Computational Fluid Dynamics
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Lecture - 26
2 Level and Multi-Level Methods for First Order IVPs-2

Welcome to the second lecture in module 5 on time integration techniques in the first lecture we discussed the basics of time integration and we derived few 2 level time integration schemes and then we have to cover this multi-level methods and predict corrected Runge-Kutta in present lecture and then we would proceed the application to understand the transport problems in the next lecture, plus a recap of what we did in the last lecture.

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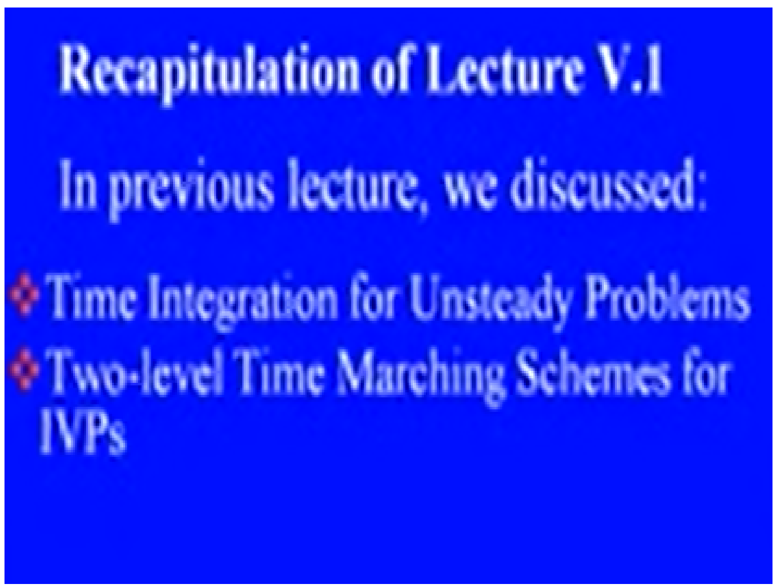

We discussed the time integration and unsteady problem the accounting for this special nature of the time coordinate which permits us to derive what we call marching time schemes wherein we start from a given time instant and obtain the solution slightly ahead in future and then use that solution obtains at that instant as a initial condition to get ahead of the next time step and we derived few 2 level time marching schemes for first order initially value problems.

We will continue further and we will derive 2 level and multi-level methods for first order initial value problems, we will focus on the multi-level methods based on what we call Lagrange interpolation polynomials and we will discuss few 2 level methods of Runge-Kutta family. So southlands lecture we will derive.

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Multi-level methods of Adams Family then we would have a look at what we call a predictor corrector methods and one special method of this family called Runge-Kutta methods. We will also have a look at the methods which can be derived directly by finite difference approximation of the time derivative. This recap of our initial value problem.

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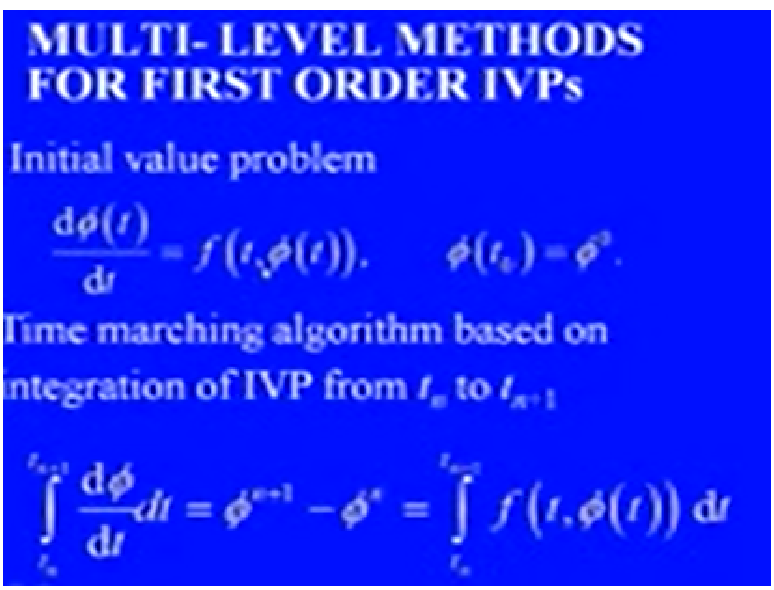

We have stated it as d phi t/dt=f t, phi. So this right hand side function could be a function of time as well as a function for unknown variable phi, initial condition at time t=0 is described phi(t0) = phi0 and we said we can obtain a time marching and algorithm based on the integration of the IVP from tn and tn+1, that is to say if you know the solution well time level tn , we should be able to obtain the solution at time level t and +1 by simply integrating or IVP.

Tn to tn+1 /dphi/dt which gives us phi n+1-phi n=tn to tn+1 in t, this finite integral of the right hand side f t, phi t dt and we said all that we need to find out is if you can get the value of this integral on the right hand side we have obtained into time marching algorithm and we did obtained a few time marching algorithm which involved the values of the variable phi at time instants t and n tn+1 , now in multi-level methods, what do we do.

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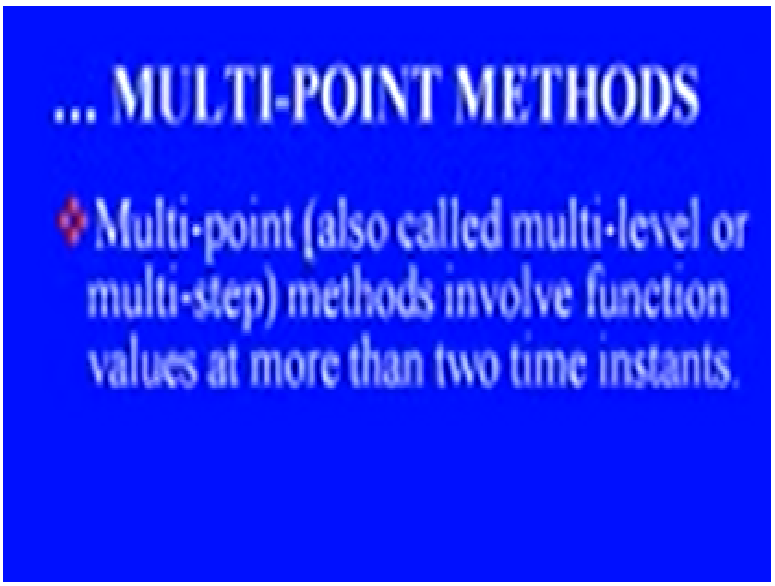

They are also called Multi-point methods to multi-step methods sometimes the difference is made a bit in Multi-point and multi-level methods for instants some people prefer to call Runge-Kutta methods which is simply 2 times tool level or one step method but they involve the value at many points in that time interval is why the multipoint in the same time in tool, but we will use these as soon on its multipoint multilevel or multi-step methods.

