Computational Fluid Dynamics
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Lecture - 28
Introduction to Finite Volume Method

Welcome to module 6 on Finite Volume Method. We had discussed earlier one special discretizing technique that was based on finite difference method. We had a detailed look at how do we approximate the derivatives in a partial differential equation at a given point and thereby obtain a discrete form of a continuum problem and we also learnt different approximation techniques, we saw its application to two scalar transport problems.

In the beginning of this course, we are you going to have a look at three prominent discretizing techniques, i.e., finite difference, finite volume and finite element method. So, today is second such discretizing technique which we are going to discuss in the next few lectures.

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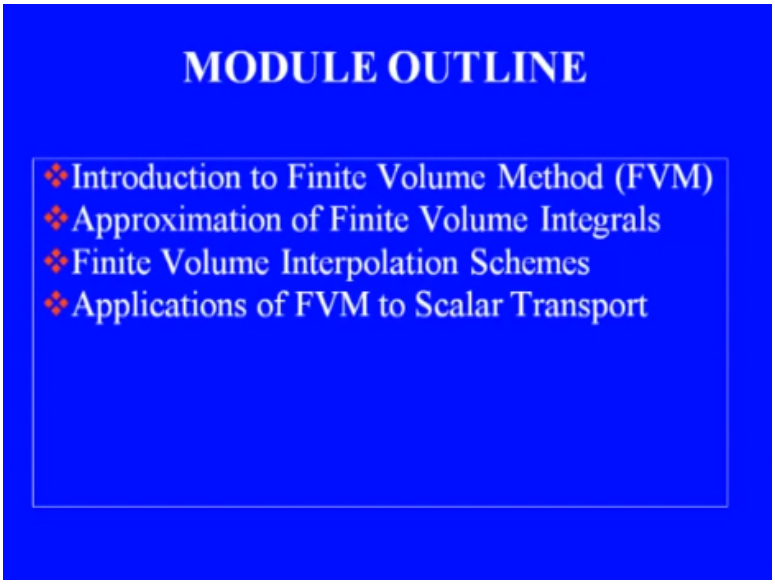

So, the outline of this module, we would have a look at basics of Finite Volume Method. It involves approximation of set of integrals which we call Finite Volume Integrals. So, we would have a look at the approximation techniques for Finite Volume Integrals. The evaluation process also involves the interpolation of the function values at competition nodes. So, we will see which of the interpolation techniques which are commonly used in the context of Finite Volume

Schemes.

Towards end of this module, we will have a look at applications of Finite Volume Method to scalar transport problems.

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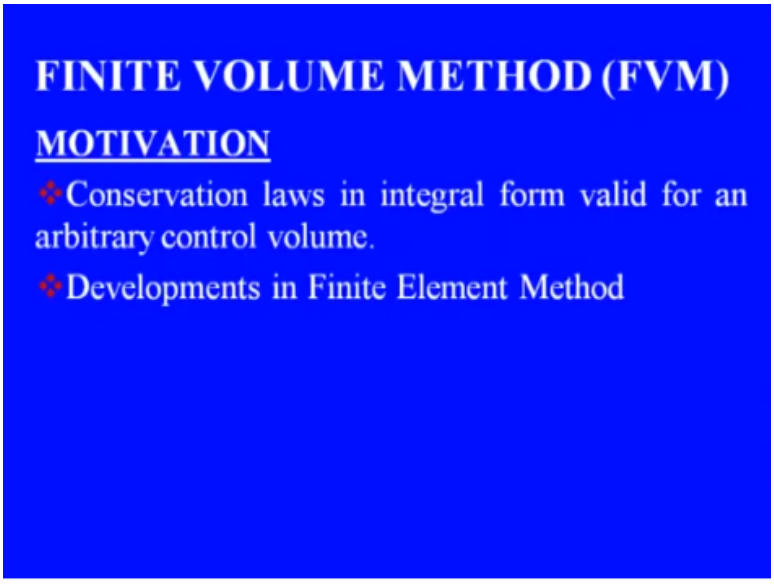

The first lecture in this series, we will focus on the basics of Finite Volume Method. Now, what is the basic motivation of Finite Volume Method. If you remember, we learnt in our mathematical modelling that we have got the basic conservation laws of physics expressed in an integral form as how we derived it starting from the conservation law expressed for a system to the corresponding integral form for an arbitrary controlled volume.

So, we have already got a form of equation which is available and the choice of control volume (()) (02:48) is arbitrary. Can we extend it further and divide a problem domain into non-overlapping set of control volumes; and if we can do that, our conservation laws would be valid in each one of those control volumes. Can we make use this feature to come up with a discretization scheme which can help solve our problem.

If you can do that, we would not be limited to the Cartesian grids which we require for Finite Difference Method or the other way which we had for complex geometry for Finite Differences, we said we have to go for a map or complex problem domain into a conceptual rectangular

competition domain. That mapping was fairly complicated, may not be possible for all set of geometrics.

So, now here we have got an opening and look we have got an equation expressed in an integral form. Now, that integral by the basic law of calculus as it could be broken down into small-small integrals over smallest subset of Finite Volumes; and in fact, what we can say that conversion law has to be valid over each one of those small control volumes which could be of arbitrary shape.

Complex domain can be represented by a union of such non-overlapping control volumes. So, that was a starting point. Further motivation is provided by developments in Finite Element Method. Let us have a brief look at it. On our board, what happened during the early rise of development.

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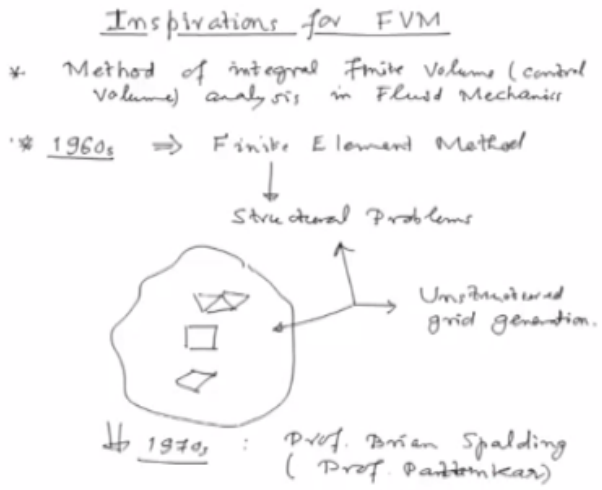

So, let us call it inspiration of Finite Volume Method. Now you can recall one more thing from the fluid mechanics that we had what we call method of integrals finite volume or rather we call it the control volume analysis in fluid mechanics and this is on e method which is used in obtaining the design variables or to get a gross overall features of a feed problem. By the end of 1960s, a Finite Element method was well established for structural problems.

So, irrespective of the complexity of solid domain or structural domain, infinite element method what we can do is we can divide the domain into small-small subdomains of what we call Finite Elements, and these elements could be of any shape, which could be triangulars or they could be rectangles or they could be of arbitrary quadrilaterals. So, we can any choice of these infinite element analysis and to deal with such problems, there were lots of development in unstructured grid generation.

So, unstructured grid generation techniques were available to decompose our complex problem domain into a set of non-overlapping finite elements of the chosen type. So, in early 1970s, there was a group at Imperial led by Professor Brian Spalding whose most famous student is Professor Pattenkar. They promoted this Finite Volume Method. They said look our integral forms, they were expressed for an arbitrary controlled volume that is over which we express our conservational laws.

