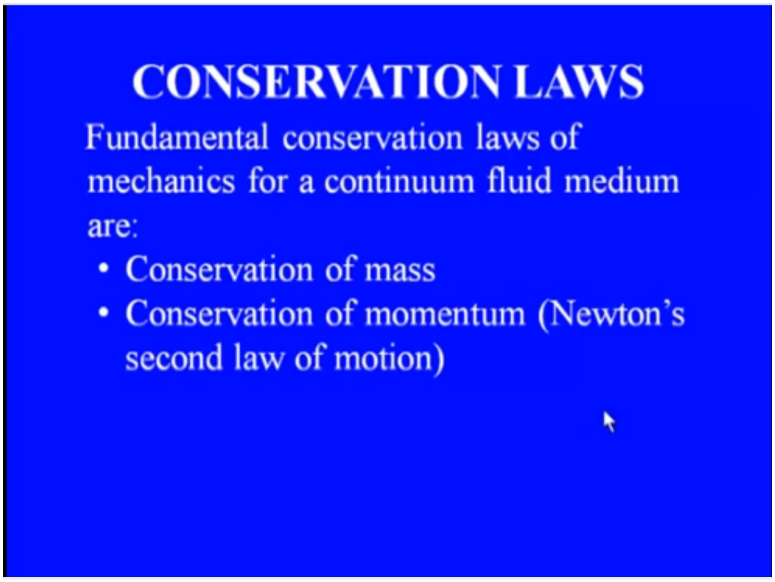

The fundamental conservation laws of any medium whether it is fluid or solid it remains the same. It is the basic conservation law of mechanics and these are the conservation of mass. For any system in a non-relativistic framework, the mass of the system remains constant. So that will be the first law, which we would use to derive an appropriate governing equations for flow problem.

The next basic law is the conservation of momentum, which essentially equivalent to the Newton's second law of motion, which would be the most important law in the fluid flow. The next one would be conservation of energy again in a non-relativistic framework. This is essentially the first law of thermodynamics.

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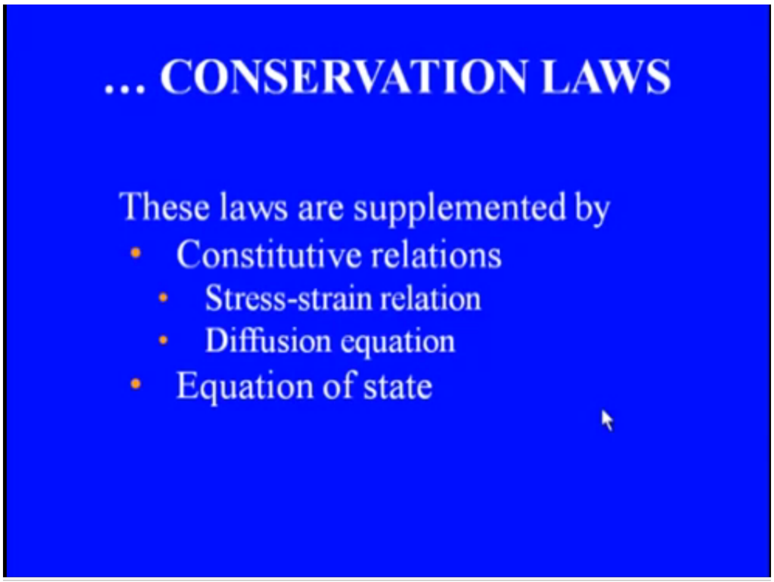

Now these laws they are not sufficient by themselves. They require certain supplementary equations. They are referred to as constitutive equations. For instance, we would need equations to relate stress and strain rate in the fluid mechanic problems and stress and strain relation for a solved mechanics problem. Similarly for diffusion of a species or a scalar transport, we would need something equivalent to Fick's law or Fourier's law.

Then we would also need the equation of the state specifically for compressible fluid flow.

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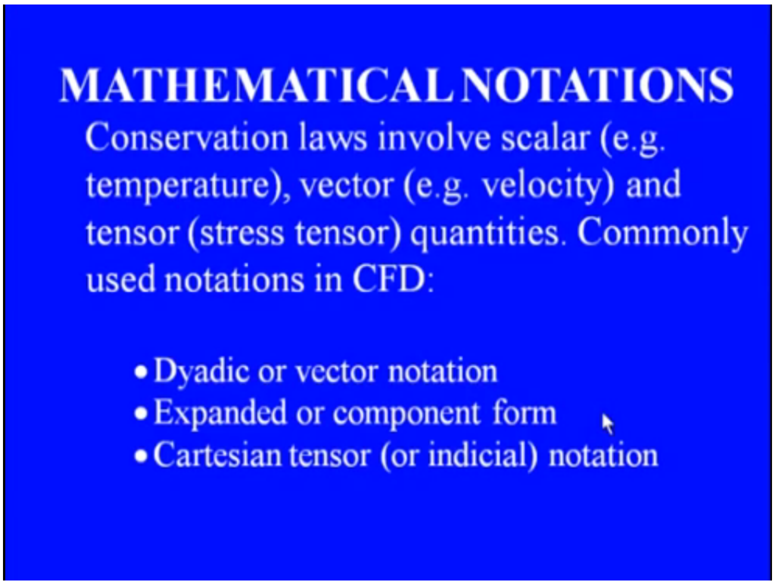

Next, let us have a look at the mathematical notation, which we commonly use in fluid dynamics specifically in computational fluid dynamics. The conservation law they will involve scalar quantities for example temperature, pressure and density, vector quantities for example velocity and forces and tensor quantities like stress tensor. The commonly used notations in CFD are dyadic or vector notation, expanded or component form or Cartesian tensor notation.

Now they are noticed in which are preferred by the fluid mechanicist or the engineers. There is yet another notation which is commonly used in research literature that is by mathematicians. They do not make any difference whatsoever with regard to the quantities involved.

All the quantities are taken as if there is a tensor of a specific order and context would make it clear whether we are referring to a tensor of order 0 that is scalar or tensor of order 1 that is a vector quantity or tensor of order 2 that is a second order tensor, but engineers they prefer to adopt different notations for different quantities so that equations become very clear at the first glance.

So let us have these 3 notations which are used by the engineers or physicist in fluid dynamics one by one. Let us first have a look at what we call dyadic notation. This is also referred to as a vector notation. In dyadic notation, we would use normal type face for scalar quantities.

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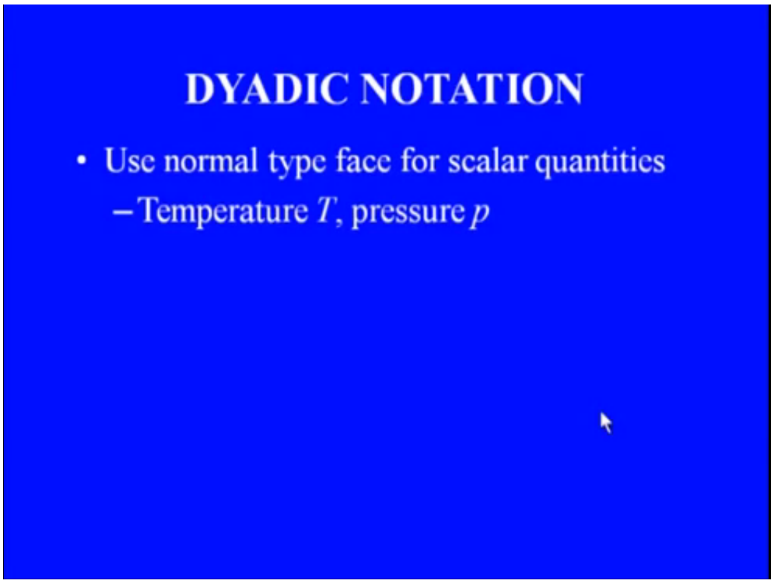

For example temperature would be denoted by simple italics T, pressure by P, density by rho and so on. For a vector or tensor quantity, we would use a bold face type in the printed material and in handwritten material we would use normally an arrow mark or under bars to denote the tensors of different order. For example, velocity vector would be denoted by simply boldly and stress tensor by bold tau or a bold capital T.

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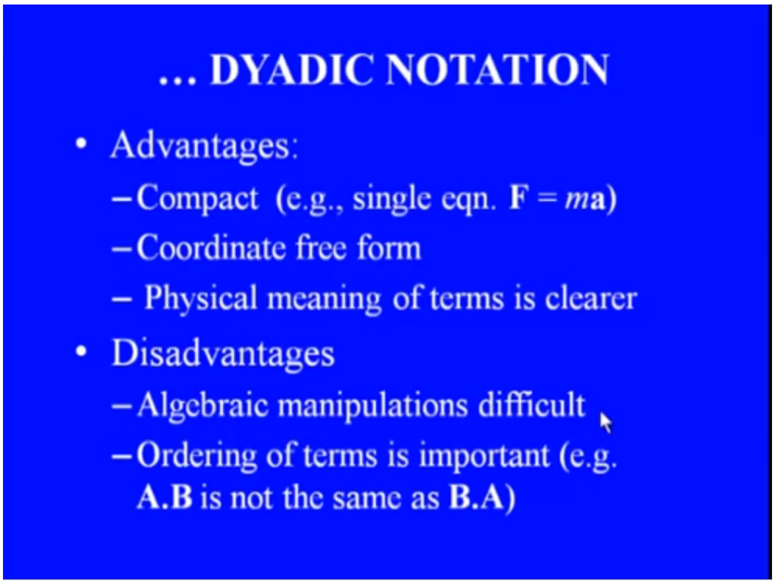

Now what are the advantages of a dyadic notation? The advantages are we get a very compact form. For example, for Newton's second law of motion in the vector form we would simply write F=ma whereas if you are supposed to write it in the component form we would need 3 separate equations, Fx=max, Fy=may and Fz=maz if we have chosen a Cartesian reference frame.