And they involve functional values at more than 2 time instants and the most popular multipoint methods are what we call Adams methods which are derived by fitting a polynomial to the derivative that is our f (phi, t) at a number of points in time and what choice of polynomial usually be Lagrange interpolation polynomial and we can have 2 families of this Adams method the first one is what we call Adams Bash forth method.

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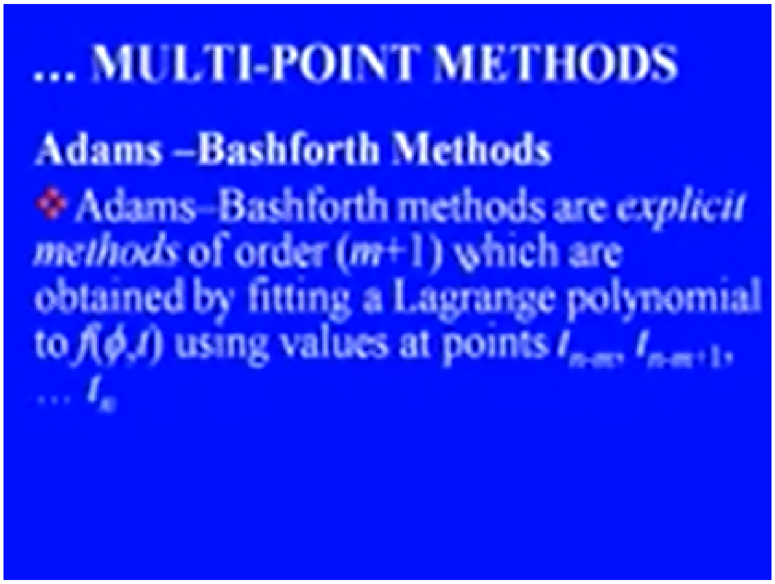

And these are explicit methods of order n+1 which are obtained by fitting a Lagrange polynomial to this function f phi(t) at points tn-m, tn-m+1, so on up to tn forfeit the value at these m points and there by obtain a polynomial approximation of f in terms of the functional values of these time instants and once these are known not all of these time instants or what we call previous time instants so a solution phi would be known at tn-m tn-m+1 and tn so on.

So the integral that you would obtained that would contain all known terms this why these methods are called explicit methods. , So for instance this for first order Adams Bash forth method this is explicit Euler method.

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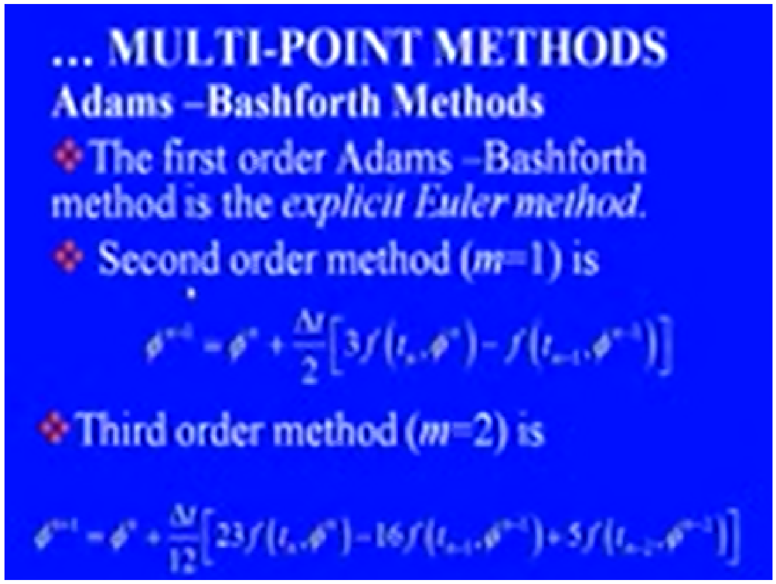

Which we can easily show the second order method m=1 is given by phi n+1 is phi n+delta t/23ftn,phi n-functional value at n-1 phi n-1 , now let us see how do we obtained this method starting from our basic premise of fitting Lagrange interpolation polynomial.

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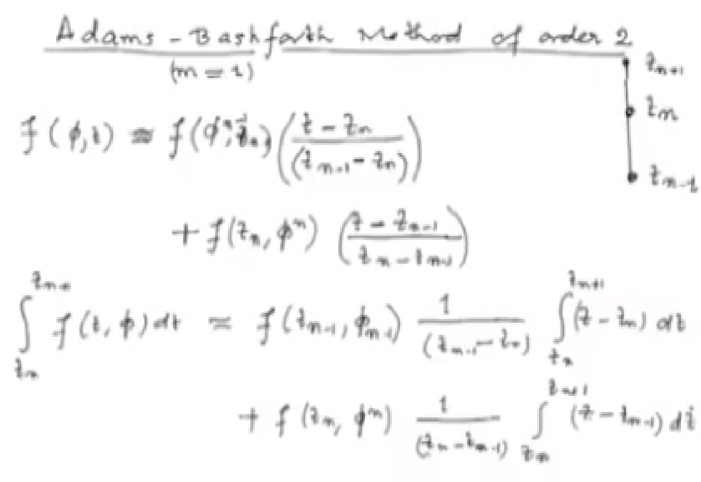

So we want to derive this Adams Bash forth method of order 2, here our m=1, so basically this would involve the interpolation of the function in terms of the functional values at time instants tn and t (n-1) and we want to obtain the solution at the time instant tn+1, So now let us write polynomial approximation for f phi t, so this can be expressed in terms of 2 Lagrange interpolation polynomials.

So first one which is based on the use of the function value at phi n tn t-t/tn-1-tn, So this f time instant and -1 and then without those t n phi n t-t(n-1/tn-tf-1), so basically we have used what we call linear Lagrange interpolation polynomial to approximate our function f phi t or f t phi, now what we want to phi n t out if you want to phi nt all this integral tn to tn+1 f(t,phi) dt, so now let us put this previous approximation so it would be f(tn-1,phi n-1).

