Computational Fluid Dynamics
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Lecture – 27
Applications to Unsteady Transport Problems

Welcome, to the last lecture in Module 5 on Time Integration Techniques. In this lecture we shall focus on the application of the time integration schemes which we learnt in the previous lectures to unsteady transport problems. So let us have a recapitulation of what we did in the last lecture. We discussed multilevel methods for time integration of initial value problems.

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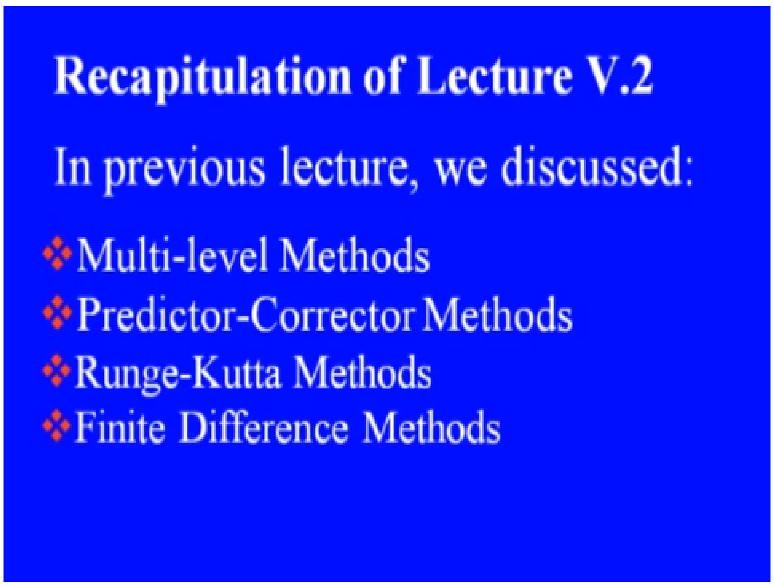

Then we looked at predictor corrector methods which included Runge Kutta methods. We also looked at few methods which we can derive using finite difference approximation of the time durative. Now let us apply a sample of these schemes to generic transport equation. So that is what we will focus on this lecture application of time integration techniques to unsteady transport problems.

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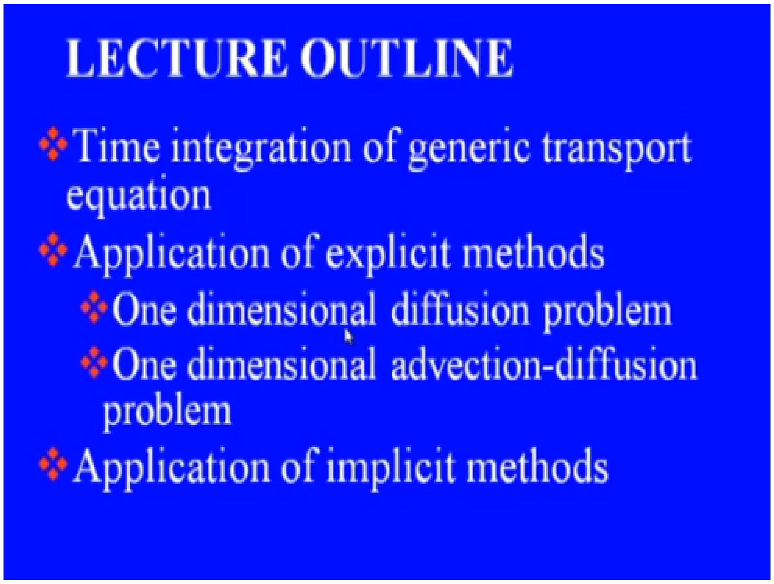

So we will have a look at how do we apply these schemes which you learnt from initial value problems for time integration generic transport equation. You will look at application of explicit methods to both diffusion problem and one-dimensional advection diffusion problem. Their extension to multidimensional problems is fairly straightforward and will also outline the application of implicit methods to diffusion and advection diffusion problems.

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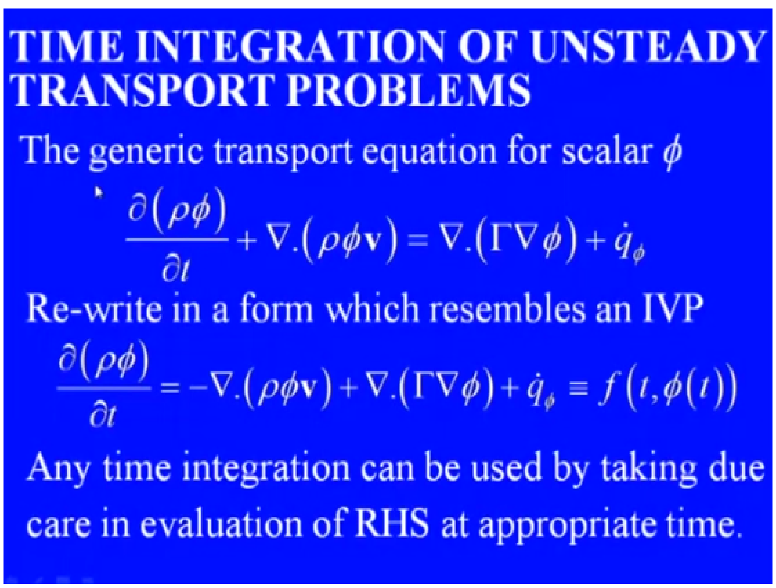

Let us have a look at our generic transport equation for a generic scalar phi. So let us recall the governing differential equation which we had obtained for the transport of scalar phi. We had the time derivative term del of rho,phi/del T+ this convicted term divergences of rho V phi=divergences of gamma times gradient phi this was for the diffusive term plus the last term was our source term, Q dot phi.

Now what we can do is we can try and rewrite it in terms of or in the form fairly similar to that of an initial value problems by keeping the time derivative on one side, transfer the remaining terms on right hand side that is say we will rewrite it as del rho phi/del T=-divergence of rho phi V+ divergence of gamma times gradient phi+Q phi dot. Now this right hand side is basically our generic function F which we have increased in the RHS of an initial value problem F T gamma phi T.

So on the left hand side we have got this time derivative and on the right hand side we have got a function which depends on t and phi t. So now we can use any time integration scheme which you have learnt in the previous two lectures for integration of initial value problem for this generic transport equation and we have to be just careful how do we evaluate this f t phi t because it involves these divergences terms.

So they have to be evaluated at appropriate times step and these derivatives ought to be replaced by corresponding finite difference approximation.