Whereas in the case of dyadic notation, we just need to write a single simple equation F=ma which clearly tells us that it is not dependent on the coordinate so that is why this dyadic notation also referred to as coordinate free form and the physical meaning of terms are very clear. The capital bold F indicates it is a force and m simple without bold it says this is a simple scalar quantity mass and bold a that denotes it is a vector quantity acceleration.

So the physical meaning of the terms in equation written in dyadic or vector notation is very clear, but what are disadvantages? The algebraic manipulations are pretty difficult. We have to remember various different formulae to manipulate the equations written in vector or tensor form. Ordering of terms is very important for example, if we have 2 tensor quantities A and B, A dot B that is dot product of A and B is not the same as B dot A.

So the order of terms is very, very important when we write in dyadic notations and we have to be very careful in manipulating the equations written in dyadic or vector notation. The next form is our expanded form which would depend on the choice of the coordinate system that is to say whether we have chosen a Cartesian reference frame or a cylindrical polar coordinate system or spherical polar coordinate system.

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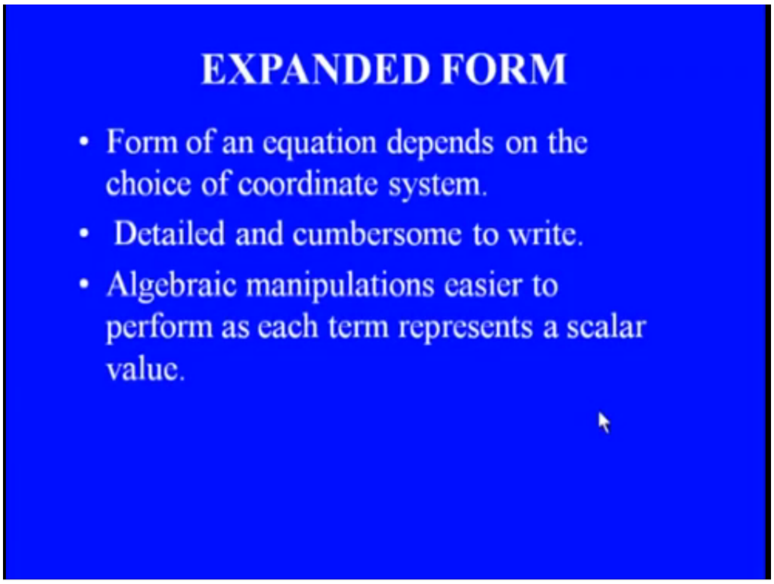

The equations are detailed and they are cumbersome to write. At the same time, algebraic manipulations are very easy to perform as each term in the equation represents a scalar quantity and order of terms in a particular equation they are not really important and for final numerical discretization or computer programming, we would actually write all the flow equations in expanded form.

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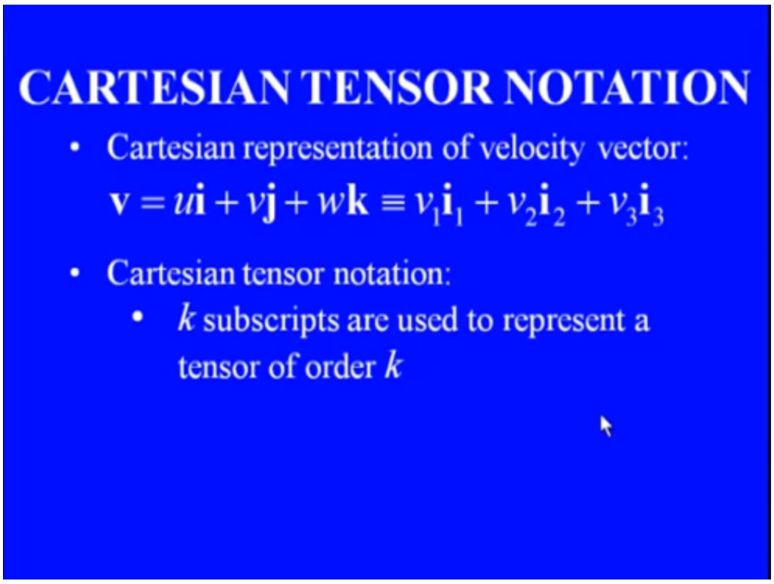

The last one which we will have a look at is what we call Cartesian tensor notation. This is primarily used in manipulation of the different equations or schematic representation of different algorithms. Let us have a look at 1 quantity. For instance, if we have chosen a Cartesian coordinate system, velocity vector v would be represented by 3 Cartesian components u, v and w where i, j and k they denote the unit vectors in x, y and z directions.

Now this could also be written equivalently as v1i1+v2i2+v3i3 whereas i1 represents the unit vector i at unit vector x direction, i2 represents the unit vector in y direction and i3 represents the unit vector in z direction. So in fact, in Cartesian tensor notation, we would represent our coordinate system o, x, y, z as ox1, x2 and x3 and how would you represent a particular quantity?

We would use case subscripts to represent a tensor of order k. So if we want to represent a scalar quantity, there is no subscript required, simple italic symbol would do. If you want to represent a vector quantity, we will use one subscript for instance v subscript i that denotes the velocity of vector.

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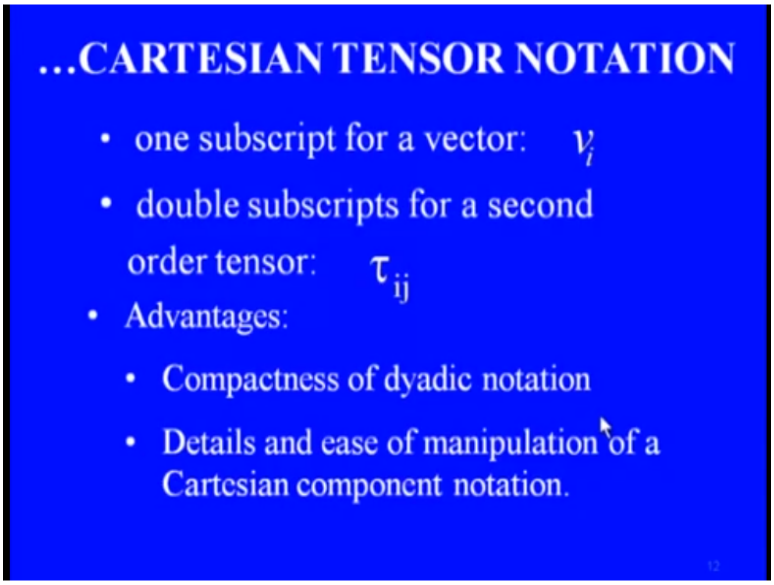

Similarly 2 subscripts they would denote a second order tensor for instance tau subscript ij indicates a tau is the tensor of order 2. The advantages of the Cartesian tensor notation is we get the compactness of the vector or dyadic notation and details and ease of manipulation of a Cartesian component notation since all the terms in a Cartesian tensor notation they are scalar quantities.

So we can easily manipulate equation written in Cartesian tensor notation. Now there are certain conventions which we need to be aware of and we want to use Cartesian tensor notation and the very first one is what we call summation convention which is primarily first order of with Einstein so it is also known as Einstein summation convention. So whenever we have a repeated index in a term, it implies summation over the range of that index which is 3 in a 3-dimensional space.

Similarly if you are dealing with n dimensional vector space the indices will run from i=1 to n.

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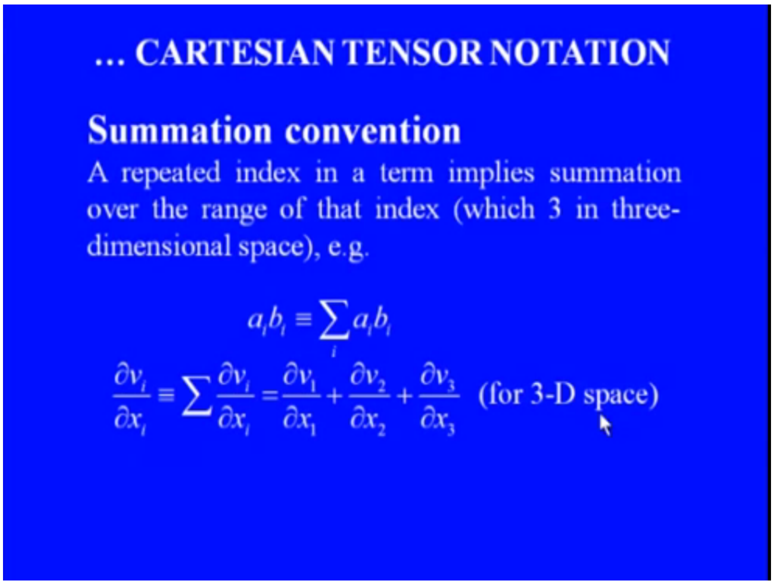

So if you write for example, ai, bi this would represent the sum over in ai, bi. So in fact ai, bi it represents a dot product of 2 vectors a and b. Similarly del vi/del xi, this represents a summation of del vi/del xi that is del v1/del x1+del v2/del x2+del v3/del x3 in 3-dimensional Cartesian space. We would frequently need a specific tensor quantity which is called Kronecker delta in our manipulations.