Since this a value which does not depend on these so we can take it out of the integral sign, similarly this tn-1-tn that can also be taken out of the integral sign t(n-1-tn) and what we need to integrate is t-tn, so tn2 tn+1 dt, So first term and the contribution from the second term f(tn, phi n) 1/tn-t(n-1)integral from tn to tn +1 of t-t(n-1) dt , now let us have a look at the integrals, how do we obtain this integrals to obtain the different integrals substitute t-tn.

Let us call it an intermediate variable v, so then our integral tn to tn+1 t-tn dt this would become integral o to delta t v dt which is v square/2, 0 to delta t that is delta t square /, how about the next integral?

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Our next integral our limits tn to tn+1 , now we have got this term-t (n-1)dt, so we can express at as tn to tn+1 , t-tn +tn-t-1 dt , now we are ready to perform the substitutions t-tn and thus =v and tn-tn -1 that is about delta t , so this is 0 to delta tv+delta t dv which is v square/2 ,0 to delta t+delta t times v, 0 to delta t , so that the first term will give us delta t square/2 and second term would simply give us delta t square.

So we get 3/2 delta t square and now the denominators tn -1-tn that is simply-delta t and the denominator in the next expression is simply delta t , so now let us substitute these back , so this integral tn to tn+1 f(t phi dt this would be – delta t square /delta t that is what we get tn-1-tn that is-1/delta t and the integral was delta t square/2, so we get –delta t square /2 delta t f(tn-1,phi n-1+ next term that 1/tn-tn-1 that is delta t.

So we get 3 delta square/2 delta t times f(tn,phi), so this gets further simplified and this we can write it as delta t /2 3f (tn, phi n)-f(tn-1,phi n-1), so you can substitute back and obtained the finite marching scheme, so we had this t phi /dt , dt integrated from tn to tn+1 this was =phi (n+1-phi n) and tis was equal to our integral which was just obtained delta t/2 (3f tn ,phi n-f(tn-,phi n-1) and we get our terms and we get our time integration scheme.

Phi f n+1 =phi n+delta t/23f tn and phi n-f (tn+1) sorry tn-1 phi f and minus 1, so this is our second order Adams bash forth method as you can easily say on the right hand side we get only the functional values phi n and phi n-1 which are known values, so right hand side can be evaluated explicitly so that is why it is an explicit method of order 2.

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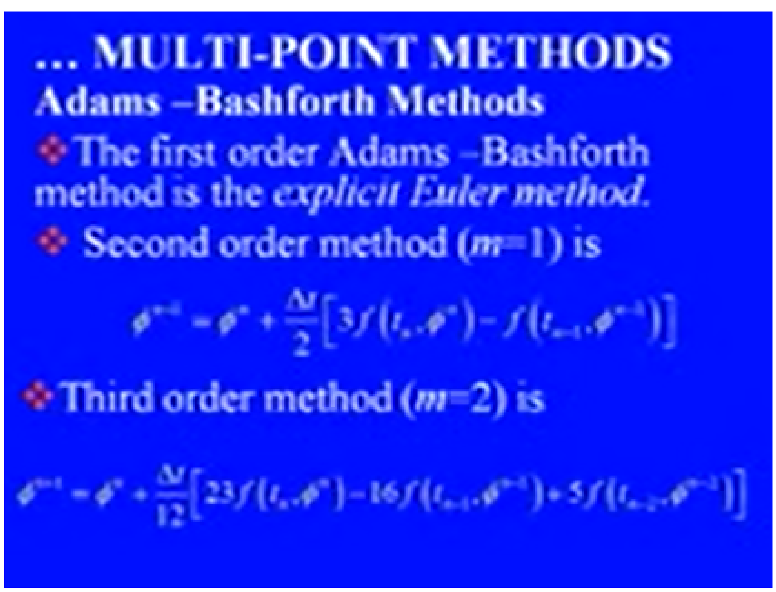

Similarly you can take m=2 that is you say you can phi t Lagrange interpolation polynomial using the functional values at tn, tn-1 and tn-2 and you can derive a third order Adams Bash forth method, So this I would leave as an exercise derive third order that is we will take m=2, third order Adams Bash forth method, we will follow the same procedure that you use a in fact when we take m=2 you will have quadratic Lagrange interpolation polynomial.

By making the appropriate substitutions t-tn=v that is the only stuff reason which we need to make and thereby you can easily obtain this third order method given by phi n+1=phi n+delta t/2 23 ftn, phi n-16f at t-n, phi n-1+phi(ftn-2) and phi n-2 , to check the accuracy if you original one this simple way whether we have derived our things correctly or not look at the co efficient it is friends in the first second order method, the coefficient f at tn that is 3.

And the next term the co efficient is -1, 3-1 that gives us2, 2/2 that is what equate to1. Similarly, here we have got 23-16that gives us 7, 7+5 is 12, 12/12=1, so that is how the things would be that the co efficient of all this functional multiplier divide and whatever we have got in denominator delta t when these 2 numbers are divided we should get 1 and that to confirm the correctness of our derivations.

Okay so this each word about explicit methods of Adams Bash forth family, I will just make one remark that we can obtain third order, 4th order, fifth order, we can find out or we can keep on increasing the accuracy order of accuracy of this method but all of these basis what we call methods because they have got on the right inside the function where there is a linear combination of the functional values of different time and steps.

And there is one important result in math literature there we say that we cannot have what we call a stable methods of order more than 2 so this one small limitation which we need to keep in mind when we go for higher order methods next people want to.

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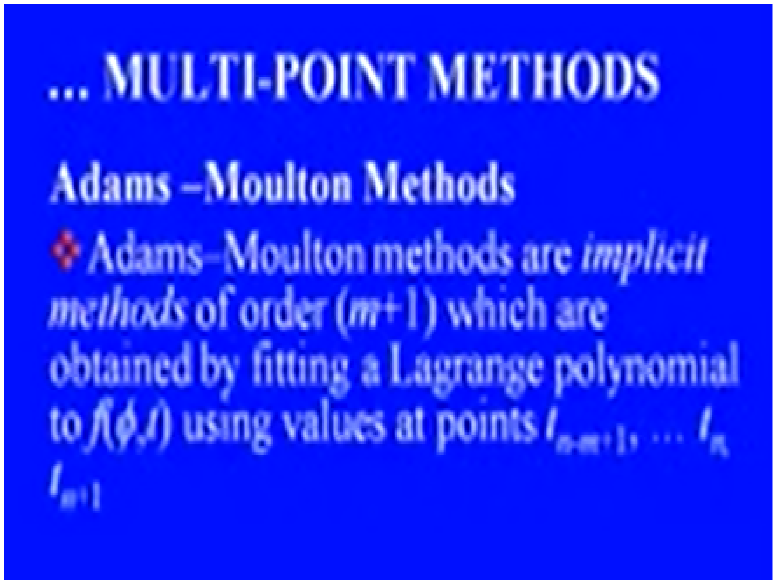

Adams Moulton methods these are do an implicit methods so they are implicit methods of order m+1 which are obtained by fitting a Lagrange polynomial to function value f phi t using values at points tn-1+n+1, so on so forth and we would include the value of it unknown function at tn+1 as well, so this poisons our interpolation for f we would use the function of the value of the function phi or unknown phi at tn+1, the result would be and implicit method.