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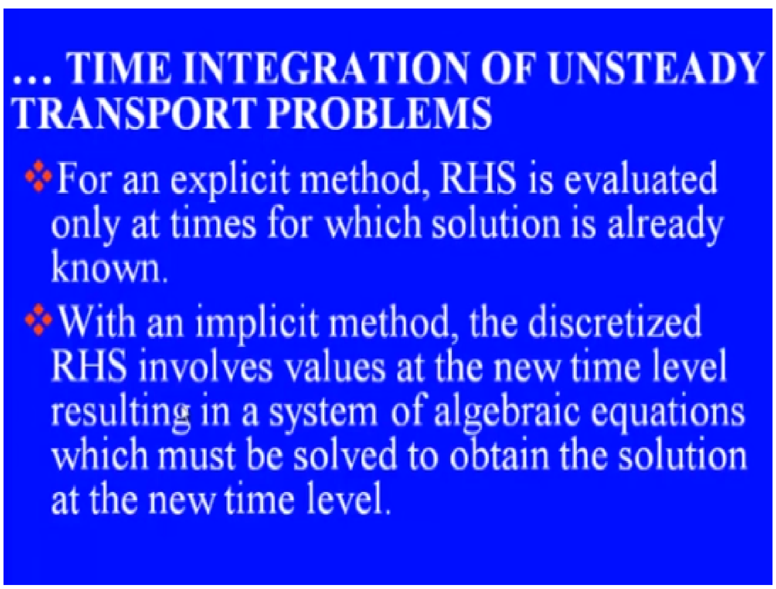

Now if you use an explicit method everything in the right hand side is evaluated only at times for which solution phi this already known and if you use an implicit method the discretized right hand side would involve values at new time level which would result in a system algebraic equations which must be solved to obtain the solution at new time level. So now there is a trade off, if you want to use an explicit method the evaluation per time step is very fast but you have already learned there is some stability concentrations which limit the value of delta T which we can use.

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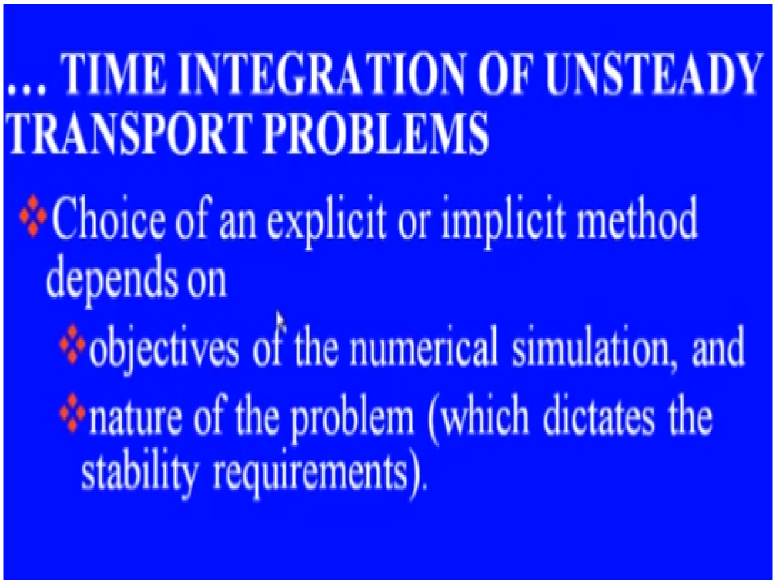

So weather we choose an explicit method or an impressive method this would depend on the objective for numerical symbolization and the nature of the problem which would dictate the stability requirements. Let us elaborate these things further.

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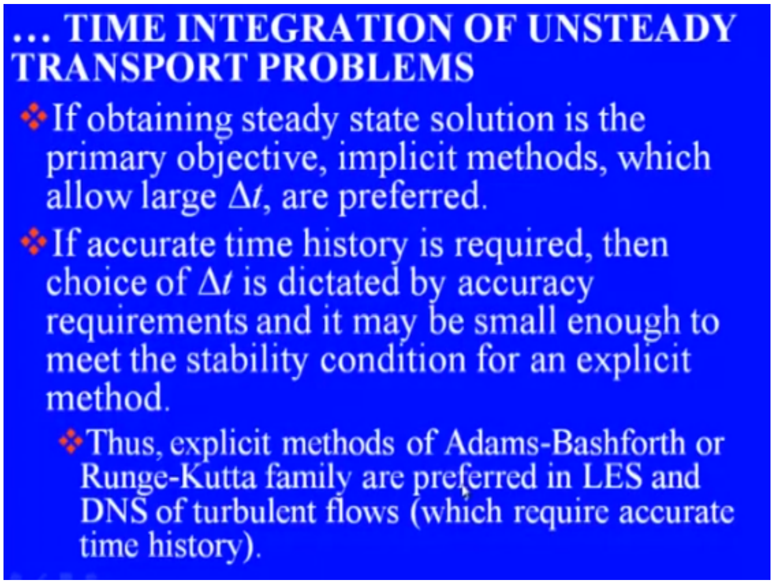

Suppose we want to obtain steady state solution that is our primary objective we have got an unsteady heat conduction problem or an advection diffusion problem but what we are ultimately interested in is obtaining what happens at T=T infinity that is state what is the steady-state solution. So now in this case if you chosen an explicit method we would be forced to use small valuable of delta T because of stability considerations. So we will have to perform or competitions for many many times steps.

So in such situation implicit method might be preferred which allow last delta T because there is no constant (()) (05:28) imposed on delta T so suppose if you use an implicit backward Euler method so we can use pretty large value time step and we can obtain the steady-state fairly easily though at each time step we have to solve the system of equations. So that is why in general if we have to obtain steady-state solution that is our primary objective, in CFT we prefer implicit methods.

On the hand if you want accurate time history then the choice of delta t would be dictated by the accuracy requirements. So as you know that smaller delta t more accurate would be your solution, so it may so happen that this accuracy requirement forces us to take a delta t which is fairly small and it is small enough to meet the stability condition for an explicit method. So, in this case it does not make much sense to solve a systemic (()) (06:24) at each time step by choosing an implicit method, we should instead go by an explicit method.

And this is one such case where for instance if you want to perform large eddy simulation or direct numerical simulation of turbulent flows these require accurate time history that we have to obtain our solution at each time step very accurately for velocity field as well as pressure field as since in such cases we prefer explicit methods of Adams-Bashforth or Runge-Kutta family.

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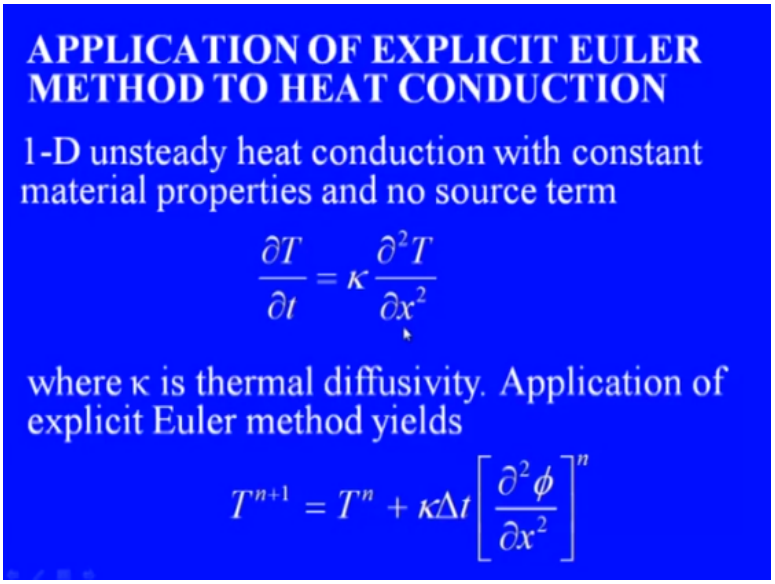

Now let us have a look at application of explicit Euler method first to heat conduction and then we will discuss its application to the advection diffusion problem. We will also discuss briefly about the stability of different schemes, explicit schemes in particular. So unsteady heat conduction equation suppose we assume that material properties are constant that is we have got constant rho CP and thermal conductivity case.