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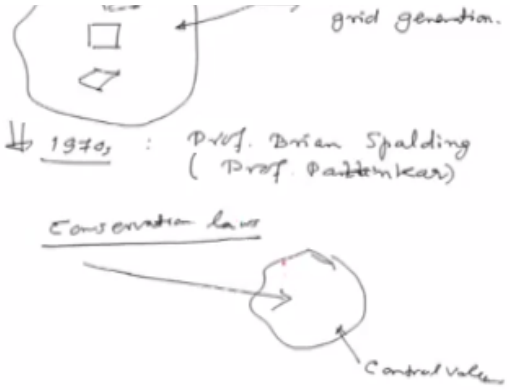

So, our given problem domain can be again divided into smaller subdomains or a smaller control volumes, similar to the way we do in finite elements. The technology for that is available in a structured grid generation techniques and make use of the basic conservation principles or satisfaction of conversation principles at each one of those smaller control volumes.

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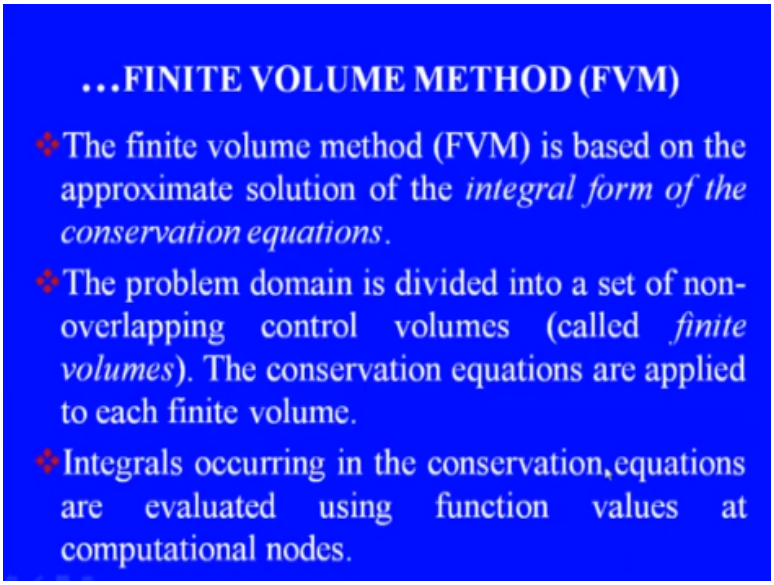

So, from the beginning, the Finite Volume Method is based on the approximate solution of the integral form of the conservation equations. On this integral form, the conservation equation is available the way we have derived in our mathematical modelling module from the first principles, and then what we can do is we have just described, the problem domain is divided into a set of non-overlapping control volumes.

These control volumes since they were finite volume, they were called finite volumes and we can now apply conservation equations to each of these finite volumes and take all of them together to obtain a system of discrete algebraic equations which can be solved in same way as we did in the case of Finite Difference Method to obtain solution to our problem and these integrals which occur in application of conservation equations to each of the finite volumes, they are evaluated in terms of the function values at what we call computational nodes .

But there could be one question here that as we derive our conservation equations in integral form, so we have the way with us to proceed with application of Finite Volume Method, but suppose we had the conservation equations in differential form.

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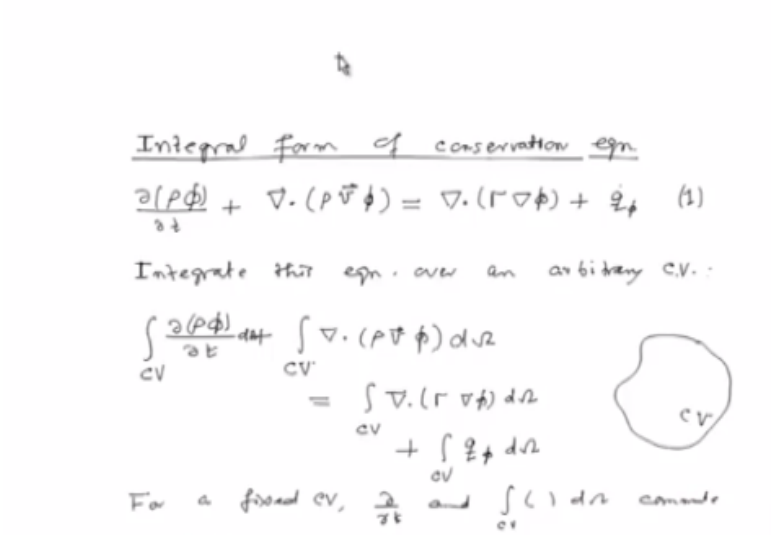

Now, supposed with did not have the integral form of the conservation equation. Suppose we were given a differential form, del of rho phi over del T+divergence of rho V phi=divergence of gamma times gradient of phi+volumetric generation right is called Q phi. So, this is our differential equation. Now, if you what to obtain an integral form, we can do the reverse process of what we followed in the derivation of the differential form of governing equations starting from integral form.

So, let us integrate this equation over an arbitrary control volume. CV could be of any shape and size. So, del rho phi/del T, this is summation here, so we can now write each one of these as separate integrals. Divergence of gamma times gradient of phi with T omega+volume integral of CV of Q phi D omega. Now, CV is fixed, we can now interchange this time derivative and integral operation for a fixed CV del/del T and integral/CV D omega, the commute.

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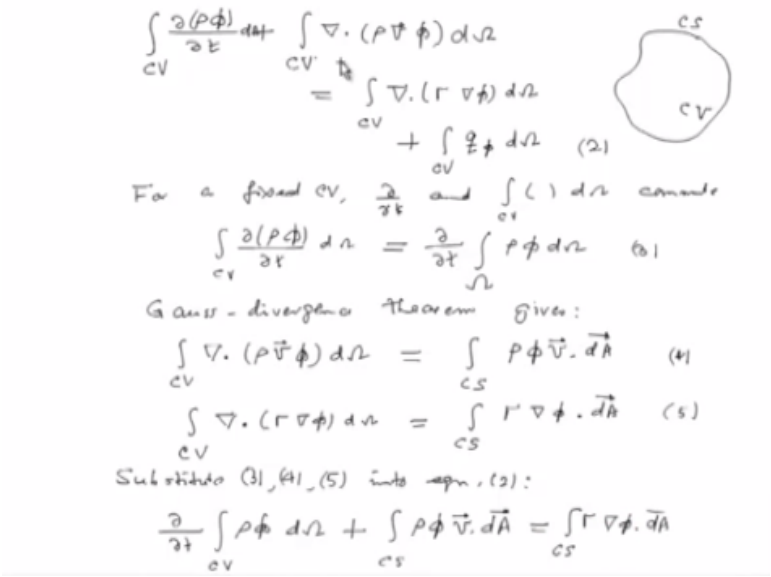

So, that is why we can write as integral of rho phi/del TD omega/CV versus time derivative of rho phi D omega/CV. Now, these two integrals which involve the divergence operators they correspond to convective and diffuse reflexes respectively. Now, these can be transformed using Gauss's Divergence Theorem. So, Gauss's Divergence Theorem gives us that integral/CV of the term divergence of rho V phi D omega.