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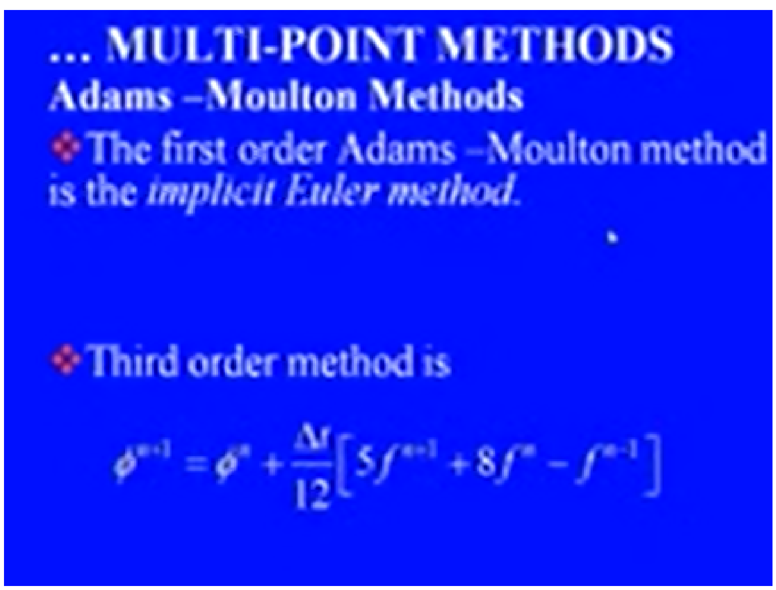

So what is the first order Adams Moulton method that is our implicit Euler method, the second one is that crank Nicholson methods, so I have not written the formula for these 2 because you have already seen these both of these earlier. The third order is given by phi n+1 =phi n+delta t/12 phi fn+1 +8fn-f(n1), now let us try and see if we can derive this third order method. So let us get back to board and try and derive this third Order Adams Moulton method.

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So now we want to use the function values at 3 points values to be used in interpolation at time instants t (n-1) tn and tn+1, so the presence of this future time instant that what leads to what we call an implicit method so tn-1 so our present tn and the future the tn+1 and please remember we are dealing with what we call uniform time step that is difference between this time labels in the same.

Now can we write the Lagrange interpolation approximation, so polynomial approximation for f t phi using Lagrange interpolation, so these 3 points involved we will have a or we would use what we call quadratic Lagrange interpolation polynomials f (t, phi) this would be approximated by fn-1 the interpolation function for this would involve is denominator, denominator would be t (n-1 –tn).

And the second term would be t(n-1-tn+1) and what we will have in numerator, in numerator we have t-tn to t-t(n+1), so that is our first term , the second term which multiplies fn not in denominator will have tn-t(n-1)x tn(-t(n+1) and in numerator we have t-t(n-1) t(n+1) similarly the interpolation function we can write for the function value at time n+1 for the sake of simplicity you have used here.

Now r shorthand notation that fn-1 actually represents the functional value at tn phi n and so on , so this would be the denominator here would be t(n+1) –tn and tn+1-t(n-1) in numerator we have t-tn and t- t(n-1) so now you have to evaluate the integral and f(t phi) and for that let us find out the different integral source , the 3 interpolation functions which we have used for the same time interval so let us find it out one by one and we would again you the same process that we would substitute use substitute t-tn=v in evaluation of integrals.

Now the choice of this change of variables that will lead us every interval would become so this gives us or integration interval this tn to tn+1 that gets transformed to 0 to delta t.

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So let us evaluate each integral one by one , let us call the first term I1 , so I1=integral of tn to tn+1 , (t-tn)x(t-tn+1)/tn-1-tn)x(tn-1-t(n+1) dt , so as far as denominator goes that is constant so let us see what are the terms in denominator are t(n-1-tn)is simply –delta t and t(n-1)- t(n+1) that would become twice of C -2 delta t now in numerator we had t-tn would be our variable V , how about t-t(n+1), this we can write as t-tn +tn-t(n+1) in terms of was substituted variable would become v-delta t.

So now in the change to variables at ideal Il items, I1 is 0 to delta t , in numerator we will have Vx v- delta t and denominator we get – delta t-2 delta t dt which is equal to 1 over 2 delta t square v cube/3- v square/ by 2 times delta t , o to delta t. So it would become 1 over 2 delta t squared delta t cube/3- delta t cube/2 or that is=-delta t/12.

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So our next integral which multiplies is fn, IE2= tn to tn+1 (t-tn+1) x (t-tn-1) /tn-t (n+1) and tn-t (n-1) dt. So once again let us have a look at the terms in denominator, so denominator terms are this tn-t (n+1) this would become –delta t and tn-t (n-1) this is simply delta t, in numerator the terms would become the C Eastham numerator t-t (n+1).That we have already seen earlier this translates to v-delta t never changed variable and t-t (n-1) this would v+delta t.

So this in the changed variables the integral I2 would become 0 to delta t v square –delta t square /-delta t square dt which =-1 delta t square v cube/3 –delta t square v 0 to delta t-1 over delta t square (delta t cube/3 –delta t cube) = +2/3 delta t, the last term the integral of the last term which is co efficient of fn+1.

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Let us continue this over Adams Moulton method, let us call the integral as I3 this is integral of tn to tn+1 (t-tn) (t-tn-1)/t(n+1-tn)x(tn+1-tn-1) dt , so denominator of the first term is simply delta t that is tn+1-tn, so delta t and (tn+1-tn-1) that would become 2 delta t t-t(n-1) this is become v+ delta t , so change the variables I3 would become 0 to delta t vxv+delta t dv/delta tx2 delta t , so equal to ½ delta t square x v cube by 3+v square/2 delta t 0 to delta t 1 over 2 delta t square 3+6 v delta t cube /6 that is 5/12 delta v.

So now if you substitute all these value then we get integral +tn to tn+1 of a function t phi t dt.