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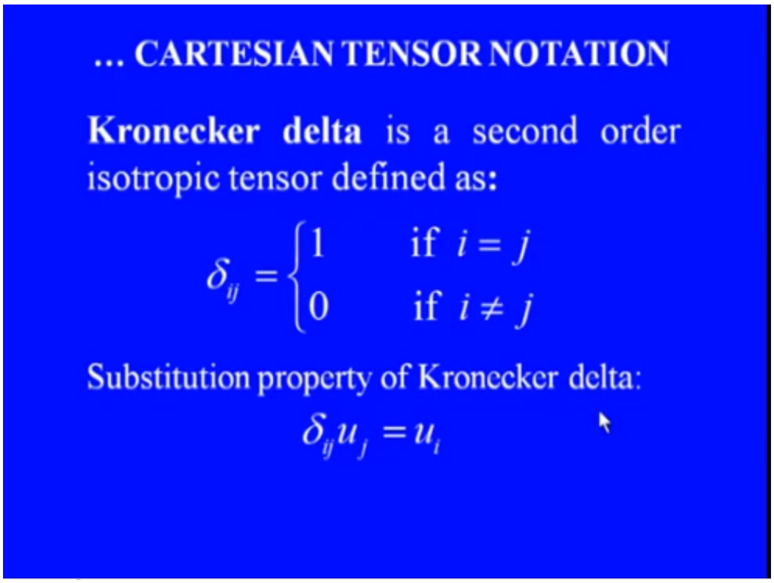

So Kronecker delta is a second order isotropic tensor, which is defined as delta ij=1 if indices i and j are equal and it is equal to 0 if i and j are not equal. This particular tensor has got a specific property, which is called substitution property of Kronecker delta that is if delta ij uj=ui that is the subscript j has been replaced or substituted by the index i.

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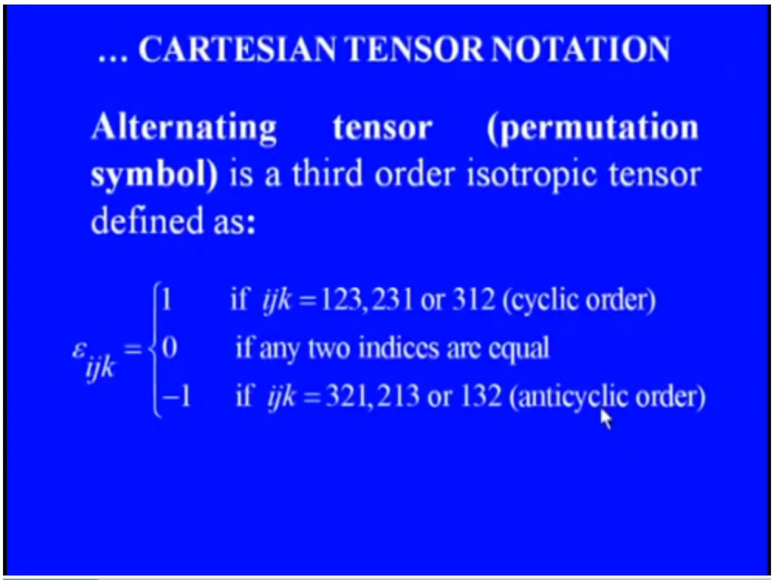

The next important tensor quantity, which is used in Cartesian tensor notation is alternating tensor or permutation symbol, which is the third order isotropic tensor, which is defined as epsilon ijk=+1 if ij=123, 231 or 312 that is indices i j and k follow a cyclic order and it is equal to 0 if any 2 indices are equal, it is equal to -1 if the indices i j k they follow the anticyclic order that is ijk=321, 213, or 132.

This alternating tensor we will primarily need in cross product of vectors.

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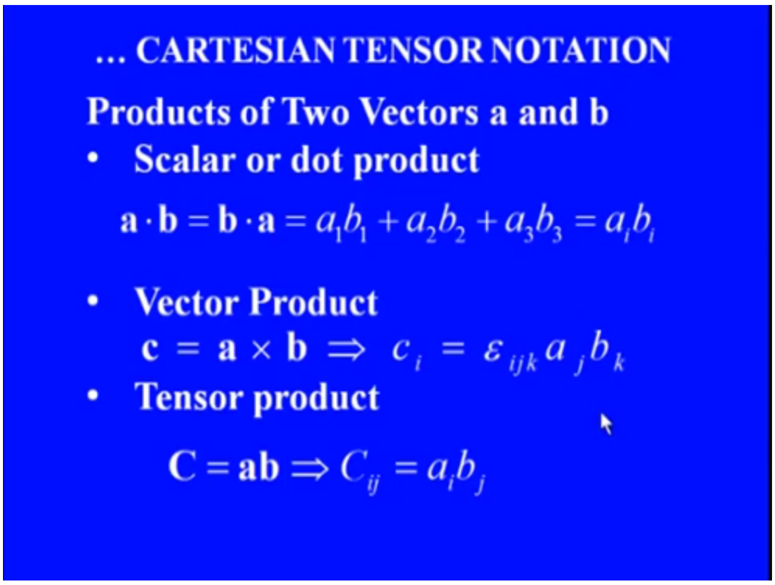

Now let us have a look at different products, which we can form with 2 vectors and how do we represent them in Cartesian tensor notation? Suppose let us deal first with scalar or dot product in this case the order of terms is not very important. So a dot b=b dot a=a1b1+a2b2+a3b3, which we can compactly write as ai bi in our Cartesian tensor notation.

Vector product C=a cross b this is written in our Cartesian tensor notation as ci=epsilon ijk aj bk where ci represents the highest component of vector C and aj and bk are corresponding components of vectors a and b respectively and epsilon ijk is our permutation tensor. Similarly we can also form a tensor product of 2 vectors for instance we have got 2 vectors a and b, C=ab this we can write indicial notation as Cij=ai bj.

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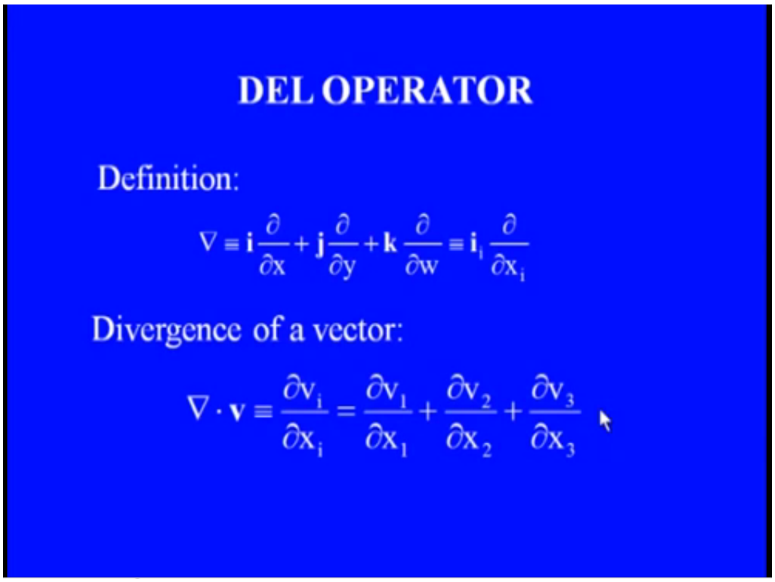

We would very often use a differential operator, which is called del operator and it is defined as del=i del/del x+j del/del y+k del/del w where i j and k these are unit vectors in the Cartesian x, y and z directions respectively. In indicial notations or Cartesian tensor notation, we can write as i subscript i del/del xi. Now this differential operator can be used to form both a dot product or it can be applied directly to a scalar quantity or any tensor quantity.

For instance, if we take dot product of this del operator with a vector v so del dot v this is equivalent to del vi/del xi or in expanded form it equates to del v1/del x1+del v2/del x2+del v3/del x3 where v1, v2 and v3 are the components of vector v in x1, x2 and x3 directions respectively.