This is equal to the surface integral of rho phi V dot DA with control surface and similarly divergence of comma, gradient of phi, D omega/CV. This is equal to this surface integral gamma times gradient of phi dot DA. So, if you substitute of these equation 3, 4, and 5*2. So, substitute 3, 4, 5*equation 2, then what do we get. Del/del DF rho phi D omega/CV+surface integral/CS of rho phi V dot DA=Surface integral/CS of gamma gradient of phi dot DA+volume integral of CV of Q phi D omega.

So, now this is the integral form required for Finite Volume analysis. So, if we have the integral form per se available with us, that is fine. If not, if you only had the partial differential equation given to us in the form of equation 1, we can take this partial differential equation, convert it into a conservative form, and then integrated over a controlled volume to obtain the integral form of the conservation equation suitable for Finite Volume analysis.

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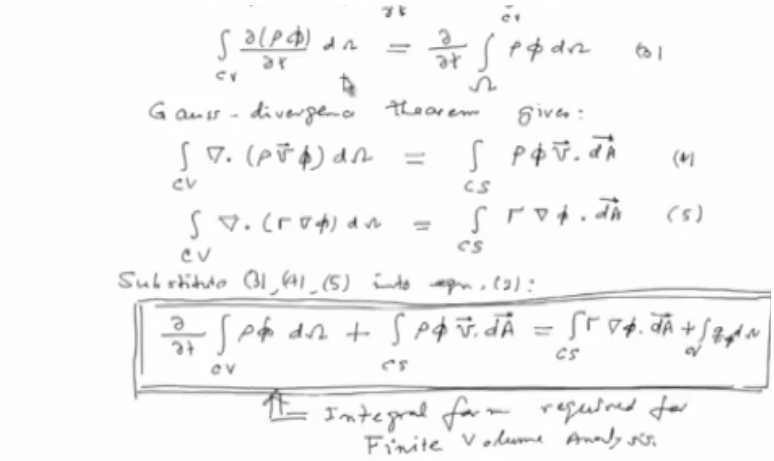

Okay, now let us see what are the attractive features of Finite Volume Method. Now, what we saw from the very beginning that look we can now have any type of decomposition of a problem domain, that is why the Finite Volume Method can accommodate any type of grid and hence it is naturally suited for complex geometries.

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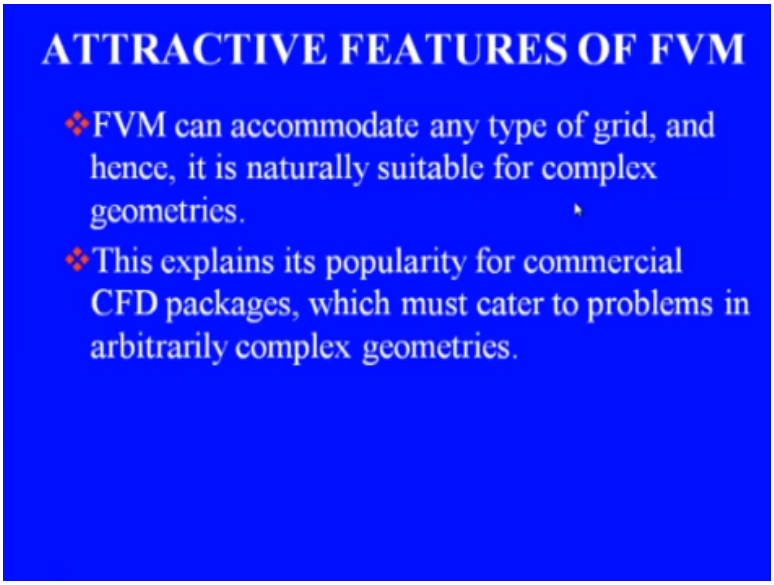

This explains its popularity for commercial CFD packages. You take any commercial CFD packages Fluent or Star-CD or similar softwares, they are all based on Finite Volume formulation because they must cater to the industrial problems which are defined in complex geometries. The advantage in unstructured gird generation method they have made the application of Finite Volume to flow problems in arbitrary complex geometries very simple.

Similarly, for large-scale flow problems which ought to be sold on parallel machines, there are mesh-partitioning techniques developed for finite element analysis. These can be applied as such to finite volume grids to make a finite volume code run on a parallel machine. So, these two developments, development in unstructured grid generation method and mesh-partitioning techniques which were originally developed for Finite Element Method.

They can be directly used in Finite Volume Method and they have contributed greatly to the popularity of Finite Volume Method for solving flow problems in complex geometries. That is the reason the why the industrial strength, commercial CFD packages are based on Finite Volume Method wherein we can import a mess which is generated using an unstructured grid generator for finite element probe analysis and solve our problem.

Now, let us have a brief look at the overall solution process which is adopted in Finite Volume Method. We have already see part of it. So, first part of the solution process using any discretize scheme, whether it is finite difference, finite element or Finite Volume Method, discretize the solution domain by a grid or a mesh.

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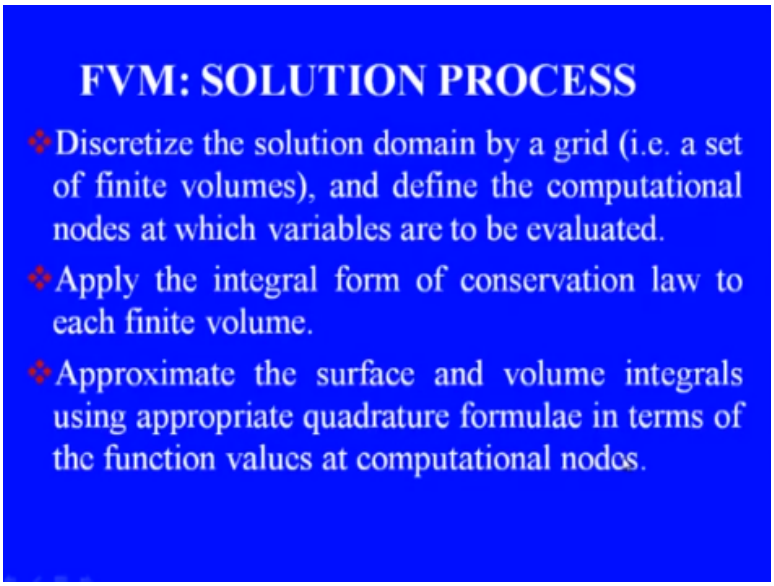

So, this is what we will do in the case of Finite Volume Method, discretize the solution domain by a grid, i.e., set of finite volumes and we have to define what we call computational nodes at

which our problem variables are to be evaluated. Now, this computational nodes, they need not coincide with the vertices or the corners of the finite volumes which we will see shortly. Now, once we have done this discretization.

The next step is apply the integral form of conservation law to each finite volume. Now, you have already seen, there are two types of integral involved. There are surface integrals and volume integrals which are in terms of our unknown variable. So, we cannot evaluate those interiors exactly. So, what we need to do is, we have got to approximate those integrals. So, approximate the surface and volume integrals using appropriate quadrature formulae in terms of the function values at computational nodes.

Because these are the once which are of interest to us in our solution process, we want to obtain the function value at computational nodes and in terms of these values, we would approximate the surface as well as volume integrals.