In that case we can dump everything in our parameter which we call thermal diffusivity. So del capital T/del T where capital T is our temperature=kappa times delta T/del x square. For the time being we have ignored the source term, if you want we can just put it does not really affect time integration procedure or the stability of the scheme. So now for the sake of simplicity of the formula we have just noted down the case where there is no source term.

Note down here that kappa is thermal diffusivity. Now let us use our explicit Eula method. So capital T and +1=capital TN+ kappa times delta T. Del 2phi over del x square, now this derivative has to be evaluated at the known time instant and now we should choose an appropriate finite difference approximation to this second-order derivative with a special coordinate x.

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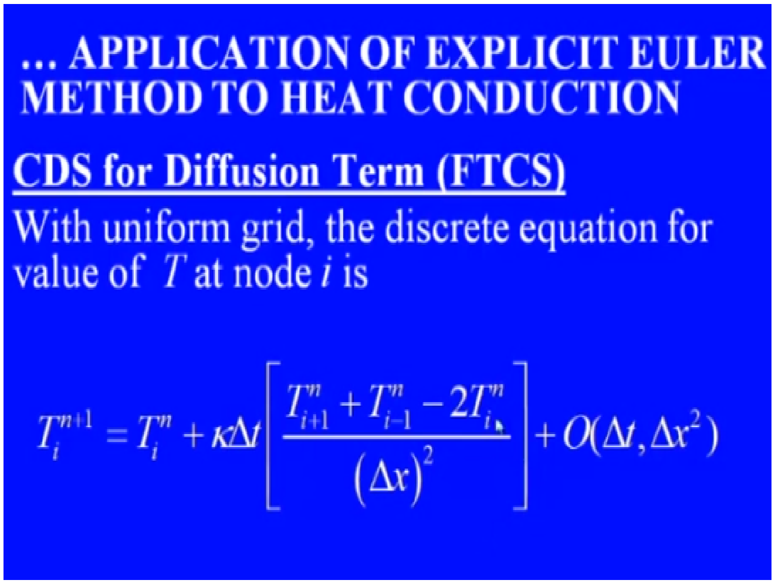

So suppose we choose central different scheme which is called CDS for the diffusion term that is delta T over del x square so the resulting scheme is popularly called FTCS, Forward Time Central Space. So this is (()) (08:50) is forward Eula time and central difference scheme space and if you chose a uniform grid then the discrete equation is simply this TIN+1=TIN+ kappa times delta T. The terms in the brackets [TI+1N+TI-1 at N-2TIN/delta x square], this is our central difference approximation for the second-order derivative and time.

And note down this particular scheme it is first order accurate on time (()) (09:24) would be first-order because we have used forward Eula method which is first order accurate in time and we have used central difference approximation in space so that is why its accuracy or (()) (09:37) in a space would be of the order delta x square. So this FTCS scheme it is first-order accurate in time and second-order accurate in space.

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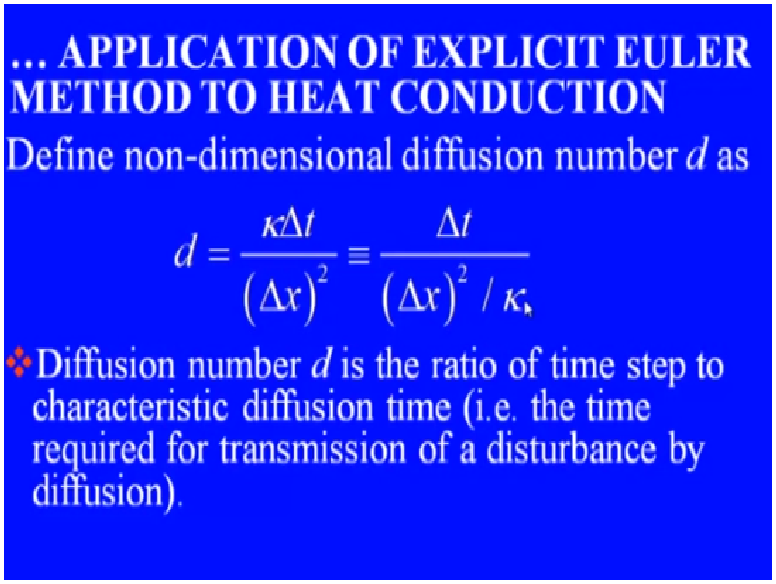

Now let us introduce a non dimensional which would call diffusion number D to simplify our expressions so D is kappa delta t/delta x square. We can rewrite it as delta t/delta x square/k. So on the numerator what we have got, this is our time step and this delta x square/k, this would again give us the time scale.

So now this particular time scale it gives us what we call characteristic diffusion time that is the time which would be required for transmission of a disturbance by diffusion. So this diffusion number is the ratio of time step to the characteristic diffusion term. So in terms of this non-dimensional parameter our algorithm can be written in a very simple form.

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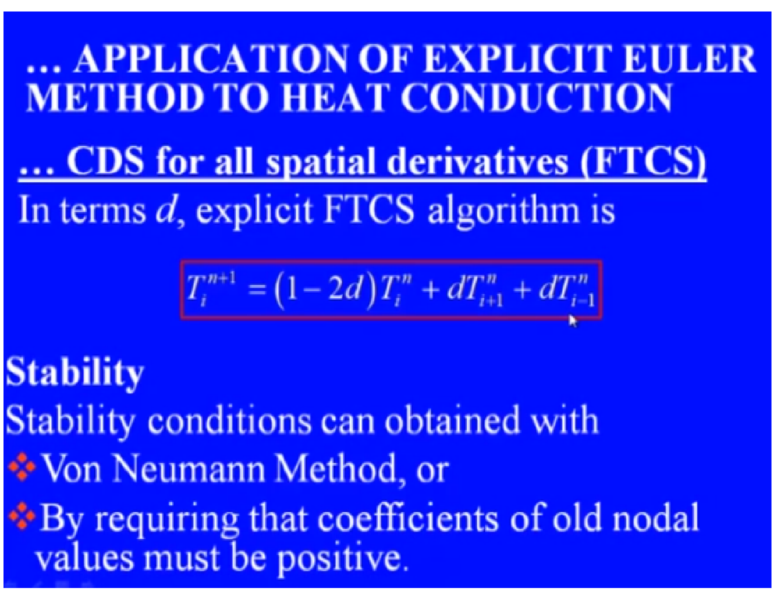

So explicit FTCS algorithm for 1D heat conduction equation can be written as TIN+1=1-2D TI at N+D times T of I+1N+D times T at the grid point I-1 evaluated at time level N. Now let us save the stability of this algorithm we have already seen the accuracy aspect and stability conditions can be obtained with Von Neumann method or by requiring the coefficients of old nodal value must be positive.