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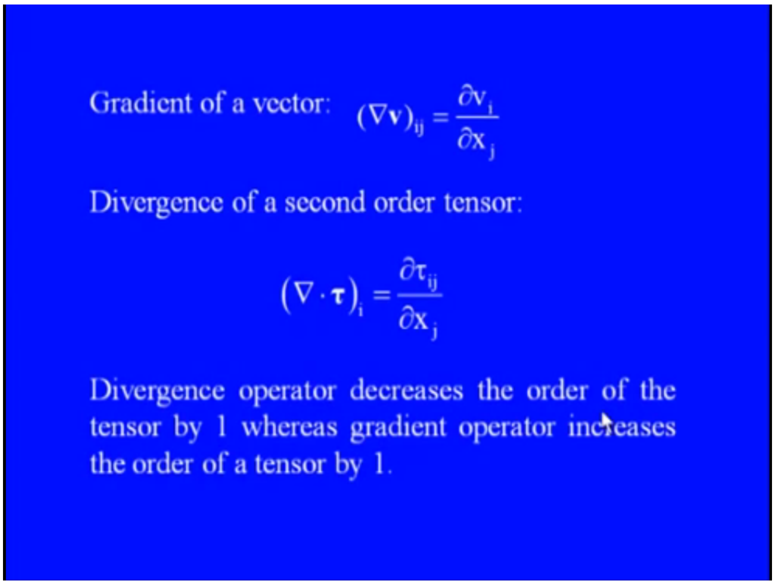

Now this operator when it is applied to vector v it leads to a tensor quantity which we call gradient of a vector. For instance, del v subscript ij=del vi/del xj. Now please note that when it is operated on a vector quantity this gradient operator gives us a tensor of second order. Similarly if you take divergence of a second order tensor, divergence of tau for instance, divergence of tau will give us a vector whose highest component will be given by del tau ij/del xj and please remember here summation is employed over the index j.

So if you look at these 2 equations which on the slide which we can clearly observe is the divergence operator decreases the order of the tensor by 1 whereas the gradient operator increases the order of the tensor by 1.

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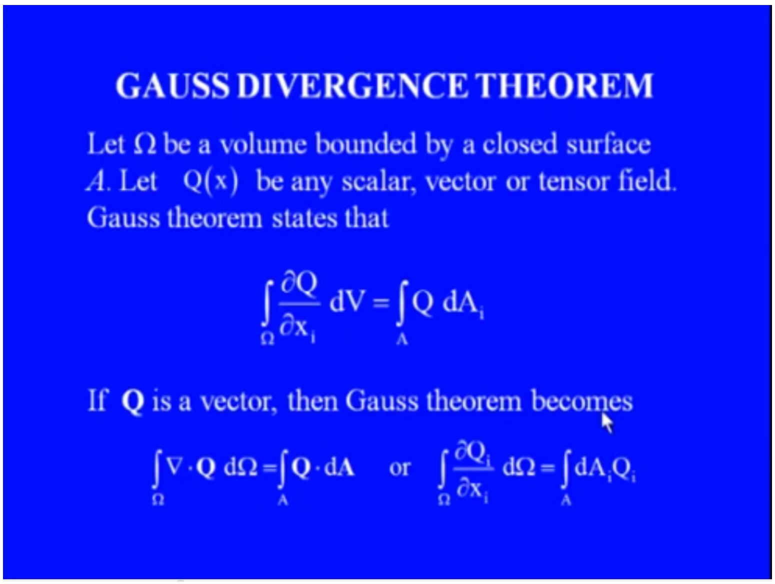

Now in CFD we would need differential as well as integral forms of the governing equations. In the derivation of these equations, we will come across volume integrals as well as surface integrals. So we would need certain mathematical tools so that we can transform the integrals in one form to the integrals in another form. For instance, we would very often require transformation of a volume integral into a surface integral.

Now Gauss divergence theorem helps us in this aspect and what is this theorem? Let us denote by omega a volume of the continuum medium bounded by a closed surface A and let Qx be any scalar vector or tensor field. Gauss divergence theorem states that the volume integral of del Q/del xi that is equal to surface integral of Q.

In particular, if Q are a vector then Gauss divergence theorem becomes so divergence of Q=Q dot A surface integral over A or we can write this as del Qi/del xi d omega=integral over dAi Qi. So we can clearly see that right hand side of this equation represents the dot product of vector Q with the area vector A. Now this theorem can be used to change the volume integral to surface integral or vice versa.

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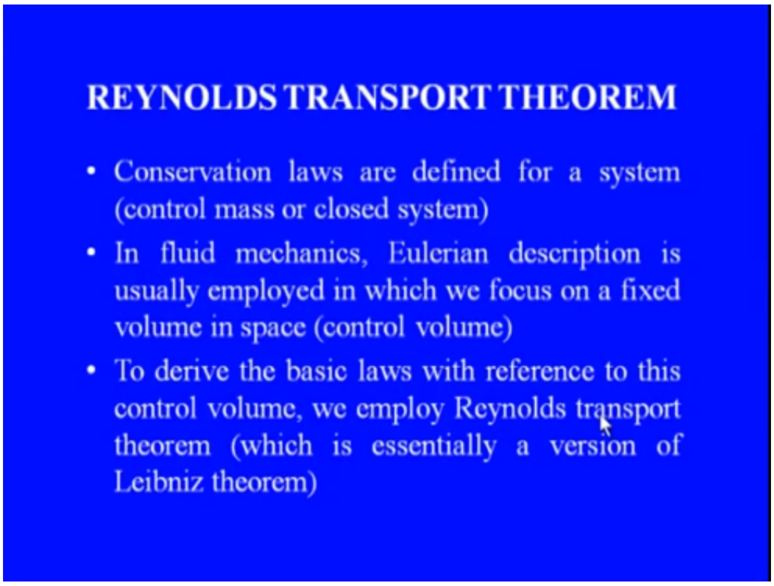

Another theorem which we would need is what is referred to as Reynolds transport theorem. What is the use of this particular theorem? The conservation laws of mechanics they are defined for a system that is a control mass or a closed system whose mass remains constant. In fluid mechanics though we would prefer what is referred to as Eulerian description in which we focus on a fixed volume in a space that fixed volume is commonly referred to as control volume.

Now how do we derive the basic laws with reference to this control volume because the basic equations are the fundamental conservation laws they are applicable to a system. So Reynolds transport theorem helps us in this task and it is essentially a version of what we call Leibniz theorem in mathematics.

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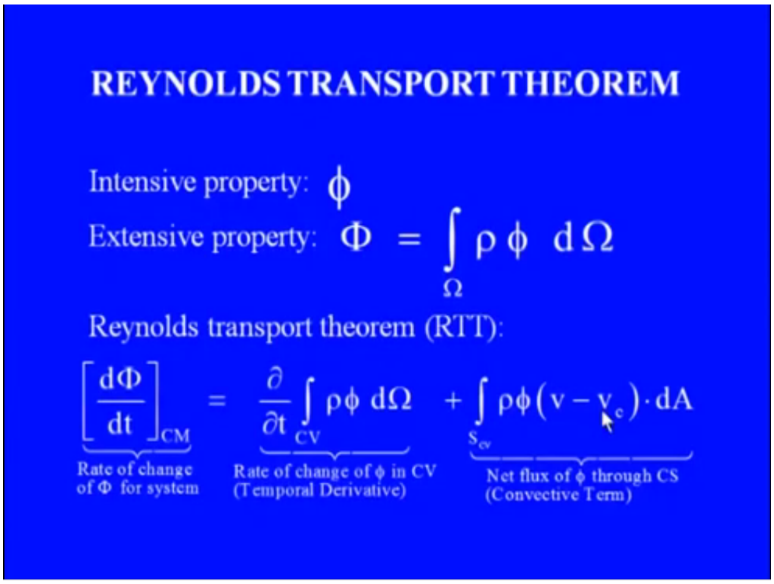

So let us have a look at Reynolds transport theorem. We will differentiate between 2 terms here, intensive property and extensive property for any given quantity of small phi, its extensive property is defined as the volume integral of phi*rho d omega or the small phi since represents the capital phi for unit mass.

So once we have got this relation between the intrinsic variable small phi and its extensive counterpart capital phi we can now write Reynolds transport theorem as d capital phi/dt that is time derivative of capital phi with respect to time for the control mass=del/del t of volume integral over CV rho small phi d omega+the surface integral with respect to surface of the control volume rho phi v-vc dot dA.

Now here vc represents the absolute velocity of the control volume or control volume might be moving in. My innocence what this theorem states is that look rate of change of capital phi for a system this is sum of 2 components that is rate of change of phi in a control volume, which we also refer to as temporal derivative+the net flux of phi through the control surfaces. Now second term on the right hand side is also referred to as the convective term.

We would use Reynolds transport theorem to obtain the governing equations of the fluid flow for an Eulerian control volume.

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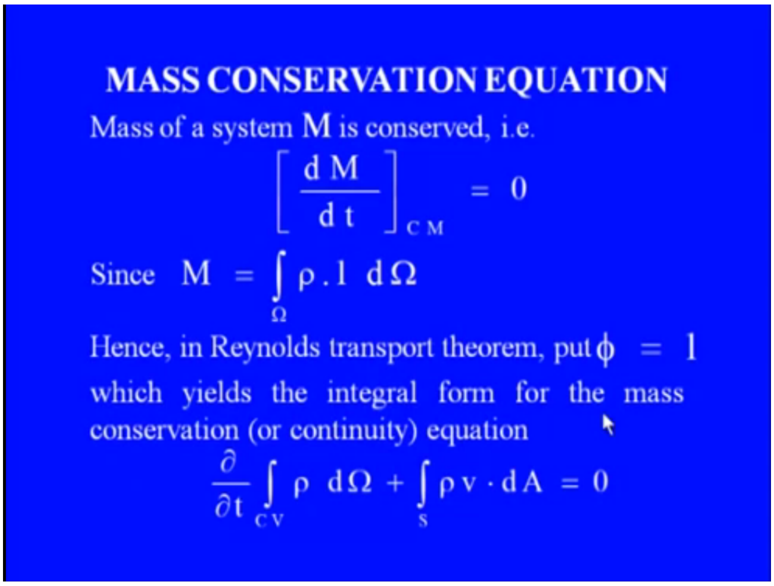

Now let us start of with the very first conservation equation that is mass conservation. In a non-relativistic framework, the mass of a system is always fixed. Hence the time derivative of the mass for any system is 0 that is dM/dt CM=0. Now this mass capital M could be defined as integral over the volume omega rho*1 d omega. So thus we can identify the intrinsic quantity linked with mass as the density rho.