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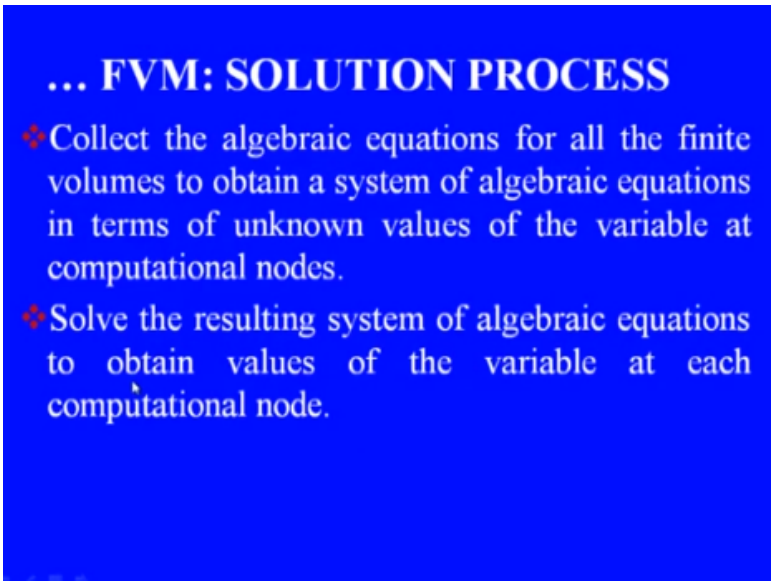

So, once we do that we would be able to convert the integral form of conservation laws applied to finite volume into a discrete algebraic equation, and once that is done, all that we need to do is collect algebraic equation for all finite volumes to obtain a system of algebraic equations in terms of an unknown values of the variable at computational nodes and depending on the nature of that system, we can use a linear solver or a non-linear solver.

So, all that we need to do is solve the resulting set of equations to obtain values of the variable at the computational nodes and if we want to obtain what we call dependent variables, for instance the ones which are defined in terms of gradients of basic variables that can be done at the post-processing styles. So, if you look back to what the way we discussed the solution process for finite differences, the similarities are obvious.

In fact, the last step would be the same for finite difference or finite volume or finite element method. It is only the first three steps which might differ from one discretization procedure to another. Now, let us have a brief look at the basic types of finite volume grids which are employed. Now, here what we mean by grids is not what type of unstructured or structured grids we are going to use.

This classification is based on (()) (22:56) the computational nodes. So, there are two common approaches to finite volume discretization basically to the choice of computational nodes.

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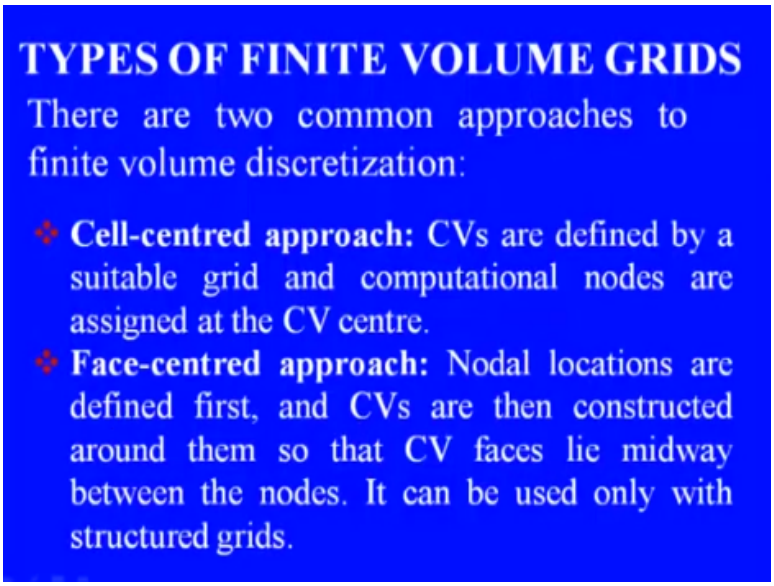

The first one is what we call the cell-centered approach. In cell-centered approach what do we do, the CVs are defined by a suitable grid and computational nodes are assigned at the CV centre. The second approach is what we call face-centered approach. Now, here what we do is we chose the nodal locations first, chose our computational nodes and CVs are then constructed

around them, so that the CV faces lie midway between the nodes.

But there is one limitation that this method can be used only with structured grids. Now, let us have a geometric look at both these approaches, get back to the board and see both of these approaches.

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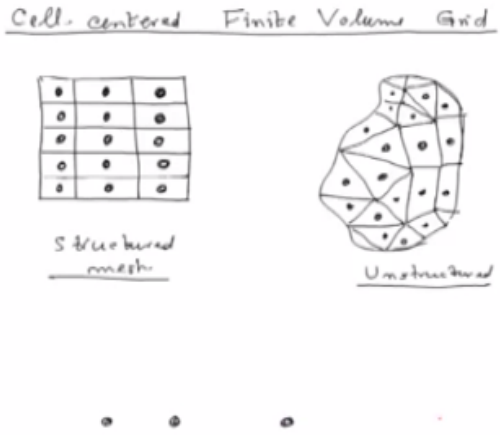

The first one is cell-centered grid, finite volume grid. So, whatever you problem domain might be. Suppose for the sake of simplicity let us take one rectangular domain, discretize it into a set of finite volumes. Suppose we have taken these rectangular finite volumes. Now, chose the centroid of each one of these. The centroids are the ones which are now our computational nodes. So, as you can see this process is fairly simple.

It is applicable to not just the structured grid. Suppose, we have a unstructured gird over a fairly complex arbitrary domain and we had used an unstructured mesh generator to divide it into a collection of non-overlapping finite volumes which could be triangular, which could be rectangular. Now, these finite volumes could be of arbitrary shape generated using our favourite unstructured mesh generator program.

We can have a collection of elements of any type. Now, to define our grid, what we need to do is take any shape and just find out where the centroid is. Now finding centroid of a geometric shape

is a fairly straightforward operation because we know the coordinates of the vertices and in terms of the vertex coordinates, we can easily determine what would be the centroid of these cells, okay. So, generating a cell-centered finite volume grid is a fairly straightforward.

So, we will have a look at two examples; one on structured mesh and another one in the case of unstructured mesh. Now, let us have a look at the second one. Second one, we said look we are going to now define our computational nodes first.

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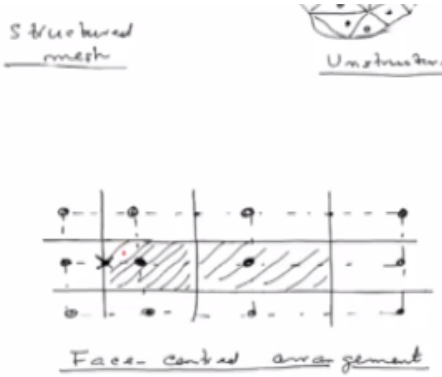

So, let us say this is our choice of the computational nodes and next what we do, draw the mid planes. For instance, between two for the time being, connect these with dotted lines. We need to find out the mid planes between these nodes which will define our control volumes surfaces. So, this is one of the mid plane, this is mid plane here. Similarly, mid plane between these two. So, these are the mid planes.

Now, the intersection of these surfaces that is what will give us the finite volume. For instance, if you look at this particular node. Now, these are the corresponding finite volume. Now, you can clearly see in this particular figure or its adjoining finite volume, the computational node is not at the centroid of our finite volume. So, this is our face-centered arrangement and you can clearly see that we can generate such faces which will intersect and lead us to non-overlapping finite volumes only if our nodal arrangements were in a structured fashion.

So, this face-centered approach can be utilised only for structured grids. The cell-centered approach can be used for both structured as well as unstructured finite volume formulations. Now, one more thing which we can say here, let us go back to our structured grid case. Let us take one computational node. The value at this particular computational node that would in a sense represent the average over the finite volume.