This is a physical requirement if you look at this formula, on the right hand side we have got one 1-2D, the coefficient of TIN, the remaining ones TI+1 and TI-1 their coefficient is D which is a positive number. Now if you want a similar sort of contribution from all the three nodes at a future time instant, the contribution coming from each grid point I, I+1, I-1 that should be positive otherwise that might lead to oscillation or instability in the solution.

So this one of the requirement or one of the way which we can look at a major stability of an explicit scheme would be to look at the coefficient of each term and require that each coefficient must be positive. Now let us try and derive this stability condition using both the approaches.

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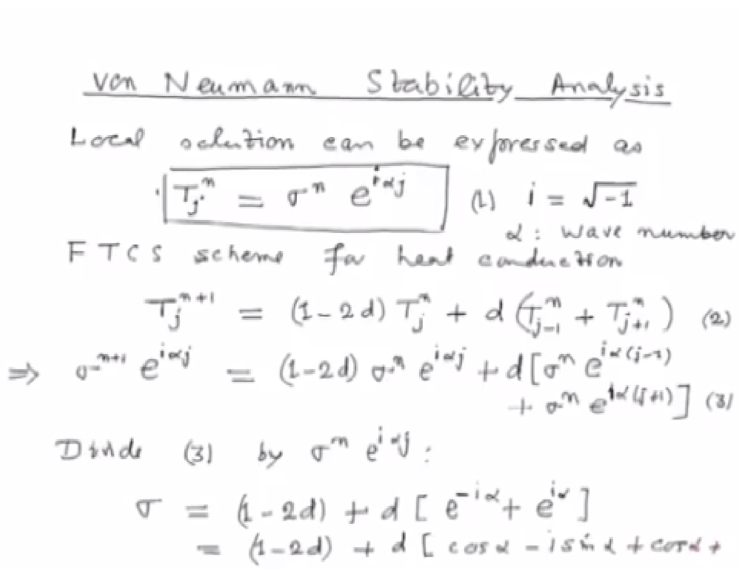

First let us have a look at this Von Neumann approach. So Von Neumann stability analysis. Now this Von Neumann stability analysis is basically related to the local solution that we are looking at a solution in the vicinity of particular node and what we can do is this local solution can be expressed in terms of complex eigenvectors. So that is the assumption which is made in this Von

Neumann stability analysis so local solution can be expressed as TJ at N sigma to the power N where sigma is I scalar quantity.

E to the power the straight I alpha J. Now this straight I which we have put in italics I this I is ever imaginary number–root 1, the square root of -1this what we call is our imaginary number. Alpha is what is known as wave number. Okay so now in terms of this TJN the way we have defined it let us write down our hat explicit schemes FTCS scheme which I have written earlier. FTCS scheme for heat conduction. So this at TJ at N+1, this was given by 1-2D TJN+D times TJ-1N+TJ+1N.

So let us have (()) (15:06) in the terms of the complex eigenvectors for all these variables (()) (15:11) now TJ N+1 would become on the left hand side we will have sigma N+1 E to the power I alpha J where I is a complex number. On the right hand side, we will get 1-2D sigma to the power N, E to the power I alpha J+D times sigma to the power N, E to the power I alpha J-1+sigma to the power N, E to the power I alpha J+1.

Now let us divide this equation, both sides of this equation 3, divide 3/sigma to the power N, E to the power I alpha J. So what we have on the left hand side is sigma, sigma=1-2D+D times E to the power-I alpha+E to the power I alpha. E to the power-phi L find it the power I alpha they can be expanded in terms of sign and cos. So this would become 1-2D+D times cos alpha-I, sign alpha+cos alpha+I, sign alpha. So now let us correct the terms 1-2D+twice of D cos alpha, the imaginary terms, they cancel out.

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$$\sigma = (1-2d) + 2d\cos\alpha$$

$$\Rightarrow \boxed{\sigma = 1 - 2d(1-\cos\alpha)}$$

Stability requires $|\sigma| \leq 1$, i.e.

$$-1 \leq 1 - 2d(1-\cos\alpha) \leq 1$$

(i) $1 - 2d(1-\cos\alpha) \leq 1 \Rightarrow -2d(1-\cos\alpha) \leq 0$

$\Rightarrow 1 - \cos\alpha \geq 0 \Rightarrow$ holds for any α.

(ii) $-1 \leq 1 - 2d(1-\cos\alpha)$

$\Rightarrow 0 \leq 2 - 2d(1-\cos\alpha)$

$\Rightarrow 2 \geq 2d(1-\cos\alpha)$

$\Rightarrow d \leq \dfrac{1}{(1-\cos\alpha)}$

Max. value of $1-\cos\alpha = 2$

$\therefore \boxed{d \leq \frac{1}{2}}$

Now let us rearrange it as sigma=1-2D 1-cos alpha. Now if we want our solution to be stable we would require that magnitude of sigma should be <1. So stability requires magnitude of sigma should be <1. So we get two possibilities that is the -1 should be <1-2D*1-cos alpha and this should be <1. So these two inequalities let us take one by one. Let us take first right inequality so 1-twiceof D 1- cos alpha is <1.

If you subtract it subtract 1 from both the sides this will lead to -2D*1-cos alpha, this should be <0/-2D so this inequality changes and we get 1- cos alpha should be>0. We have got this equal to signs here. Now remember cos alpha is always <1 in magnitude so this holds this inequality holds for any alpha. So this does not give us any requirement. Now let us have a look at the second half of the inequality is -1<I=1-2D, 1-cos alpha.

Let us add1 both the sides of the inequality so we get 0<or=2-2D1-cos alpha. Now this tells us 2 is >or =twice of D 1-cos alpha or we get D should be < or =1/1-cos alpha. Now what is the maximum value of 1-cos alpha. Minimum value of cos alpha is -1so this maximum value of 1-cos alpha this would become 2. So therefore this inequality says that D should be <or =half. So this is the stability requirement which we get from Von Noemann analysis. That is say our diffusion number should be < 1/2 which puts a restriction on the time step.

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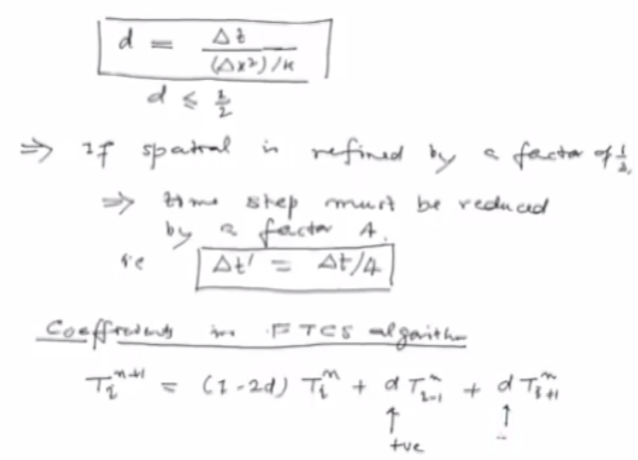

Okay now let us have a look at few more consequences of this requirement. Let us note down that D was given in terms of delta t/delta x square/k. Now for stability we want d to be < or = 1/2. So we have chosen 1 delta t for time integration and suppose in some regions we want to refine our spatial mess, so if a spatial mess is refined by a factor of 1/2, the definition of D and this requirement size that the time step must be reduced by a factor of 4 that is new delta t prime that should be delta t/4.