Hence in Reynolds transport theorem, we can put small phi=1 and thereby we can obtain the integral form of the mass conservation equation, which is also referred to as continuity equation as the temporal derivative of integral rho d omega+the surface integral of rho v dot dA=0. So the very first term tells us that volume integral of the density and its temporal derivative+the flux of the density*velocity that has to be 0.

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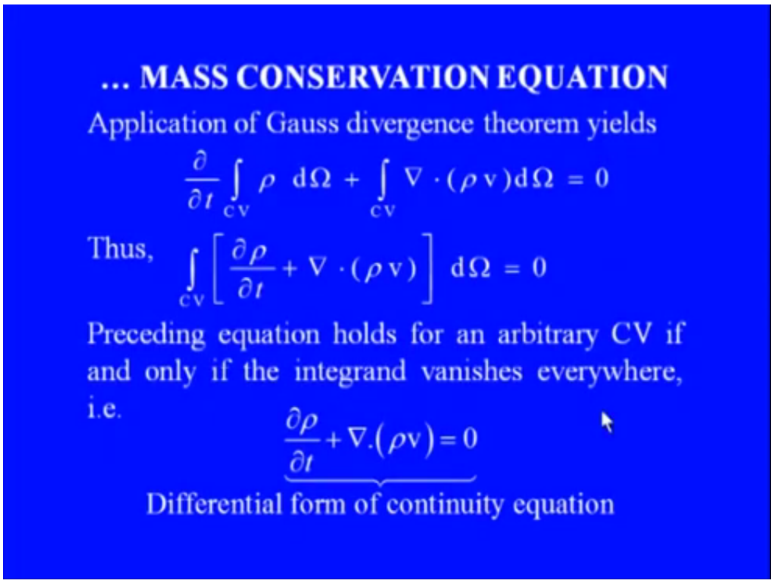

Now this form is referred to the integral form of the continuity equation. We can apply the Gauss divergence theorem and thereby we can change this integral equation into a differential equation. So let us first take the case of a fixed control volume that is a control volume does not change with respect to time so in that case in the first term del/del t this operator can be taken inside our integral operator.

So the first term of the continuity equation becomes integral/CV del rho/del t+for the second term which is a surface integral we can apply the Gauss divergence theorem and thereby we could obtain divergence of rho v d omega integrate over control volume this is equal to 0. We can combine both of these 2 terms together and thereby we can get a simple integral equation that is the integral over the control volume of del rho by del t+divergence of rho v=0.

Now this particular equation it holds for any control volume that is to say our choice of the control volume was arbitrary and this integral is equal to 0 it is only possible if the integrand is 0 everywhere. So that is why this integral equation leads us in this particular differential equation that is del rho/del t+divergence of rho v=0. Now this is referred to as the differential form of continuity equation.

Now differential form of continuity equation contains 2 terms that is first one is derivative of density with respect to time and the second term is divergence of rho v, now these 2 terms their summations would be 0 everywhere.

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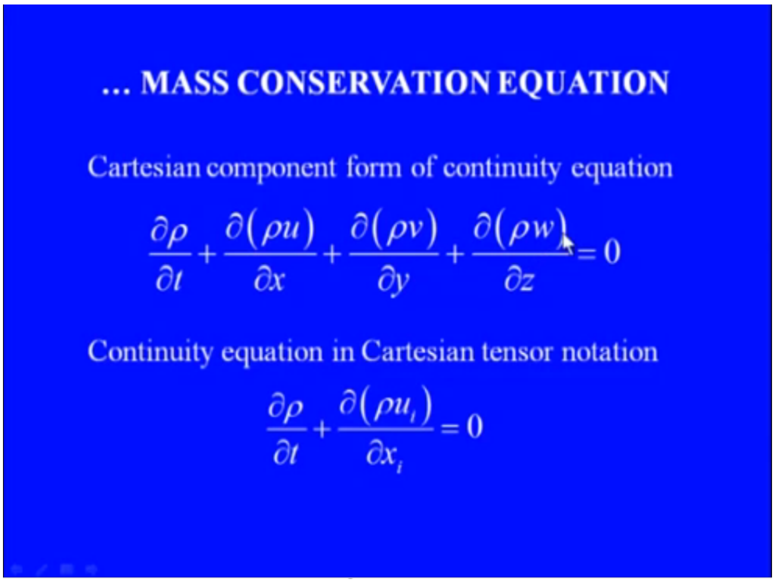

Now we can write this equation in Cartesian component form as del rho/del t+del of del x of rho u+del of del y of rho v+del of rho w/del z=0. We can also write this equation or continuity equation in Cartesian tensor notation as del rho/del t+del of rho ui/del xi where ui denotes our velocity vector. Our next equation would be momentum equation for which we will start from the Newton's second law.

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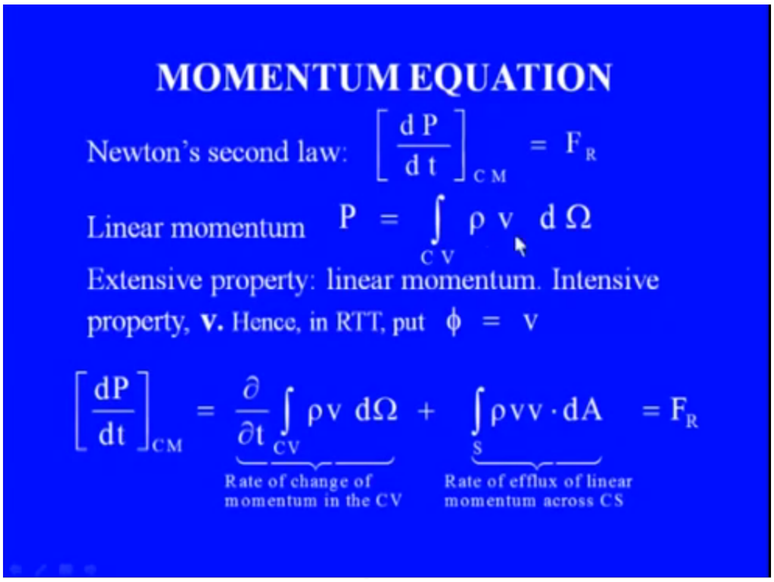

The Newton's second law of motion says the time rate of change of momentum of a system that would be equal to the resultant force applied on the system where this P represents the linear momentum, which can be defined for a system as an integral rho v d omega over the control volume. Now in this case let us identify what would be the extensive property and what would be the intensive property for using Reynolds transport theorem?

Clearly from this integral we can see this capital P are the linear momentum, which are extensive quantity and rho*v which is what we had in our definition of extensive property as v gives us our intensive property for linear momentum so thus we have got small phi=v and if you put small phi=v in Reynolds transport theorem.

We can obtain a very simple relation that d capital P/dt for a control mass system that is time rate of change of momentum for a control mass system=del/del t of the volume integral over the control volume of rho v d omega+the surface integral over the control surface of rho vv dot dA=FR. The first term refers to that is del/del t of rho v integral over the control volume this gives us rate of change of the momentum in the control volume.

And the second term that is surface integral of rho vv dA this gives us the rate of efflux of linear momentum across the control surface. Now please note this vv this is not a simple dot product of 2 vectors. In fact, vv, v is a vector hence vv denotes a second order tensor. So what we have got here is rho is a scalar quantity, vv becomes the second order tensor so we have got the scalar dot product of the second order tensor vv/our area vector dA.

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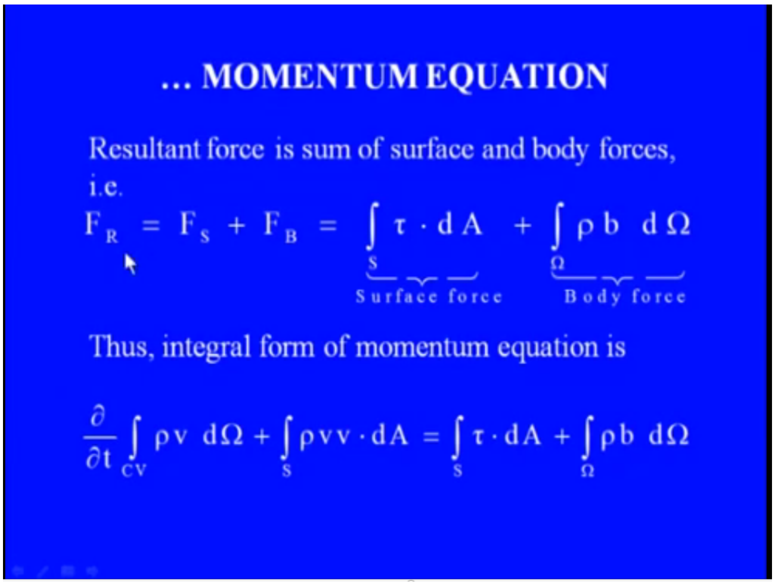

Now let us have a look at the resultant force for any control volume. The resultant force can be expressed as some of the surface and body forces that is FR=F subscript S+F subscript B where subscript S refers to surface force and B refers to the body force. Now the surface force can be obtained if we knew what would be the stress acting on the surface where stress were tau which acts on differential element dA.