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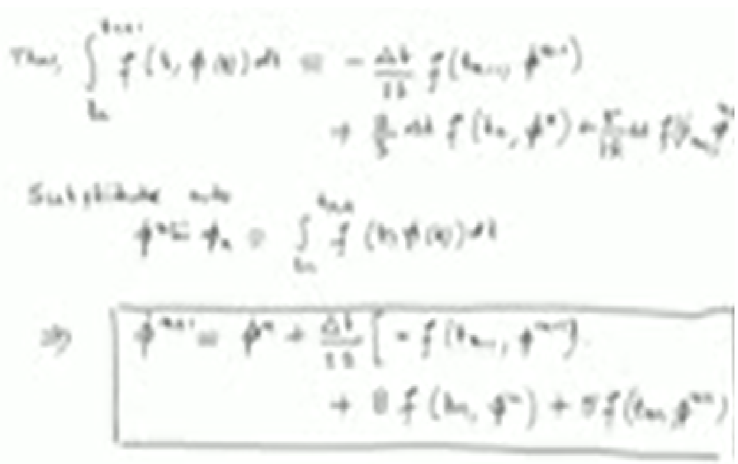

This would become-delta t /12 f(tn-1 phi n-1 +2/3 delta t f (tn, phi n)+ 5/12 delta t f(tn+1, phi n+1), so substitute it our original equation n2+phi n+1-phi n = integral tn to tn+1 ft phi t dt we will get phi n+1 = phi n, let us take delta t/12 common, so delta t/12 then we will get in within brackets(-f(t-1 phi fn-1)+ 8 times f"(tn and phi n+ 5 times f (tn+1) phi n+4. So this is our formula for Adams Moulton method.

Okay, so this is a third order method, phi n+1 =phi n delta t/12 5 f (ftn+1, phi n+1) +8f (tn, phi n)-f (tn-1, phi n-1), same procedure you can adopt if you want to obtain Adams Moulton method of higher orders the integration process and substitution which you made exactly the same what we had just discussed.

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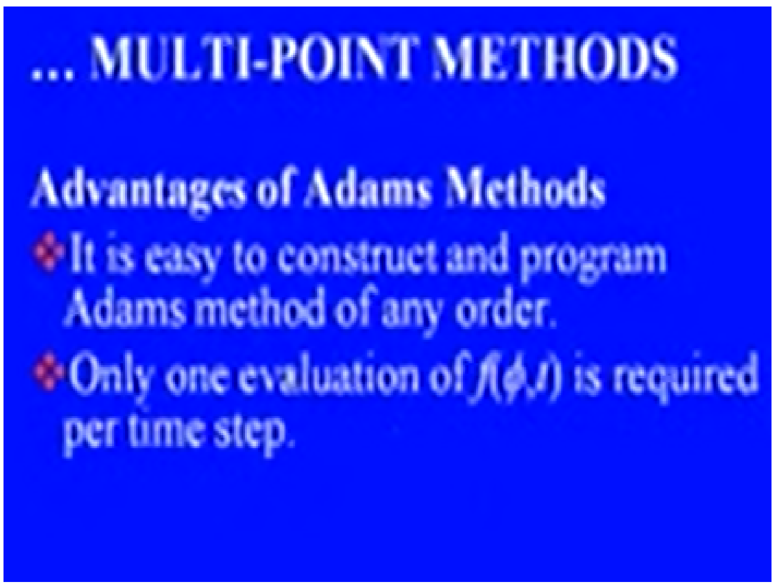

Now what are the advantages of these Adams method both implicit and explicit type they are easy to construct and program Adams method of any order we can follow exactly the same procedure of integration and at each evaluation we need only one evolution of function f phi t which are coming from the previous system they can be evaluated and saved in a temporary variable.

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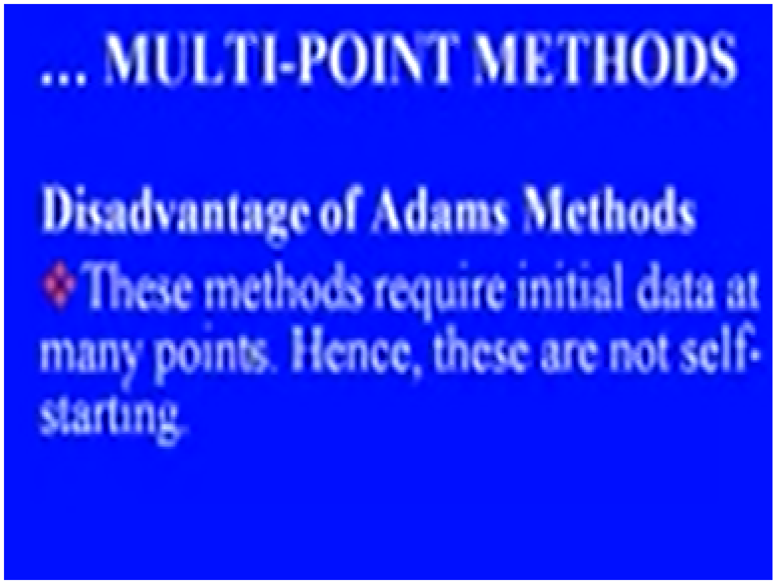

Disadvantages of these methods they require initial data at many points. If you get higher –order Adam method and they will require data not just a tn they would also require data at tn –on tn—2 and so on. Hence these methods are not self is starting in the sense that as t=t0. We have got only one set of initial later so in such situations what we have to do is at first time step we have to use a lower order Adams method for instance at t=t0.We use first order method or Runge-Kutta method.

And then change over in the succeeding time steps to a higher order at Adams method.

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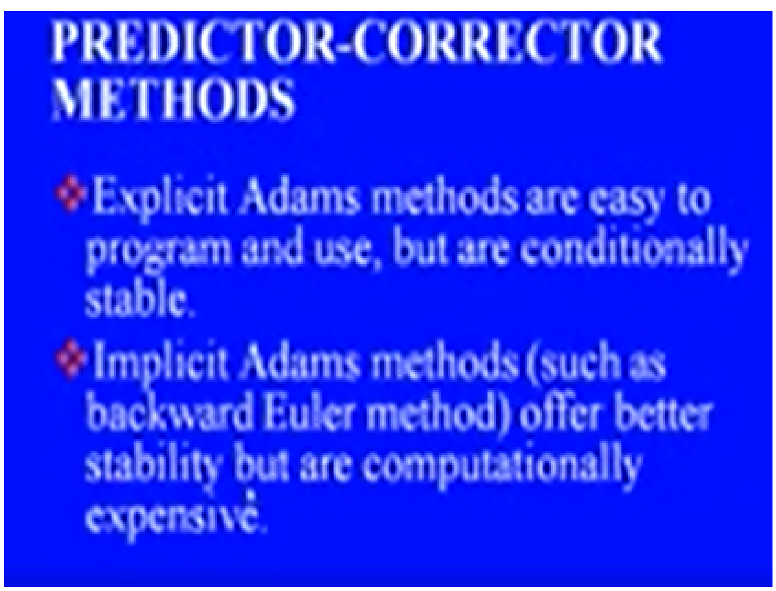

Next the family of methods we will discuss is what we call predictor –corrector methods and the rationale is very simple we looked at 2 family if Adams method explicit method and implicit ones explicit Adams method they are easy to program and use but there is stability problem they are conditionally stable implicit method specifically the backward Euler method it is unconditionally stable for any value of delta t.