So now that is a severe restriction and that is the reason why this explicit Euler method is not suitable for the problems for which you would like to reach the steady-state quickly. This particular method we are going to retain or use only for the cases where we require very accurate value of temperature at each time step that you say if we want a very accurate time history.

Now remember this particular condition which we derive from Von Neumann stability analysis, we could have also derived it earlier while accessing the coefficients in this FTCS algorithm, what were the coefficients let us have a relook at TI at N +1 was 1-2D TIN+D times TI-1N+T times TI+1N. Now the last two they are anyway positive.

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Stability requires that the first coefficient must be positive that is for 1-2D should be >0 and which straightway leads to the same condition that d< or =1/2. So whether we perform the Von Neumann stability analysis or we impose the restrictions of all coefficients for all time values of the function, this should be positive, they lead us to the same stability condition.

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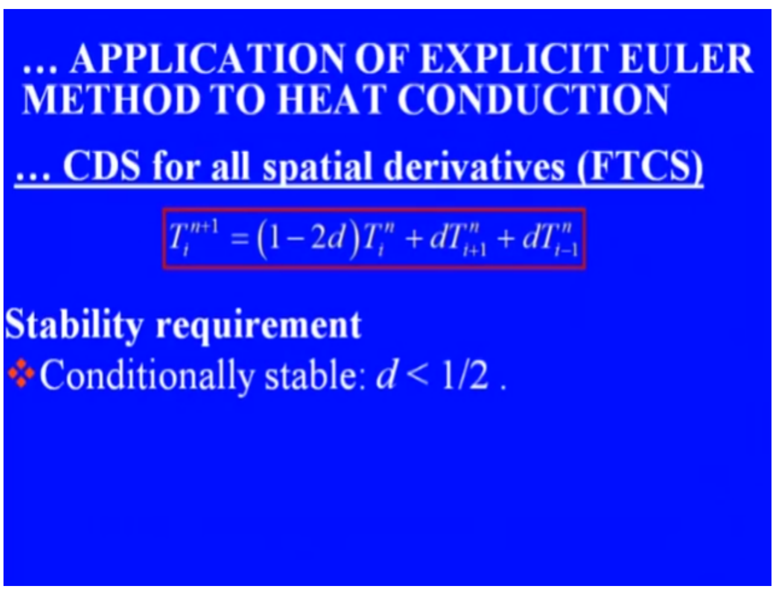

We can say this FTCS algorithm applied to the heat conduction problem, the stability requirement of the diffusion number d should be <1/2. Now let us apply our explicit equation Euler method to advection diffusion problem.

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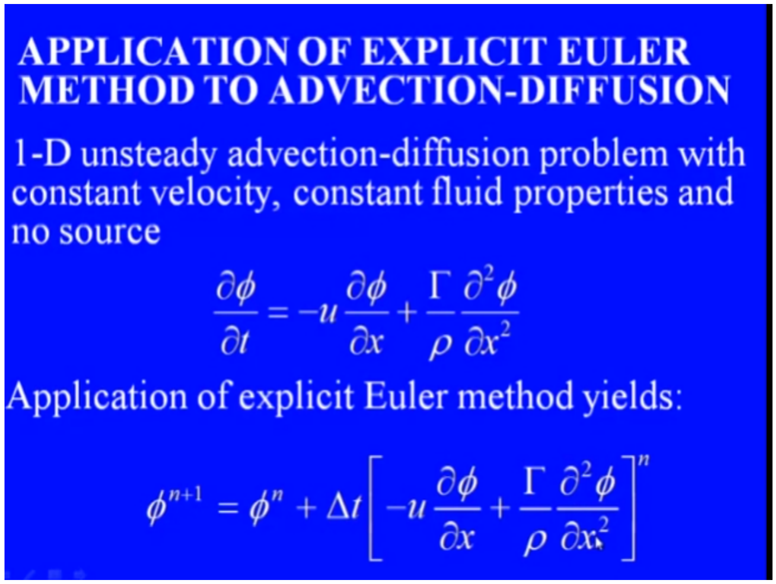

So 1-D advection diffusion problem we have only seen its generic transport equation and written in terms or reform very similar to an initial value problem where we would assume the velocity to be constant and fluid properties also to be constant and no source term, so we will write it as del phi/del t=-u del phi/del x+gamma by rho del2 phi/del x square. So this is in a form very similar toward our standard initial value problem so d5/dt and on the right hand side this is our function f.

So let us apply explicit Euler methods, we get phi n+1=phi n, delta t times is function evaluated at tn, that is -u del phi/dx+gamma/rho, del 2 phi/del x square at time instant n. Now you have got two derivatives here, one with the conviction term, del phi/del x and one with the diffusion term del 2 phi/del x square and at each grid point we need to find out their finite difference approximation.

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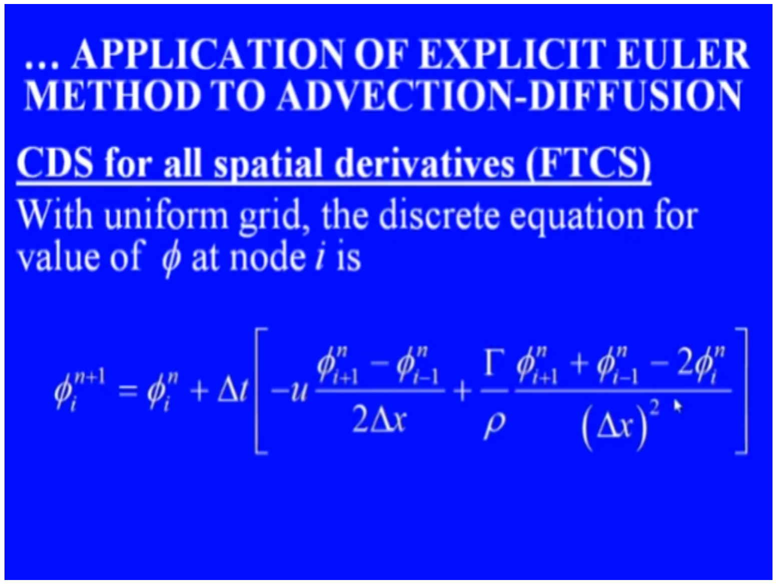

So now we choose a uniform grid and we use central difference scheme for all special derivatives that will again lead to what we call FTCS scheme so the discrete equation for the value of phi at node I is phi I n+1=phi I n+delta t-u times phi I+1n-phi I-1n/2 delta x. This is a central difference approximation for about del phi/del x2+gamma/rho. Next we have got the central difference approximation for del 2 phi/del x square term that is phi I+1n+phi I-1n-2 phi In/delta x square.