So, thereby what we expect is that the representation of our unknown variable can be done to a much greater accuracy in the case of cell-centered finite volume approach. But in our formulation of our integral equation, we need some surface integrals where we need to evaluate certain fluxes which are involved in gradient terms. So, if look at the evaluation of those gradients, they would involve the derivatives.

Derivatives can be obtained at the cell faces with greater accuracy in the case of our face-centered approach because now they have got two computational grids and the face-centered at that particular point is midway between the two. So, we can use very accurate central difference approximation for evaluation of derivative at this face centre. But never this because of the versatility of the cell-centered approach that it can be used for both unstructured as well as structured grids.

The cell-centered approach is the one which is used more commonly. So, let us summary of what we just discussed.

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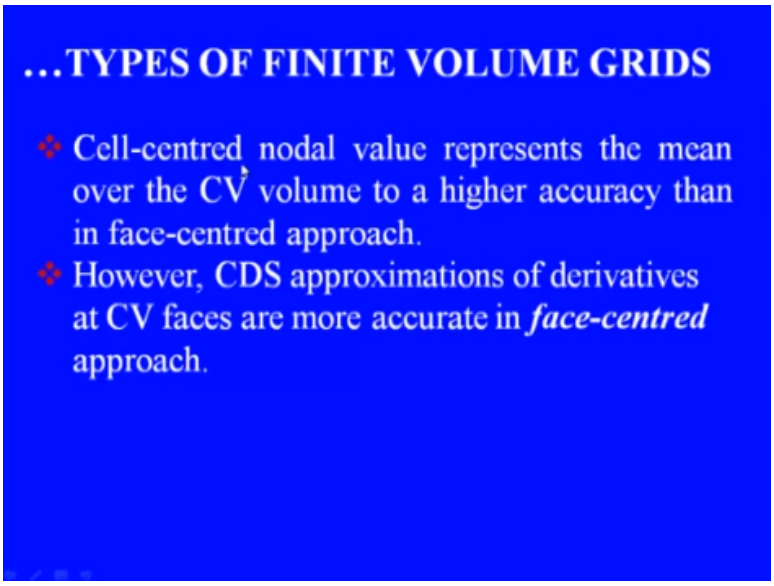

A cell-centered nodal value represents the mean over the control volume to a high accuracy than face-centered approach; however, CDS approximation of derivatives at CV faces are more accurate in face-centered approach. We have already seen the cell-centered approach can be used with structured or unstructured grids whereas the face-centered approach can be used only with structured grids, hence most of the time it is cell-centered approach which is used.

So, this cell-centered approach is more commonly used in finite volume formations.

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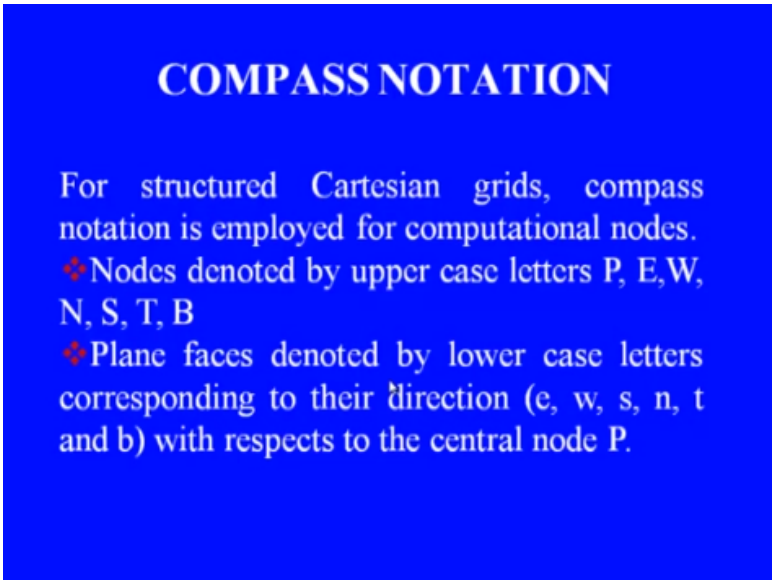

Next, we are dealing with structured grids in finite volume and also it also be used on notation fairly similar to what we had already seen earlier in the case of finite difference method when we

define our computational molecule, we use what we call a compass notation. So, this compass notation can also be used for structured Cartesian in finite volume grids and in the case of finite volume formulation, we will have two sets of things which we need to look carefully.

We have got our computational nodes. So, computational nodes are denoted by upper case letters, the P is our central node or the centroid of the finite volume of interest to us, E is the eastern neighbour node, W is the western neighbour node, N is northern node, S is southern node, T top, B bottom node, and so on. The plane faces, i.e., the faces between let us say the computational node P and E, the face for control volume or finite volume which lies between nodes P and E that is denoted by a lower case letter e.

So, similarly, rest of the cell faces, we use the corresponding symbols depending on the directions. So, plane faces denoted by lower case letters corresponding to their directions by e, w, n, t and b with respect to the central node. To explain it more clearly, let us have a look at the 2D situation for a compass notation. So, let us first draw a structured grid including some extractions in 2D. So, x direction and y direction.

Suppose, this is our central finite volume with reference to which we want to illustrate our compass notation. So, the centroid of this finite volume, that is what we will denote by P and the corresponding faces is XI, XI+1, XI-1. Similarly, this is YJ, YJ+1, XI+2, YJ +2 and so on. The computational nodes to the right which is in eastern direction with respect to it be viewed use a symbol E. The computational node on west that would be denoted by symbol W.

If we had (()) (36:13) further, the nodes to the east of this would use two capital letters EE and so on. Similarly, in the Y direction, the node in the positive Y directions will be called the northern node. This becomes NE, then is computational node NW. In negative Y direction towards southern side immediately to next, this becomes computational node S. This is computational node SW, SE, so it will become SEE and EE and so on.

Now, let us look at the faces. The finite volume face between the nodes P and E, i.e., to the east of P that would be denoted by symbol e, s, w, n and so on. If we had a three-dimensional thing

with one towards top of it, that node would be represented by symbol T and the face between P and T that would be denoted by symbol t and so on. So, this is our compass notation which would be used with structured finite volume grid.

In the case of unstructured finite volume grid, we would go back to the lessons which we would learn from finite element method. We would use the element connectivity in a state to denote these faces and surfaces.

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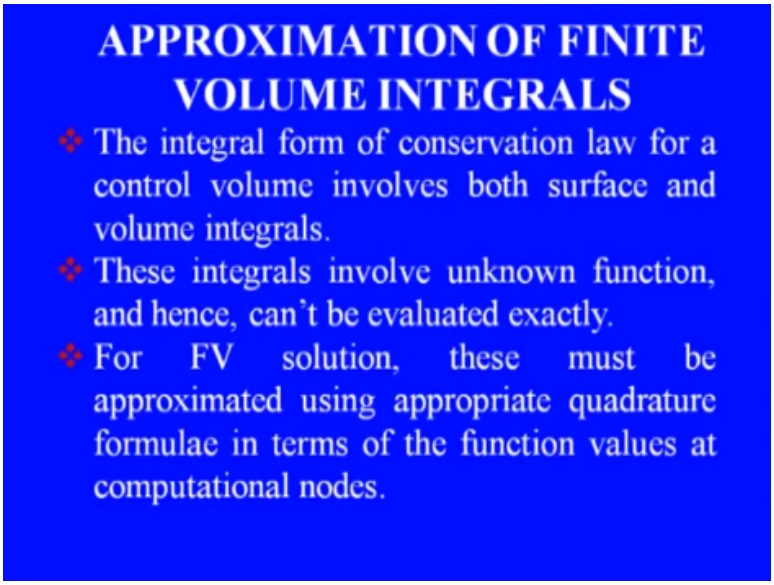

Now, next let us have at the approximations which are involved in finite volume formulation. So, what we saw in the integral form of conservation law rehab. We have both surface and volume integrals and these integrals involve unknown function values; and hence, we cannot evaluate them exactly. In fact, they cannot be evaluated because they are in terms of unknown function values.