The dot product of tau with dA would give us the differential force acting on that particular elemental surface area. This we can integrate over the surface of the control volume to get a net resultant surface force. Similarly the total body force can be obtained if we knew the body force per unit mass that is rho B d omega integrate over omega that will give us the total body force.

Now this body force normally arises because of less gravitational attraction or electromagnetic field or similar long range forces whereas the surface forces they arise from the contact with different medium or solid boundaries. Now we can substitute this expanded form of these integrals for the resultant force FR in the pre-equation, which we obtained in the last slide.

And thereby we would obtain the integral form of momentum equation as del/del t of volume integral rho v d omega+the surface integral rho vv dot dA=surface integral of tau dot dA+the volume integral rho B d omega. So this is our integral form of the momentum equation.

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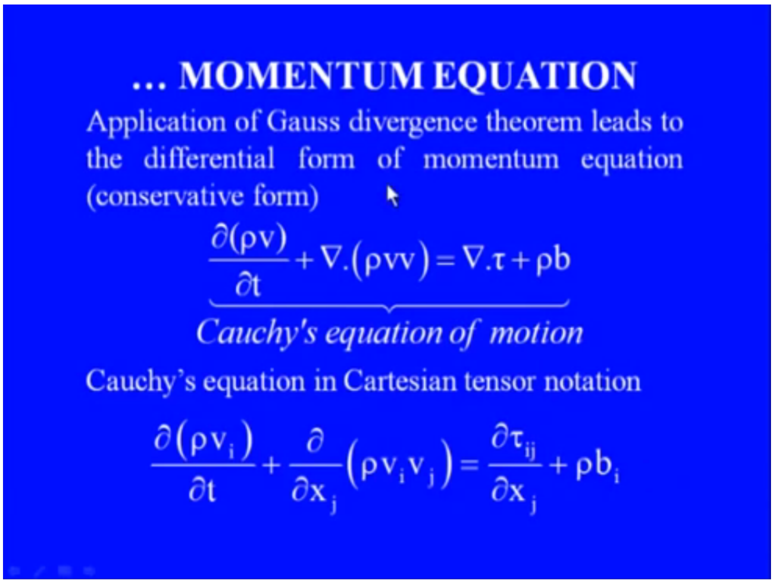

Now this integral form can be converted into a differential equation by again using Gauss divergence theorem and in this case what we need to do is first we will change the order of differentiation and integration hereby assuming CV to be a constant control volume that is constant control volume, which does not change with time. So this particular term will lead us to volume integral of del rho v/del t d omega+this surface integral can be changed into a volume integral using Gauss divergence theorem.

This would become volume integral of divergence of rho vv d omega. Similarly this surface integral tau dot dA becomes volume integral of divergence of tau d omega. This is again a volume integral. Now all of these were terms they have been transformed into volume integral. We can take them on one side and we can write them as simple volume integral and again we can argue right hand side has become 0 and that can be true only if the integrand vanishes at every point.

And that argument leads us to our differential form of momentum equation that is del rho v/del t+divergence of rho vv=divergence of tau+rho v. Now this particular equation was first derived by the French mathematician Cauchy so this is referred to as Cauchy's equation of motion. Now this is equation of motion in the vector form. We can write it as a Cartesian tensor notation as del/del t of rho vi+del/del x of rho vivj=del tau ij/del x j+rho bi.

Expanded form of momentum equation could be obtained in different coordinate systems.

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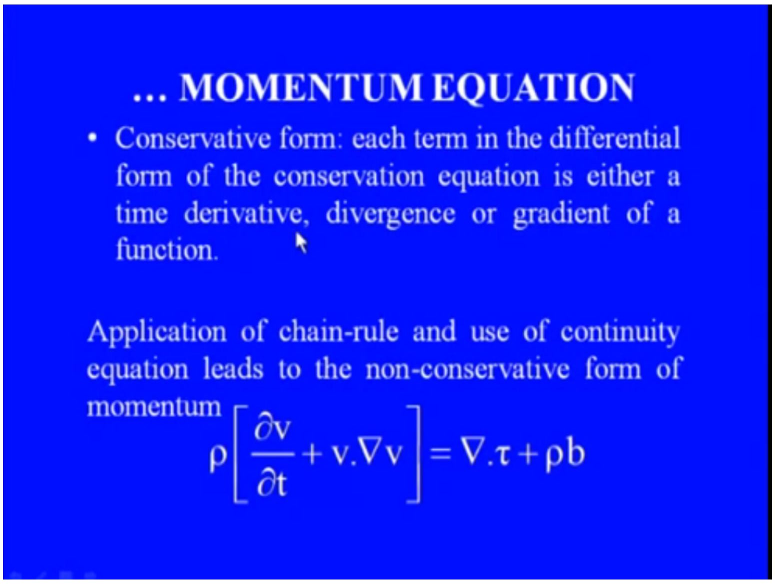

Now please note that this particular form we call as conservative form. What do you mean by conservative form? Here each term in differential equation of the conservation equation is either a time derivative, divergence or gradient of a function. So if you look carefully here the first term is the time derivative, the second term is the divergence of rho vv and the next term that is the first term on right hand side this is the divergence of the stress tensor.

This volume component b or the body force b it can be represented by a gradient of some scalar quantity. So this is again representable as a gradient so this particular form of the momentum equation is referred to as the conservative form of momentum equation. Now we can change it using the chain rule of differentiation and continuity equation. We can obtain the non-conservative form of momentum equation, which is given by rho del v/del t+v dot del v=divergence of tau+rho v.

The right hand side has remained the same only changes have been in the left hand side. If you closely observe this left hand side, the del v/del t this refers to the local change in the velocity vector that is local acceleration, v dot del v this is referred to as a convective acceleration so this whole term in the bracket this is essentially the acceleration term. So rho*acceleration=divergence of tau+rho*v.

So this is our non-conservative form of momentum equation. In CFD, we primarily focus on the conservative form of momentum equation.

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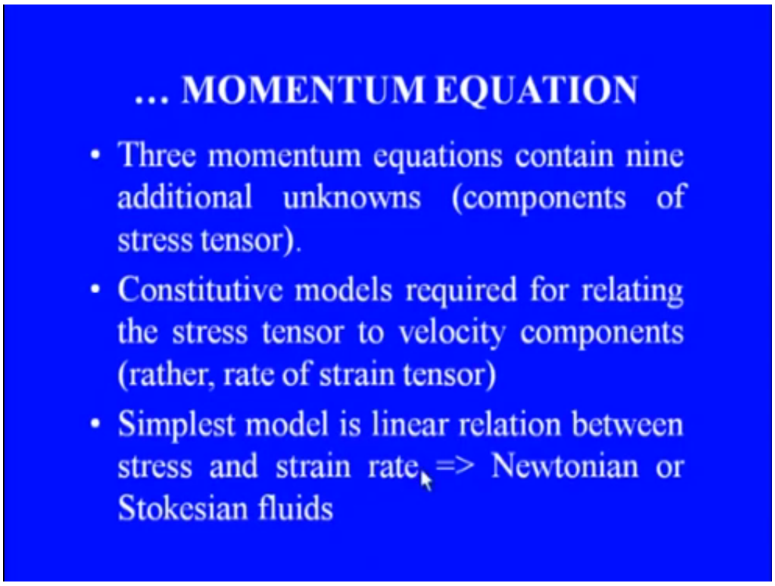

Now if you look at the momentum equation carefully we would see one thing very clearly that this vector equation is essentially a system of 3 equations and how many unknowns we have got the number of rho would be an unknown, velocity vector is unknown that is we have got 3 unknowns here and we have got a tensor quantity here, second order tensor tau and tau would have 9 components.

So you have got more number of unknowns than the number of equation. Total number of equations are 3 only for rho of course we can get continuity equation so we get 4 equations, 4 equation for 9+4=13 unknowns so there is something missing. The system equations will not complete and it cannot be used for the mathematical solution of equations. So we need what we call constitutive models for relating the stress tensor to velocity components rather we would try to relate the stress tensor to rate of stress tensor in the fluid dynamics.

Now the simplest model is the linear relationship between stress and strain rate. This is used for Newtonian or Stokesian fluids. For non-Newtonian fluid, stress and strain relationship is non-linear and there are different forms of the non-linear loss which are available which can be substituted in momentum equation to be simplified further.