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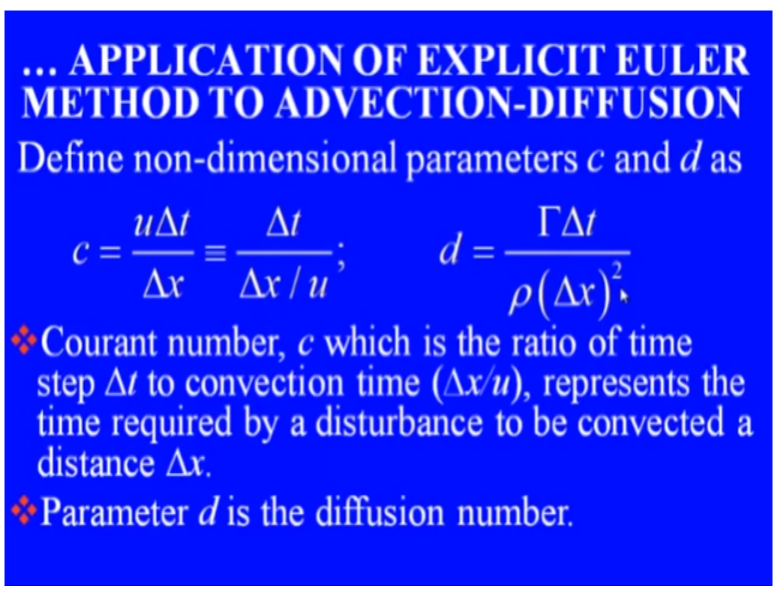

To write this equation more compactly let us introduce few numbers or 2 non-dimensional parameters, so let u s define non-dimensional parameter c and d as c is defined as u delta T/delta x. If we rearrange this definition, it is delta t/delta x/u and these defined as gamma delta t/rho times delta x square which can also be written as delta t/rho delta x square/gamma, which is equivalent to our diffusion number which have seen earlier in the context of conduction problem.

So this parameter d is nothing but diffusion number which you have seen earlier. Now let us have a look at this parameter c, now the c is called Courant number, which have been produced Courant of course and it is ration of what, it is ratio of time step delta t, now what is delta x/u, this delta x/u tells us the time which we take for a disturbance to traverse distance delta x, so this is why we call it as conviction time. So Courant number becomes now the ratio of time step to the conviction time.

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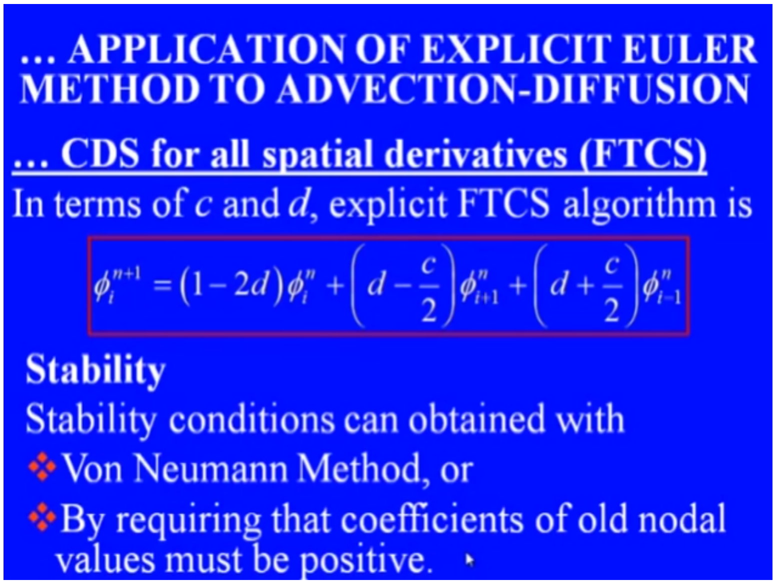

Now in terms of these two parameters, c and d, this is the shorthand form of explicit FTCS algorithm for advection diffusion problem. Phi I n+1=1-2d phi I n+d-c/2 phi I+1n+d-c/2 phi I+1n+d+c/2 phi I-1n. Now we want to look at the stability of this method so this again can be obtained by either using Von Neumann analysis which we have done earlier or by requiring the coefficient of old nodal values must be positive. Now let us try this approach the second approach for deciding on the stability of this method.

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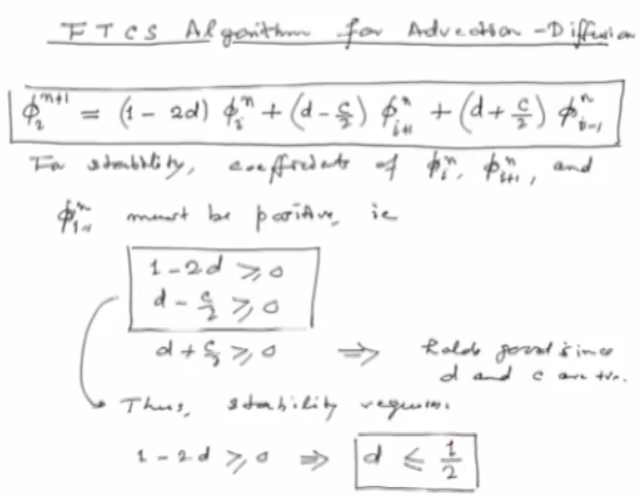

So an FTCS algorithm for advection diffusion problem. So what we had is phi I n+1 this was given as 1-2d phi I n+d-c/2 phi I+1n+d+c/2 phi f I-1n. We want all coefficients on the right hand side to be positive so for stability coefficients of phi I n, phi I+1n and phi I-1n must be positive that is 1-2d should be >or =0, d-c/2 should be > or =0 and d+c/2>=0. The last one is obvious, d and c were both positive numbers, so this holds good since d and c are positive.

So basically we have got these 2 conditions so this stability requires the first one says that 1-2d should be >=0 that is d<=1/2. This was the same requirement which we had earlier derived for the case of heat conduction equation and this is the one which is going to put a restriction on time step.

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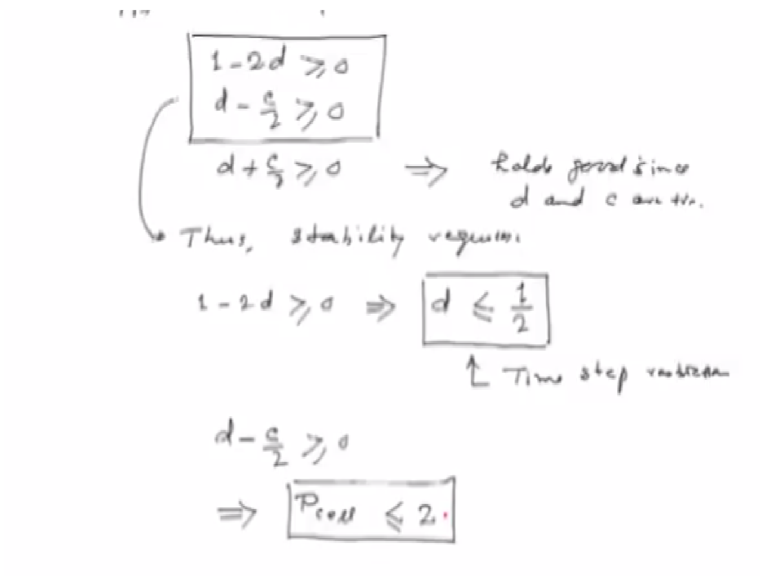

And the next one says d-c/2>=0 that translates to what we call a restriction on above. Velocity field or it leads to the local (()) (32:58) number, this should be <=2. So the first condition or first requiring inputs or limitation on time step, the second one puts a limitation on our spatial discretization.