So, for finite volume solution, what do we do. We will assume that our unknown function is a specified at computational nodes and use the function values at computational nodes to obtain an approximation for these integrals using appropriate quadrature formulae. So, now let us first have a look at the schemes for the surface integral. How do we approximate the surface integral.

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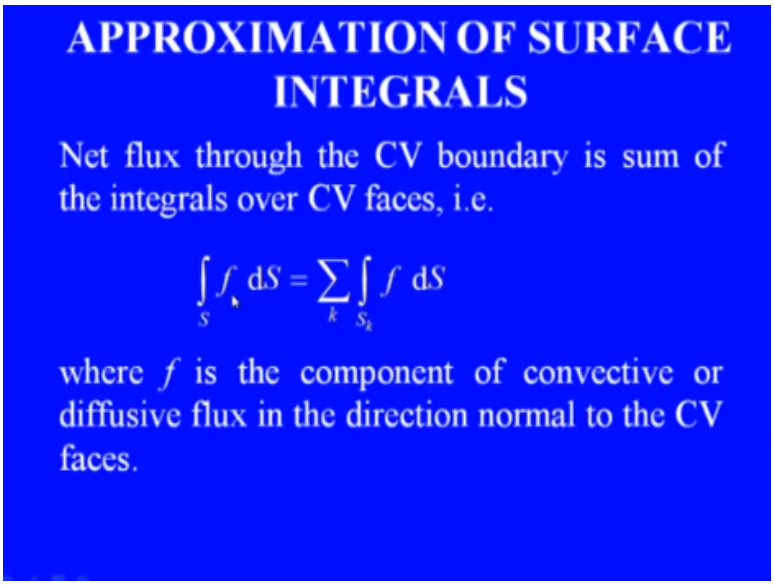

Now, let us take a generic function F and we want to find the net flux through the CV boundary. This is sum of integrals over CV faces. So, integral of SFDS=summation/K integral/SKFDS where SK is one particular face of volume or finite volume. Now, here our F could be a component of a convective flux or a diffusive flux in the direction normal to the control volume face. So, if we can obtain a quadrature formula for one of these faces.

Let us say, K=E, i.e., eastern face. We can derive derived the formulas for rest of the faces as well. So, now let us take eastern face as our explain face.

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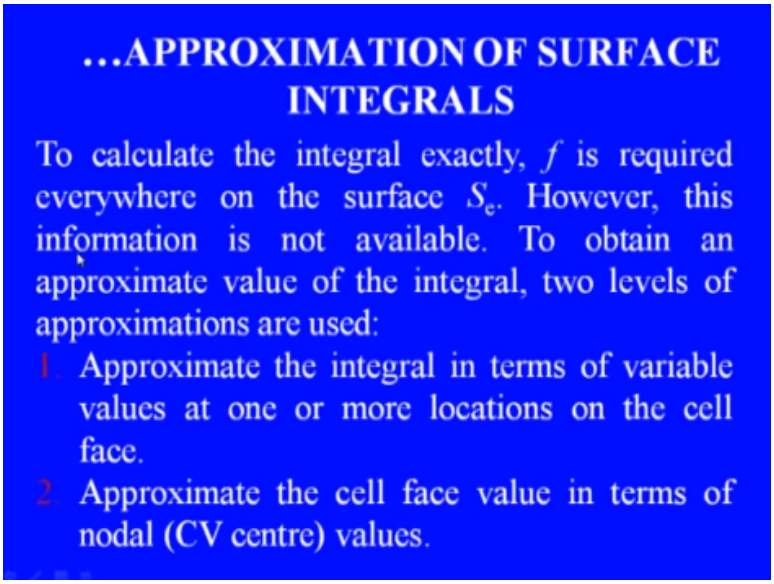

So, suppose you want to calculate the integral exactly, then we require F everywhere on the

surface of the eastern face, SE. However, this information is not available. In fact, where we know in terms of computational nodes values, that is only specified at the computational nodes. So, we have to obtain an approximation for this F. So, to obtain an approximate value of the integral, we have to use what we call two levels of approximation.

The first one is approximate the integral in terms of the variable value at one or more locations on this cell face SE, i.e., the first step which we have to perform. Now, that value itself would be unknown. So, that cell-face value has to be approximated in terms of the nodal values because nodal values are the ones which we are going to solve for. Those are available and those are the ones which will come in a final discrete algebraic equation.

So, now let us approximate the cell-face values in terms of our values at computational nodes or CV centre values. So, these are two levels of approximations which would be involved in the evaluation of all the integrals. So, we will first have a look at the first approximation that how do we approximate the integral in terms of the values of variable F at one or more locations on the cell face.

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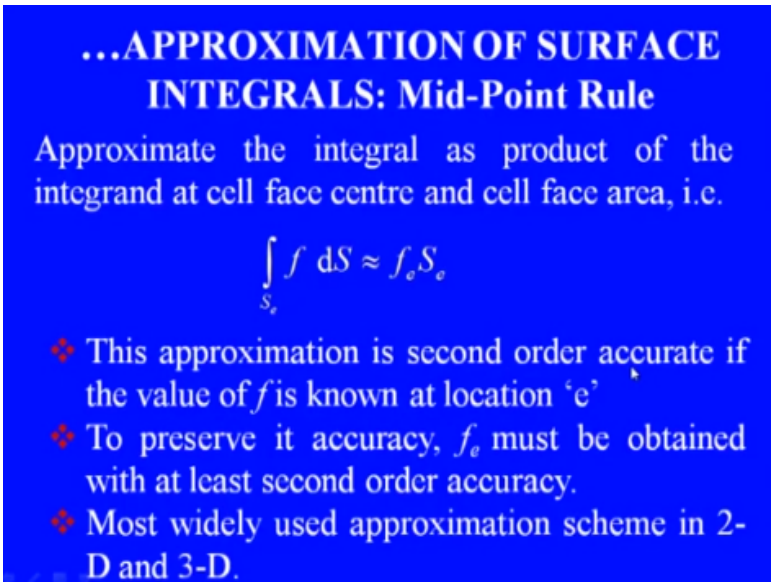

The first approximation scheme is called midpoint rule. It looks extremely simple but it is extremely useful and powerful. This is the one which is used almost exclusively in unstructured finite volume codes. So, approximate the integral as the product of the integrand at cell face

centre and the cell face area, i.e., integral of FDS at this face SC can be approximated as the value of F at the centre of this face which we will denote but f subscript e multiplied by the area of the face.

So, even this simple approximation, it is of second-order accuracy. If value of F is known at the location E where this E denotes the centre of the face SE. So, if you want to preserve the second-order accuracy, we have to obtain FE in terms of the values at computational nodes to at least second order accuracy and how would we do this. This part we will have a detailed look in our next lecture wherein we will discussed different interpolation schemes.