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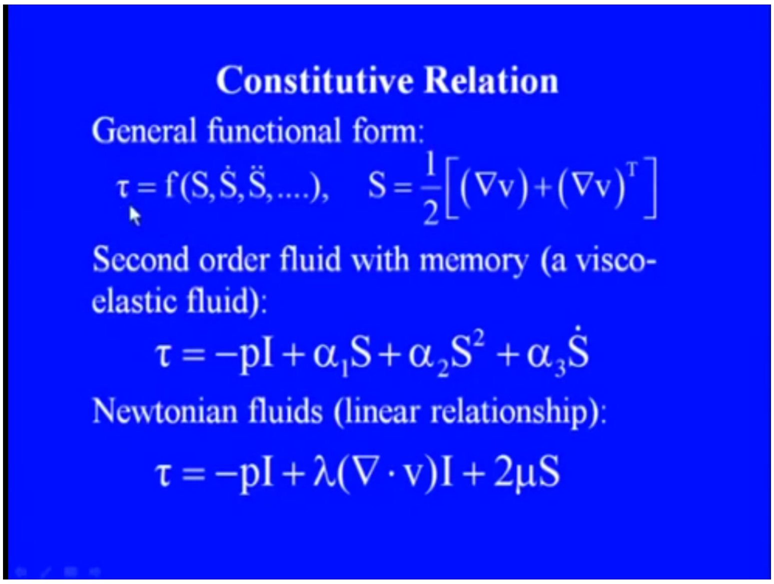

Now let us have a look at this constitutive relationship for a general fluid the stress tensor tau can be a function of this strain rate tensor S where S is given by half gradient of v+gradient of v transpose. So this S represents a strain rate tensor and a general dependence could be written as tau is a function of S the second derivative of S and so on. Now the presence of this time derivatives that would take care of the different types of fluids.

For instance, if we had a second order fluid with memory which is also referred to as a visco-elastic fluid. In that case, the dependence would be only on the stress strain rate tensor S and is time derivative that is tau would be written as –pI where this I is identity matrix, I scalar quantity alpha 1, alpha 1*by the stress strain tensor S+alpha 2 times S square+alpha 3 times S dot.

Now alpha 1, alpha 2 and alpha 3 these represents a typical material properties, which would be determined empirically by performing experiments for a particular fluid. For a Newtonian fluid, the dependence is very simple. We have got tau is a linear function of S and in this case we can use some simple algebraic manipulations to arrive at a very straight forward form for this stress tensor that is tau is given by –p subscript I+lambda divergence of v I+twice of mu times S.

Now here these coefficients lambda and mu they are referred to as coefficients of viscosity. Lambda is called first coefficient of viscosity and mu is referred to as a second coefficient of viscosity or dynamic viscosity, p is also referred to as thermodynamic pressure.

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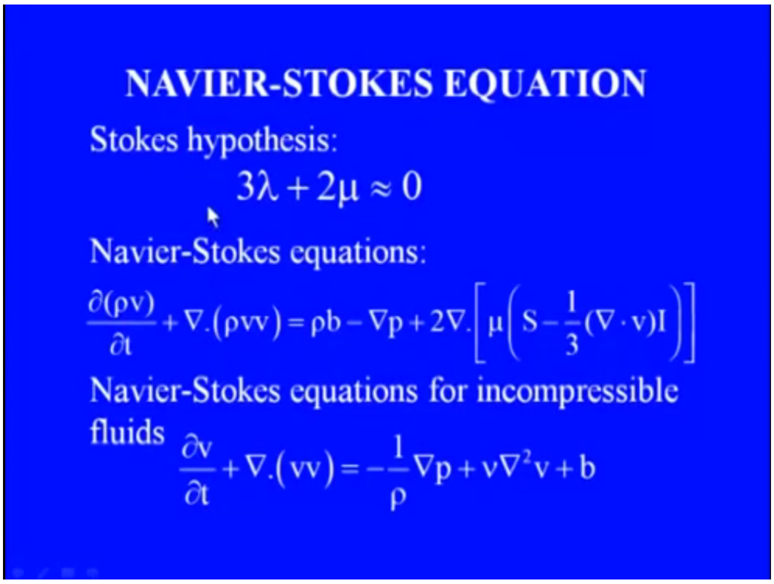

Now we have got 2 unknowns in the previous relations lambda and mu. We would like to simplify this further. So for this Stokes can be the hypothesis that for majority of fluids there is a relationship that is 3 lambda+2 mu=0. In fact, this 3 lambda+2 mu it is related to what we call the bulk modulus of a fluid.

So Stokes hypothesized that this particular combination is equal to 0 and this leads to a very simple form for the momentum equation which is referred to as Navier-Stokes equations. Now these equations were derived separately at different points of time by French man Navier and the British man Stokes. So they are referred to as jointly by as Navier-Stokes equations.

And we can write in dyadic form as del of rho v/del t+divergence of rho vv=rho*v-gradient of p+2 times divergence of (mu)*by S-1/3 divergence of v*identity tensor. Now this equation holds good irrespective of whether fluid is compressible or incompressible. That is to say whether its density is constant or variable. Now in case the fluid density were constant that is to say rho is not a function of time or spatial coordinates.

Then this Navier-Stokes equation can be further simplified and the simplified form of Navier-stokes equation can be written as del rho/del t+divergence of vv=-1/rho gradient of p+mu times del square v+b. Please note these 2 equations they are written in vector form and they can be applied to any coordinate system, but when we want to discretize these equations for CFD analysis we have to write these equations in appropriate reference frame, which we have chosen.

We can choose either a Cartesian reference frame or a cylindrical polar coordinate system or spherical polar coordinate system or curvilinear system and we have to write the expanded form of either of these two equations separately for each choice of the reference frame.

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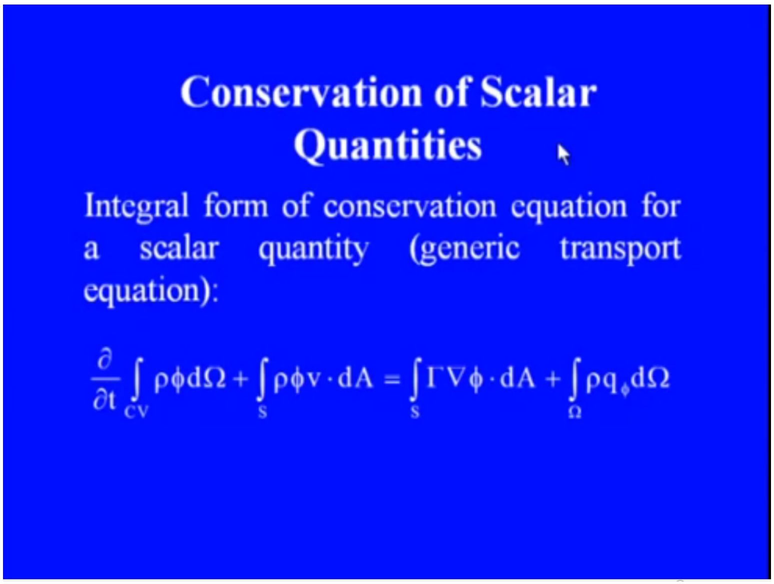

The next would be how do we obtain the conservation for an arbitrary scalar quantity? For instance, if we had a generic scalar or an arbitrary scalar phi, a transport equation for this generic equation can be written by looking at our momentum equation that is we should have a time derivative term then we should have what we call a convective term on the left hand side.

On the right hand side, will have a diffusive term and a term which is contributed by something similar to a body force terms. Now this has been derived or written in analogy with our momentum equation where we had a time derivative term, a convictive term, a surface integral of the effects on surface+whatever effect we can obtain from the body or volumes generation. So this is our generic transport equation for a scalar quantity phi.

This equation for I scalar quantity phi can also be transformed into a differential equation by using a same logic that is we transform first each of these surface integrals using Gauss divergence theorem. For instance, this particular term surface integral of rho phi v dA this would become volume integral of divergence of rho phi v. Similarly, this surface integral gamma gradient of phi would become divergence of gamma gradient of phi d omega.

So both of these can be transformed into volume integrals. We can transfer all of these terms on the right hand side combined together in a single volume integral that is that will read as volume integral of del rho phi/del t+divergence of rho phi v+(-)divergence of gamma gradient of phi-rho q phi d omega and right hand side of this integral is equal to 0, which can happen only if the integral vanishes identically at every point.

And that leads us to differential form of generic transport equation given by del/del t of rho phi+divergence of rho phi v=-divergence of gamma gradient of phi+q phi.

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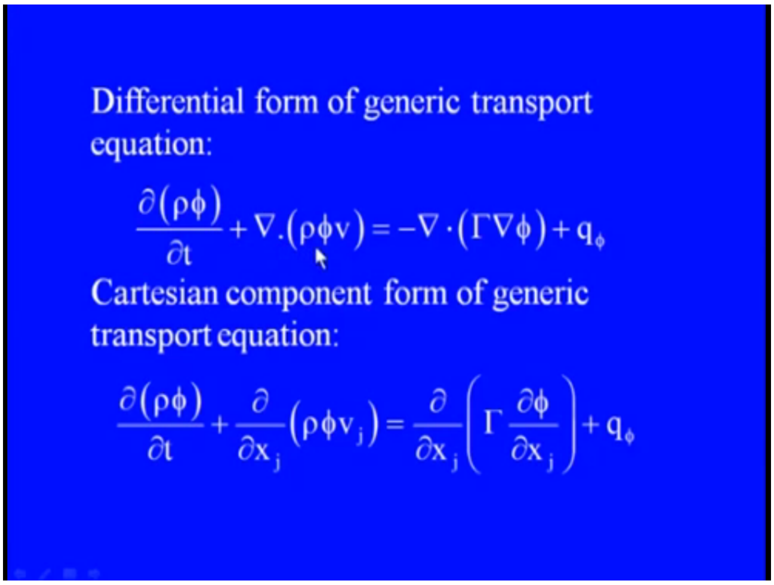

Now we can write this vector equation for generic transport equation in indicial notation as del/del t of rho phi+del/del xj of rho phi vj=del/del xj of gamma del phi/del xj+q phi. Now here this gamma represents a sort of a diffusion coefficient or diffusivity, q phi represents a volumetric generation term. The next conservation equation is that of the energy conservation and the integral form of energy equation can be easily written from our thermodynamics.