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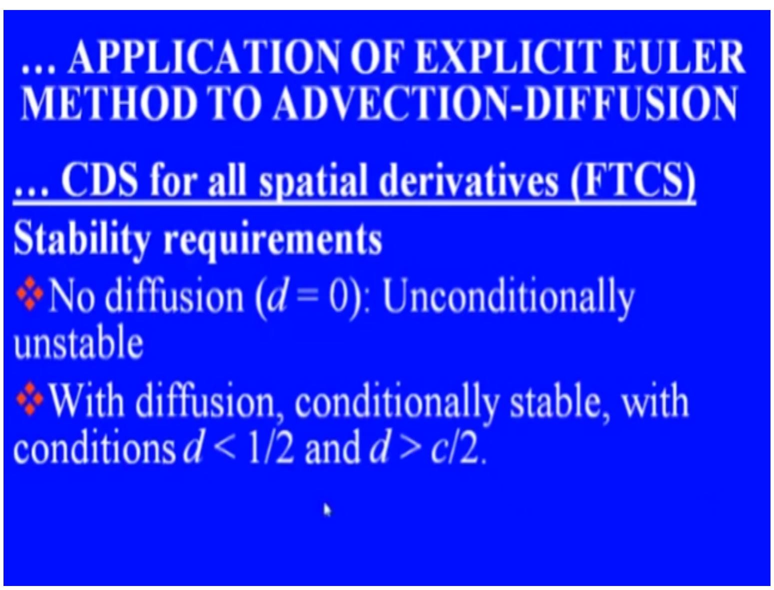

Now let us have a look at 1 more case if d=0, then what will happen. If d=0 the first coefficient 1-2d that will be positive but what happens to the second one, the second coefficient will always be negative so that means for no diffusion this FTCS algorithm for advection diffusion problem is unconditionally unstable and with diffusion we have already derived the conditions that d<1/2 and d>c/2, which tells us that local Peclet number should be <2.

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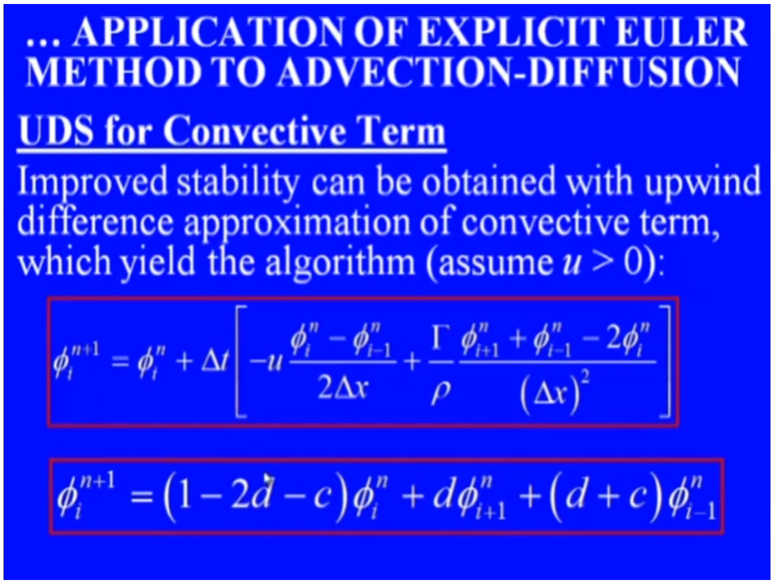

Now can we improve upon these situations, specifically when diffusion is very small, there is conviction dominated problem, you would like to have an explicit scheme which has got better stability property, so improved stability can be obtained with what we call upwind difference approximation of convective term. For diffusion we would still continue to use our central difference approximation but when it comes to the convective term let us use an upwind difference schemes.

If you assume u>0 that is flow is from left to right in x direction, so often difference approximation for del phi/del x would be phi of I-phi of I-1/this 2 should not be there, this should be phi I n-phi I-1n/delta x factor of 2 should be missing there +gamma/rho the second term of central difference approximation for del 2 phi/del x square so phi I+1n+phi I-1n-2 phi of n/delta x square.

Now if you introduce to an earlier term, diffusion number and the Courant number, so in terms of c and d parameters this particular algorithm which we called upwind FTUDCS scheme that becomes phi I+1=1-2d-c phi I at n+d times phi fi+1n+d+c times phi fi-1n. Now what we can easily say here, the coefficient of phi I+1 and phi I-1, they would always be positive. So there is no restriction whatsoever in terms of local cell Peclet number and the first one, this is the one which we should require to be greater than 0 for the sake of stability, this will put a restriction in our time step.

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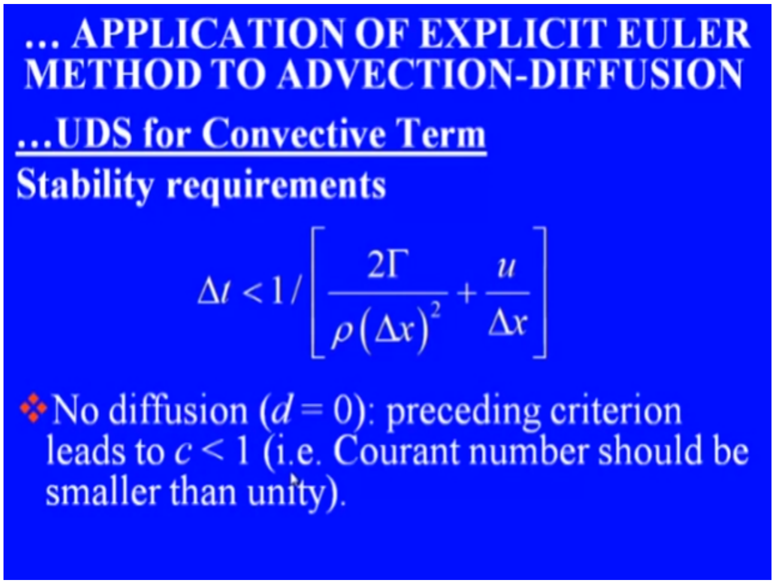

And if you put that this is what we get, so stability requirement simply say that delta t should be <1/[2 gamma/rho delta x square +u/delta x] and we can easily see that if there is no diffusion, d=0 then this particular criteria would lead us to what we call the Courant number should be <1. Okay, this first term becomes 0, so we straightway get this particular restriction. Now this condition is also referred to as Courant Friedrichs Lewy condition or the Courant condition and this is an important requirement for the use of explicit time stepping schemes in CFT applications that the Courant number should always be <unity.