Now, the so-called very simple midpoint rule, it is widely used in 2D as well as in 3D. In fact, in 3D unstructured formulations, this is the scheme of choice. Next, could be that for two-dimensional problems, we can use Trapezoid rule which will require the values of F at CV corners and this is given by integral FDS is approximately=S/2 FNE+FSE. Basically, what do we do is that we want to approximate this value over the cell face as an average of FNE+FSE multiply that by area, this what would give us the value of this surface integral.

If you want a higher-order accuracy, Trapezoid scheme has again got the second-order accuracy. If you want a higher-order scheme in two-dimensional, we can use what we call Simpson's rule.

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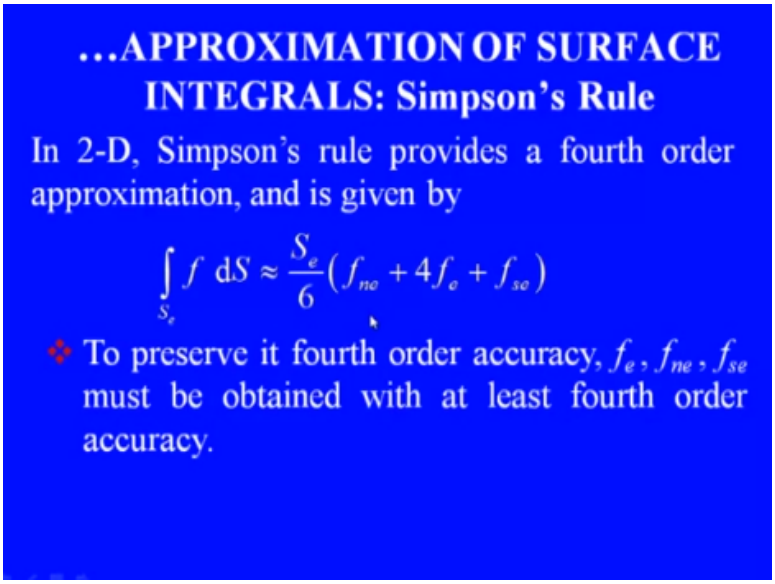

So, it provides a fourth-order approximation and here we will have values at three points involved. The function values at north-eastern point, north-eastern corner, the centre of the face E and south-eastern corner. So, in terms of these values, the integral is given by integral SE FDS is SC/6. Value of F at north-eastern corner plus 4 times value of F at the centre point of the face plus the value of F at the south-eastern corner.

Now, the FNE, FE and FSE these would have to be calculated using the nodal values at the computational nodes. So, if you want to preserve the fourth-order accuracy of the Simpson's rule formula, we have to evaluate FE, FNE and FSE with at least fourth-order accuracy. So, in 2D yes we can have fairly good amount in interpolation formula, but in 3D we would find the Simpson's rule to be too demanding to use. So, we will stick to a midpoint rule in most of three-dimensional finite volume analysis.

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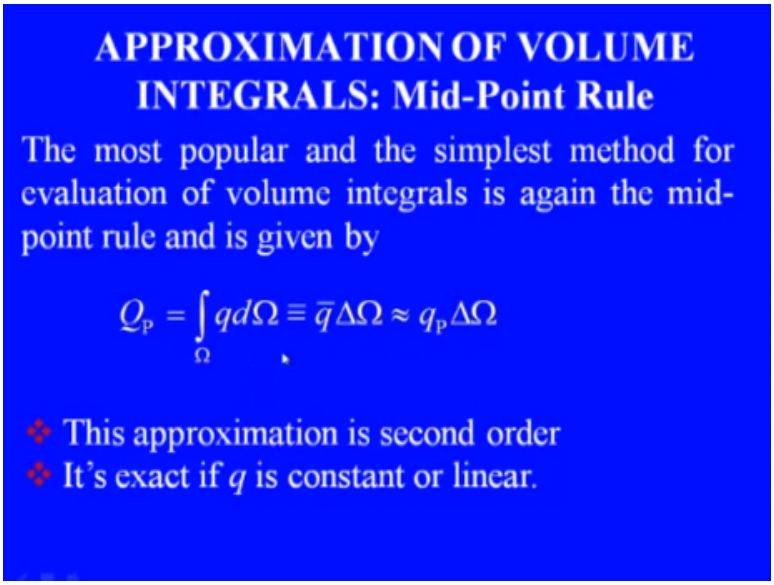

Next, is approximation of a volume integrals because we had two integrals in our conservation rule which involved volume integrals. So, the most popular and the simplest method of evaluation of volume integrals again our midpoint rule, i.e., take the value at the centroid multiply that by the volume and that gives us the approximate value of the integral which we denote by QP the integral of QD omega.

So, this is approximately equal to Q bar delta omega and this is equivalent to Q bar delta omega

where Q bar is an average value over the cell and it can be approximated as value of Q at the centroid of the cell multiplied by the volume the cell. Now, once again, the simple approximation scheme has got second-order accuracy and in fact it would lead to exact value of this integral if Q were constant or linear.

So, once again I would like to emphasize that this particular formula which looks utterly simple is the one which is used very widely in finite volume analysis. But suppose you are interested in higher-order methods and if you want to develop fairly complicated code, yes you can do that.

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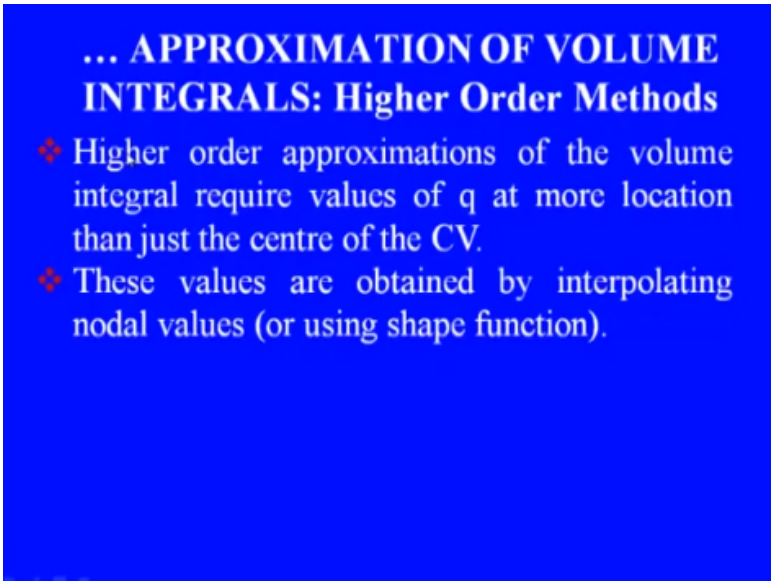

You can obtain higher-order approximation of the volume integrals. In terms of the value of Q at more locations than just the central of the CV and these values can be obtained by interpolating the nodal values or we can use the shape functions very similar to what we will see in our finite element module later on.