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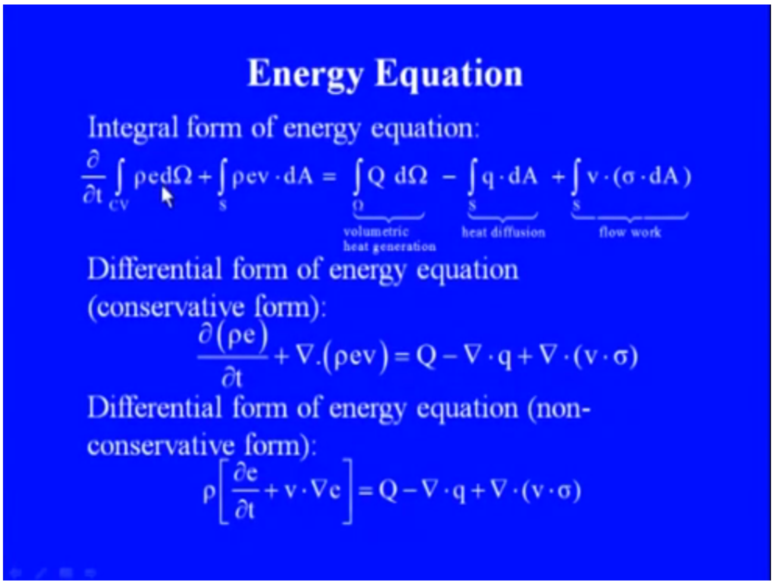

If a small e denotes the specific energy that is energy per unit mass then del/del t of rho e d omega that is the volume integral of this energy and is time rate of change+rho ev dA a surface integral this will give us the efflux of energy. So rate of energy generation in the control volume+efflux of the energy=the total amount of energy which is supplied from the external sources which could be due to the transfer of heat from the surface or volumetric heat generation given by capital Q or the generation of heat because of the flow of a viscous fluid.

So we have got few terms here the first one refers to the time rate of change of specific energy in the control volume, the second one refers to the efflux of the energy from the control volume across the control surface. The first term in the right hand side that tells us the volumetric heat generation. The next term tells us the heat diffusion across a surface and the last terms tells us the generation of heat because of the viscous dissipation.

Now once again this integral equation we can transform into a differential equation. We need to keep doing this because this integral forms would be used for one particular methodology, which we call control volume approach whereas for finite difference method or for finite element method, we need a differential forms of the governing equations. So let us interchange the terms here.

That is let us interchange the differentiation and the integration so this term will become integral over the control volume of del rho e/del t. The next term which is the surface integral we can transform using Gauss divergence theorem into volume integral as divergence of rho ev d omega. The next term the solid volume integral, this surface integral can be again transformed into volume integral as divergence of q d omega.

The next one can be written as divergence of v dot sigma d omega. We can combine these 2 in a single integral equation. All the terms are transferred on the left hand side and the right hand side of the integral equation will get 0 and then we can again use the logic that volume integral can vanish only if the integrand is identically 0 everywhere in the volume and thereby we get a simple differential form for energy equation.

And this is our conservative form of energy equation del rho e/del t+divergence of rho e v=Q-divergence q+divergence of v dot sigma, where sigma is the viscous component of the stress tensor. Now this is conservative form of equation because the first term is time derivative, the second term is divergence of something, the third term is a constant function. This term is divergence of flux and the last term again is divergence.

So all the terms are expressed in the form of either time derivative or divergence of some quantity so that is why we refer this particular form as a conservative form of energy equation. Now left hand side can be further simplified. We can use the chain rule of differentiation this term can be written as del rho/del t into e+rho times del e/del t similarly this term can also be broke into 2 parts by using chain rule of differentiation.

We can combine these 2 terms, make use of continuity equation and some of the terms will vanish del rho/del t into e that term will vanish and we get so called non-conservative form of energy equation given by rho del e/del t+v dot del e=Q-divergence of q+divergence of v dot sigma. So this is our non-conservative form of energy equation, but let us make it very clear that most of the time specifically in finite volume formulations and finite element formulations we would use the conservative form of energy equation.

The equations we have derived so far they hold good for any fluid whether the fluid is compressible irrespective of fluid density, fluid velocity and so on. We can derive different simplified forms. For instance, all the flow equations can be written in a much simpler form if even the fluid may be compressible but the flow can be assumed to be incompressible.

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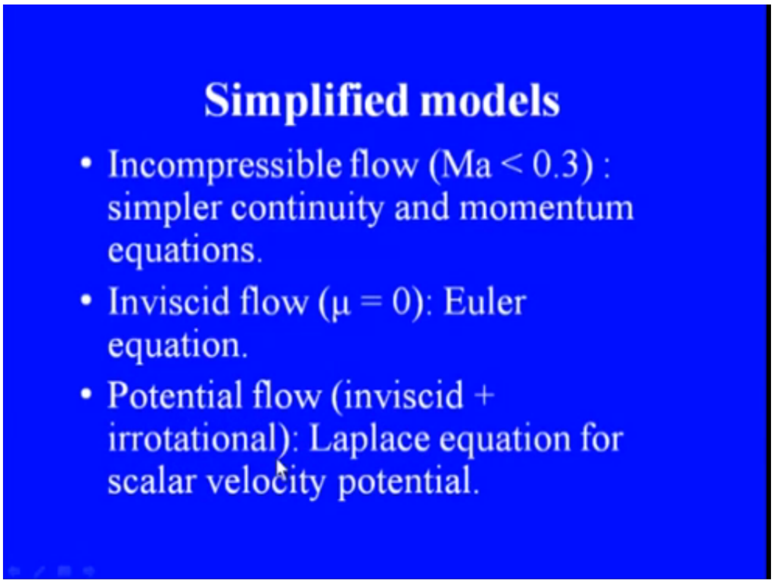

This can happen if the Mach number of the flow is less than 0.3. When the Mach number is less than 0.3, in that case compressibility effects can be neglected and even for a working fluid like air or similar gases all the equations which we had the continuity equation, momentum equation or energy equation, density can be assumed to be constant, which will lead to simpler forms of continuity, momentum and energy equations.

Similarly in sudden situations, the viscosity of the fluid can be assumed to be very small. This happens at high speed aerospace fluids where the velocities are very high and we can neglect the contribution of the viscosity to the stress tensor and this simplified form is referred to as Euler's equation. Yet another simplified form could be if we assume the fluid to be inviscid that is we can neglect the viscosity of the fluid as well as any rotationality effect present.

So in this case we need to only worry about a single scalar equations in terms of a scalar velocity potential and this scalar velocity potential is referred to as this particular equation Laplace equation for scalar velocity potential is called as potential flow equation. So these are few simple forms.

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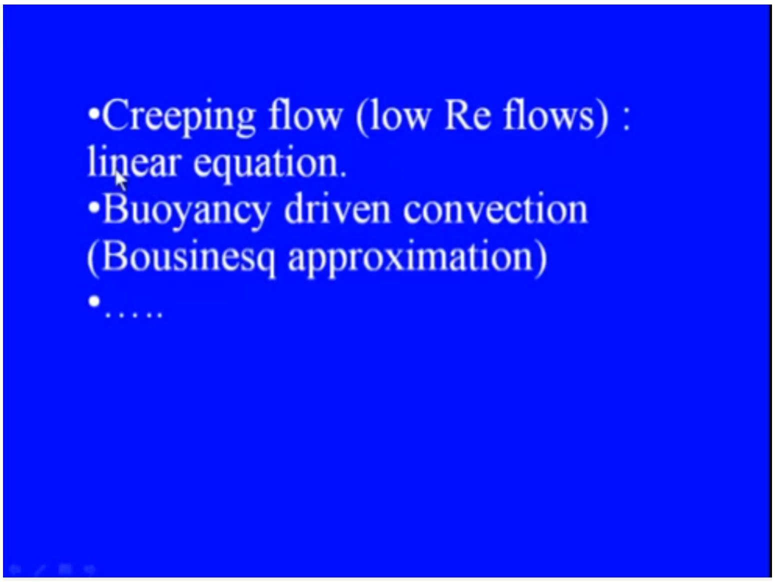

Further simplified forms could be obtained for low Reynolds number flows in which case we can neglect the time derivative term and this leads to what we call the creeping flow. In this case, our nonlinear inertial terms on the left hand side momentum equation they vanish so we get a linear equation. Similarly for buoyancy driven convective flows, we can use what we call Bousinesq approximation.

Thereby we can assume density to be constant in continuity equation in energy equation as well as in momentum equation and we would only incorporate a small change, which reflects the effect of buoyancy in the body forces. So there are various other simplified forms, which are possible and which are utilized in approximation of different flow problems. So this where we would stop.