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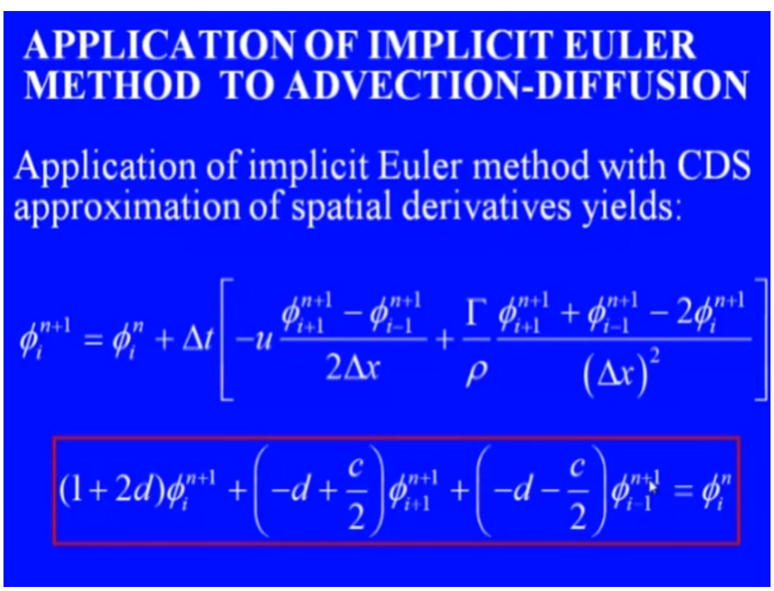

Now let us move onto some implicit methods and let us apply our backward Euler method to the two problems which I have discussed so first application of implicit Euler limited with central difference scheme for a spatial derivative for our heat conduction equation so we had Ti n+1=Ti n+kappa times delta t, Ti+1, n+1+Ti-1n+1-2Ti n+1/delta x square. Introduce of diffusion number in terms of diffusion this algorithm could be written in simpler form 1+2d Ti n+1-t Ti+1 n+1-T times T of I-1 at n+1=Ti n.

So now this is an equation which has to be solved, not just Ti at n+1 is unknown, Ti+1 at n+1 and Ti-1 at n+1 also unknown. So we will collect such way all the nodes and we will get a system of linear algebraic equations which must be solved at each time step to opt in our solution. In the same way we could have also applied our Crank Nicolson Method and we would have obtained an algorithm which looks fairly similar to this.

Next if you apply implicit method with CDS for advection diffusion problem where we have used the CDS for this convicted term as well as diffusion term. So phi I n+1 becomes phi I n+delta t-u times phi I+1 at n+1-phi I-1 at n+1/2 delta x+gamma/rho, phi I+1n+1+phi fi-1n+1-2 phi I n+1/delta x square or in terms of diffusion number and Courant number and rearranging those terms we get 1+2d phi I n+1+-d+c/2 phi I+1 n+1-d-c/2, phi I-1 n+1=phi In.

So this is again a couple system of equations, this is what we will get if you collect the equation for all the nodes which would again be linear equation so we ought to solve a system of linear equations to get our solution while using our implicit Euler method.

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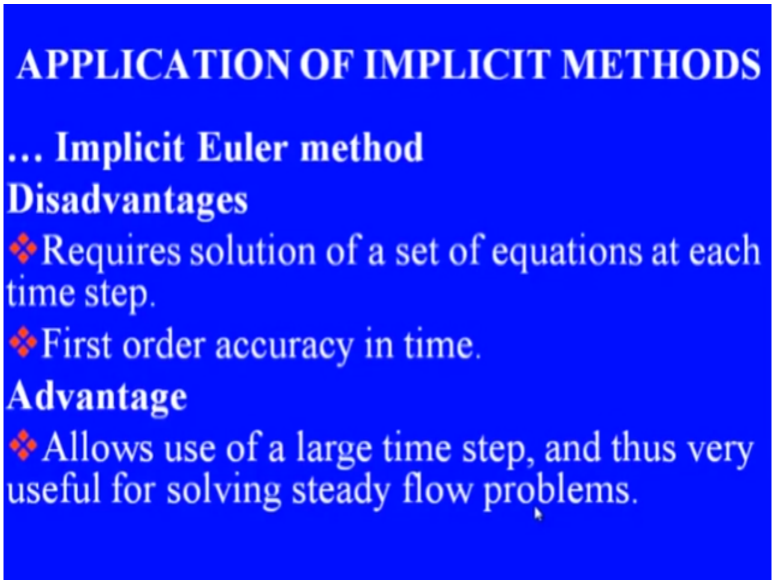

So now let us summarize the disadvantage which we will get that require solution of a set of equations at each time step and please remember our implicit Euler method was first order accurate in time, so that is the disadvantage which we have got with this method. Of course if you want higher order equation we can move onto the Crank Nicolson Method or higher-order Addons Molten method in time.

Now as far as implicit Euler method is concerned, it has also got a very big advantage, that this method is unconditionally stable for any choice of delta t, so it allows use of use of a large time step and thus it is very useful of solving steady flow problems whether you are dealing with a steady our aim is to obtain a steady-state solution of a heat conduction equation or an advection diffusion equation or a full Navier Stokes equation, we can use implicit Euler method to quickly obtain using large time step barriers our steady-state solution.

So that is the big advantage which our implicit Euler method provides. It stability properties were so good that this is the preferred choice for solution of nonlinear time dependent equations.

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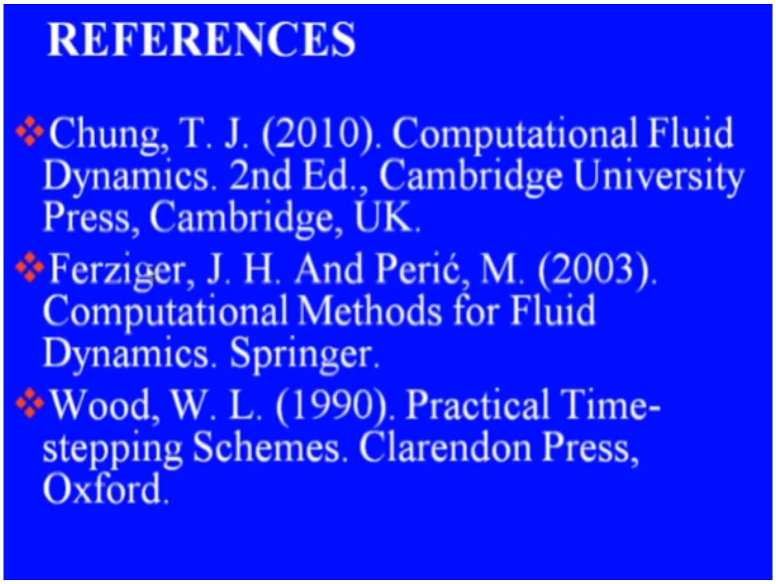

So that is where we would stop as for our applications are concerned. For more applications and results please refer to the books which we have discussed earlier Chung's book on computational fluid dynamics, it contains many algorithm, their stability analysis and so on. Similarly, you can also look at the book by Ferziger and Peric which gives further detailed applications of time integration schemes to advection diffusion problem with some examples.

And for looking at various time integration schemes which you can extend to our heat conduction or advection diffusion equation you can look at a book by Wood.