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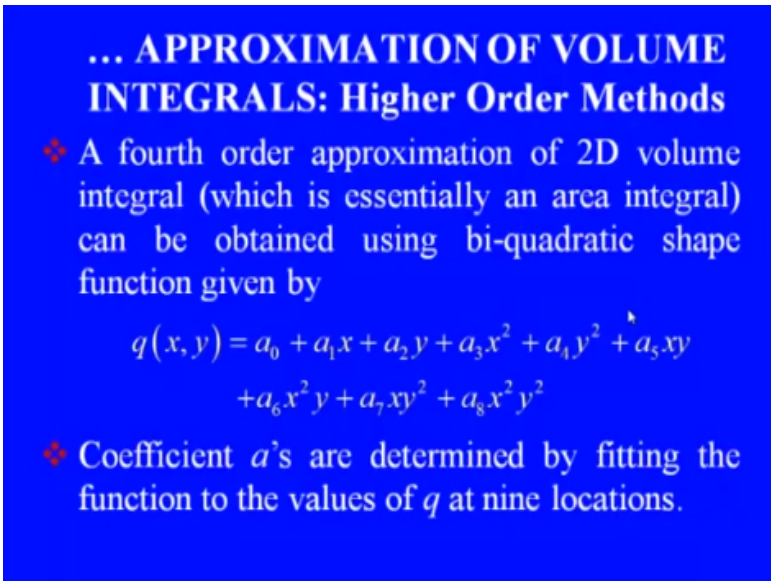

For instance, let us say we are dealing with 2D volume integral which is simply an area integral. So, we can use a bi-quadratic shape function for approximation of variable Q. QXY can be given in terms of quadratic values. So, A0+A1X+A2Y+A3X square+A4Y square+A5XY+A6X square Y and so on. So, now these coefficients which you have got A0 to A8, they are determined by fitting the function to the function value at nine locations.

We have got these nine unknowns, so we will fit them at nine locations and we can obtain the value of these coefficients in terms of the nodal values of the function.

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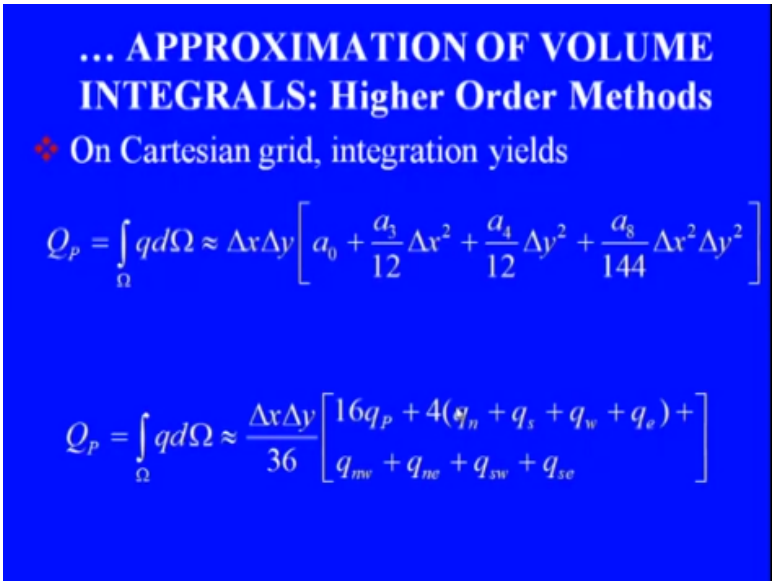

On a Cartesian grid, if perform the integration, you will get a simplified form that QP is given by

delta X*delta Y [A0+A3/12 delta X square+A4/12 delta Y square+A8/144 delta X square delta Y square], and if we had a uniform Cartesian grid, the equation becomes even simpler.

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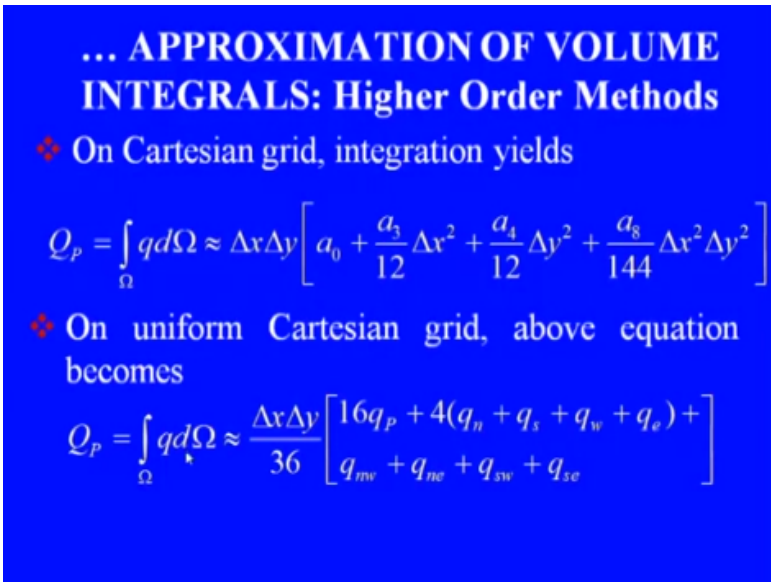

So, it can be reduced to this form. So, QP=QD omega that is approximately=delta X delta Y/36[16Q at P+4 times Q at N+Q at S+Q at W+Q at E]. So, Q n, s, w, e, they are the values with respect to face centres plus Q at north-eastern corner plus Q at north-eastern corner plus Q at south-western corner and Q at south-eastern corner.

You can easily appreciate that obtaining each one of these in terms of the (()) (49:19) coordinates would fairly involve task because if you want to retain the fourth-order accuracy of this formula, we have to approximate each one of these Q values to at least fourth-order accuracy which is not a main task. So, in practical analysis, we would basically restrict ourselves to midpoint rule integration.

So, this is where we would stop in this lecture. In the next picture we will have a look at the interpolation schemes whereby we obtain the values of variables at face centres of the corners in terms of the nodal variables.

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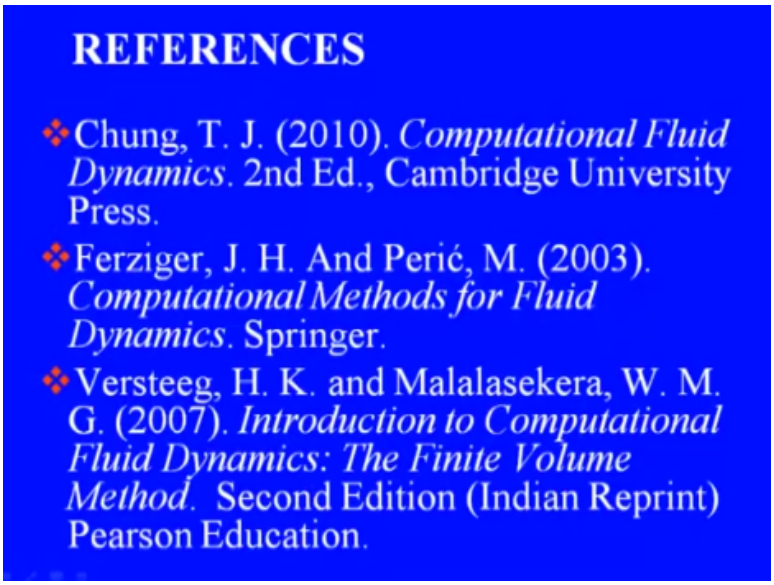

In the meanwhile, if you are interested, you can look at these references, book by Chung. It also gives you some details about Finite Volume Methods. Similarly book of Ferziger and Peric, it contains two chapters on Finite Volume Analysis and there are two books which are exclusively devoted to Finite Volume Method by Versteeg and Malalasekera's book. This is introduction to CFD based on Finite Volume Method.

Similarly, you can also pick up the classic book of (()) (50:29) anchor on CFD which is again based on Finite Volume Method.