So these implicit methods they offer better stability but they are computationally expensive expensive because we have to solve a system of equation at each time step it is something which we can get either compromise something which is as easy to evaluate as explicit address methods and has slightly better stability properties so this what we get from a particular character may that this particular character meet offered it compromise between these 2 choices.

Or a list of elevations and slightly beat of a few stability then explicit Adams method and has slightly better stability properties, so this is what we get from predictor-corrector methods, this predictor-corrector method offer a compromise between these 2 choices, there are less evaluations and slightly better efficient stability then explicit Adams methods and there is wide variety which you can find.

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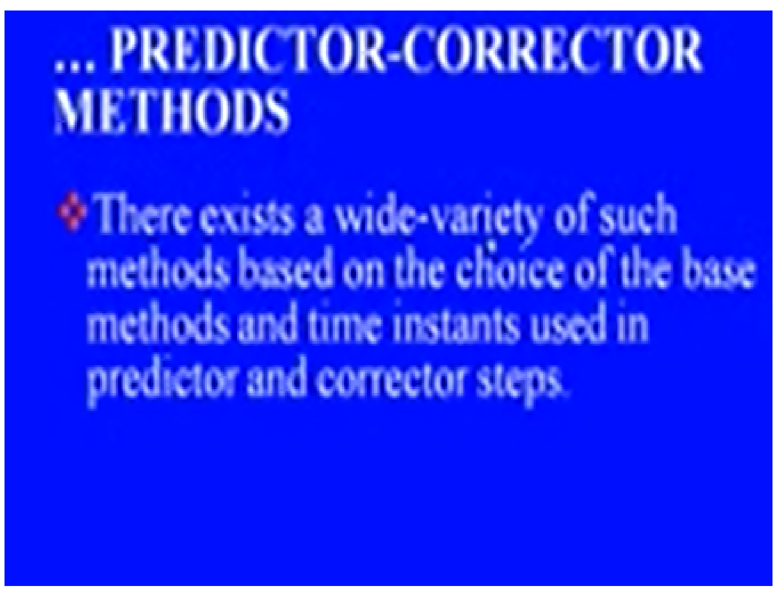

In the mathematics on numerical methods for initial value problems, now this choice of variety depend upon the choice of what we call best methods and time instants used in predictor and corrector steps, now if you use Adams Bash forth methods a predictor and Adams Moulton scheme as corrector method via refer to as multi-level predictor corrector methods or Adams Bash forth Moulton+m schemes and the possibilities would be good.

Or popularly known as Runge-Kutta methods which are essentially 2 level methods Multi-point methods. Let us have a brief look at a few methods of both times, so first one multi-level predictor corrector methods.

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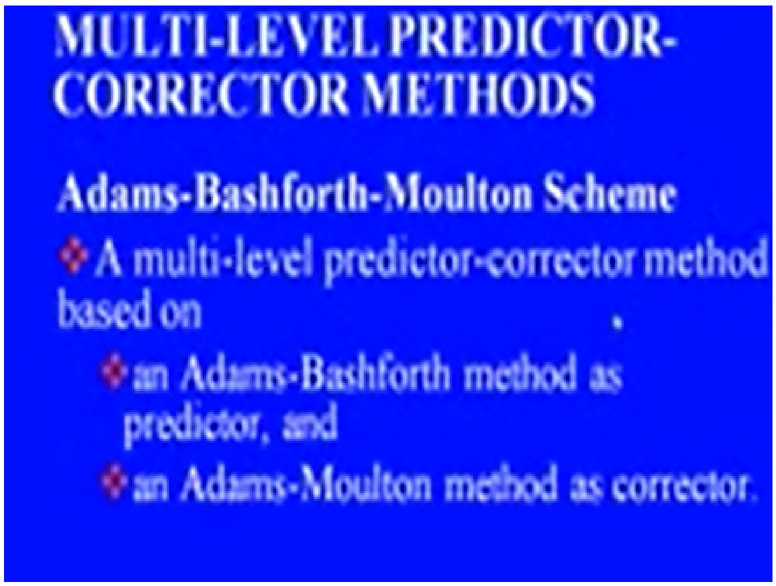

That is our Adams Bash forth Moulton scheme, so it is a multi-level predictor corrector method because it requires the initial conditions are quite a few times step, so here would use an Adams Bash forth method which is an explicit method as predicted and Adams Moulton method as corrector, so this one scheme is called 4th order.

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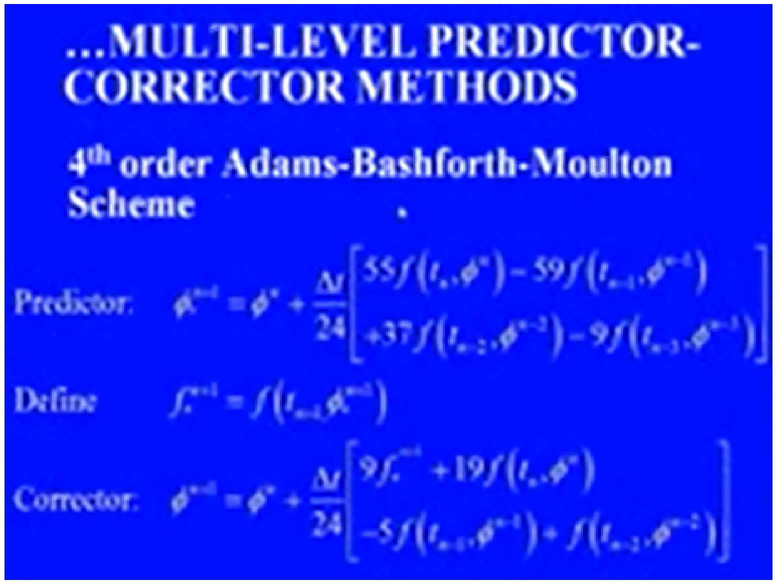

Adams Bash forth Moulton scheme, so predicted, you would predict the value at time n+1 using an explicit Adams Bash forth scheme , Let us call it phi star n+1 this =phi n+delta t/24(55 f at tn phi n-59 value function f at tn-1 phi n-1 +37 ft n-2 phi n-2 -9ft -3 phi n-3 , now let us define a variable f star n+1 , the value of function evaluated at tn+1 using the predicted value phi at tn+1 is called 5 star n+1 and now we can chose Adams moulton scheme as character.

So phi n+1 using the scheme becomes phi n in place of the unknown value at tn+1 you are going to use f star n+1, so phi n+1 becomes phi n+delta t/24 9 f star n+1 , this is based on the predicted values +19ftn phi-f(tn-1,phi n-1)+f(tn-2 ,phi n-2), So this predicted corrected methods they can give us high accuracy but their stability is much poorer compared to the implicit Adams Moulton scheme.

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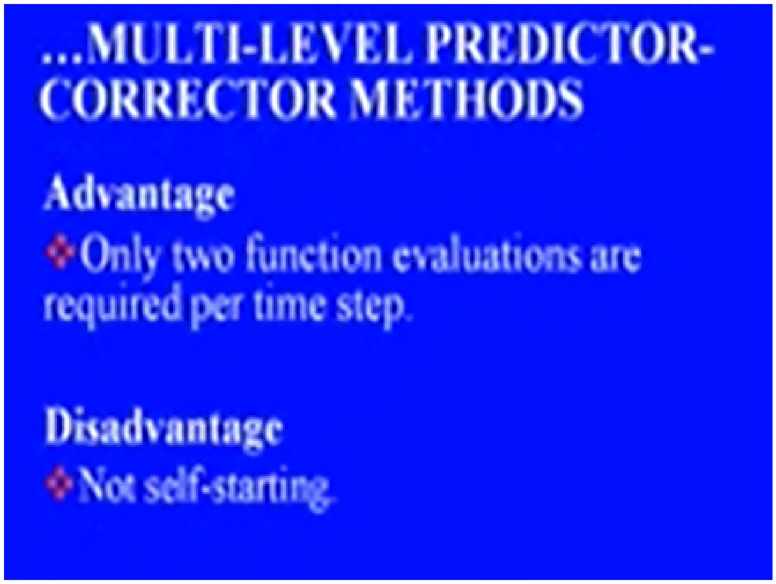

The advantage would be in this case you would require 2 functional evaluations at each time step disadvantage again these Adams family they are not self is starting because of the first time step we require either a lower method or use Runge-Kutta could turn.

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Next our Runge-Kutta methods we will have a look at 2 of these such a good time methods so these are 2 level methods multi-point methods, we are going to evaluate the value at time level tn+1 using the values t and few time instant in between these 2 so these are easy to use and self is starting but they require more computational effort per time step as require compared to multi-level Adams methods which require only one functional evaluation per time step.

The advantage with these Runge-Kutta method are they are more accurate and stable than multi-level Adams method of the same order, so that is why they are very popularly used in CFD.

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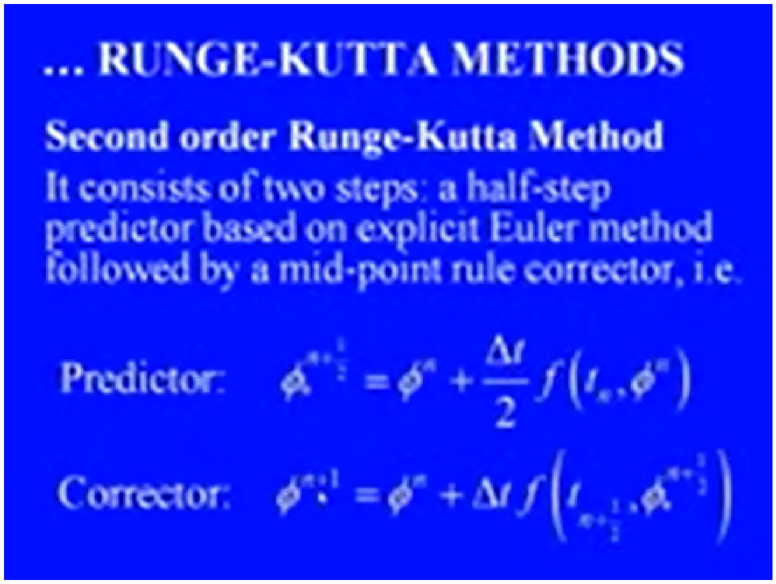

Now let us have a look at 2 of these second order Runge-Kutta method the second order method consists of 2 steps a half step predictor based on explicit Euler method followed by a midpoint rule as corrector, what do you know half a step we are looking at the value predicted value at tn+1/2, so our time step delta t/2, so that is why call half a step, the predictor step is, let us compute an estimated value of the function phi at times step instant n+1/2.

So 5 star n+1/2=phi n+deltat/2 f tn, phi n everything is known at time instant n, so we can easily compute phi star n+1/2 then compute a correct value of function at n+1 using mid-point rule, now midpoint rule requires the functional value at time instant tn+1/2, this what we have computed at using predictor step 5 star n+1/2 so this what we are going to use in our midpoint rule phi n+1 is phi n+delta t.

The value of function f at tn+1/2, phi star n+1/2, so this is what we have used here in the midpoint rule in place of phi n+1/2 which is unknown we had used the predicted value so now as a result we get a scheme which is second order accurate in time, that is what the accuracy of the midpoint rule and it is explicit So this is what we call explicit second order Runge-Kutta method.

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The next one is what we call 4th order Runge-Kutta method, please note that there is a family of 4 Runge-Kutta method, it is available in literature, both of explicit type and implicit type so if you are interested in details, you can pick up any book on the numerical solution of initial value problems and you can see variable order Runge-Kutta methods in detail, yet we will look at one more, that is 4th order Runge-Kutta method.

So it consists of 4 steps a half step explicit Euler Predictor followed by an implicit Euler corrector at tn+1/2 followed by an a midpoint rule perfect for full time step and the last one is Simpsons rule corrector for full steps. So let us look at each of these steps one by one.

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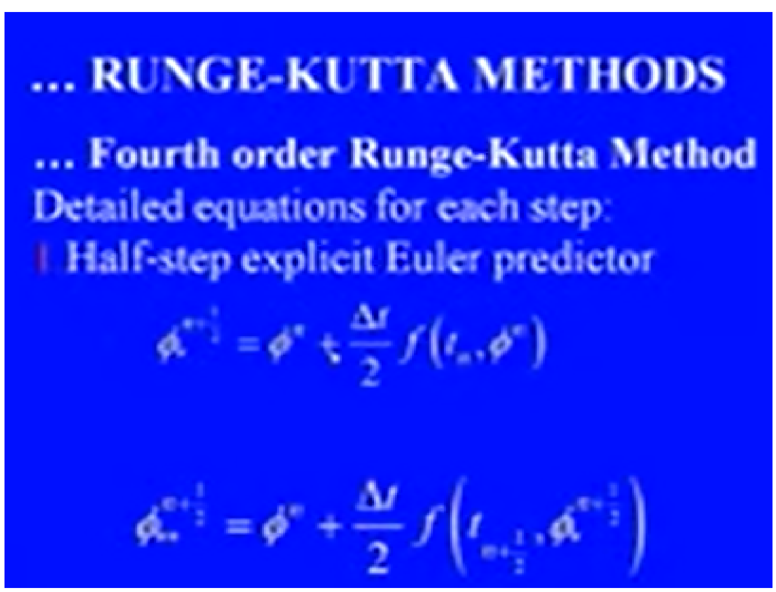

Half a step explicit Euler predictor that is to say we want to compute the functional value n+1/2 of tn+1/2, let us call it phi star +1/2, so this is=phi n+delta t/2 ftn phi n. So this is the value predicted by explicit Euler method, now let us correct it using implicit Euler at time instant tn+1/2, so let us call it corrected value phi star star, so phi star star n+1/2, this is =phi n+delta t/2 function f Evaluated at tn+1/2, phi n+1/2, in place of unknown phi n +1/2.

We use the value predicted in the first step, soon phi star star n+1/2 can be computed explicitly because on the right hand side everything is known phi n is known. Tn+1/2 is known and phi at n+1/2 that has been replaced by the predicted value in the first step of the Runge-Kutta method, Third step let use.

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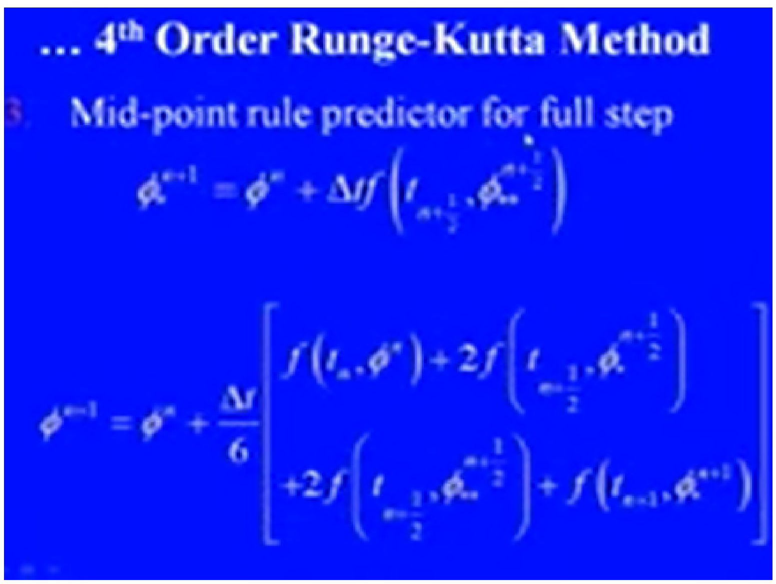

Midpoint rule predictor because you have already got a fairly good estimate of the functional value of phi at tn+1/2 so that can be used in our midpoint formula so phi star n+1/2 is 1 =phi n+delta t , f of tn and +1/2,phi star star n+1/2 and we will still call this is not as final value but this phi star and n+1 is a predicted value and no we would use Simpsons rule corrector for the full step.

And this final step would involve all the predicted values which have computed so far so that in the first 2 steps we computed the phi star n+1/2 we got this functional evaluation done ftn phi n in the next week did the curve functional evaluation tn+1/2 phi star and +1/2 so all these evaluations we are going to make use of in our Simpsons rule corrector so phi n becomes phi n+delta t /6 ft and phi n+twice of ft n+1/2 gamma phi star n+1/2 +twice of f tn+1/2.

Phi star star n+1/2 +f(tn+1, pphi star n+1) so in this case we would need total of 4 functional evaluations f at t and phi n f at tn+1/2 f phi star n +1/2 the third one is f tn+1/2 ,phi star star n+1/2 and f (tn+1, phi star n+1) So compared to our Adams Bash forth scheme we require 3 additional functional evaluation per time step but these Runge-Kutta methods are a lot more accurate and they are lot more stable compared to Adams family methods.

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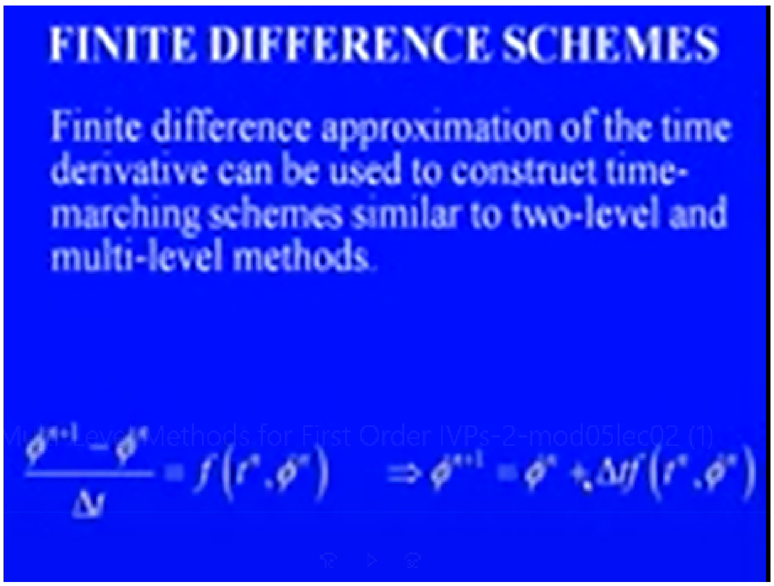

The last scheme in this time integration, you would have a look at what we call finite difference basis schemes you have already seen few such methods when we were discussing the implicit Euler and Explicit Euler methods.

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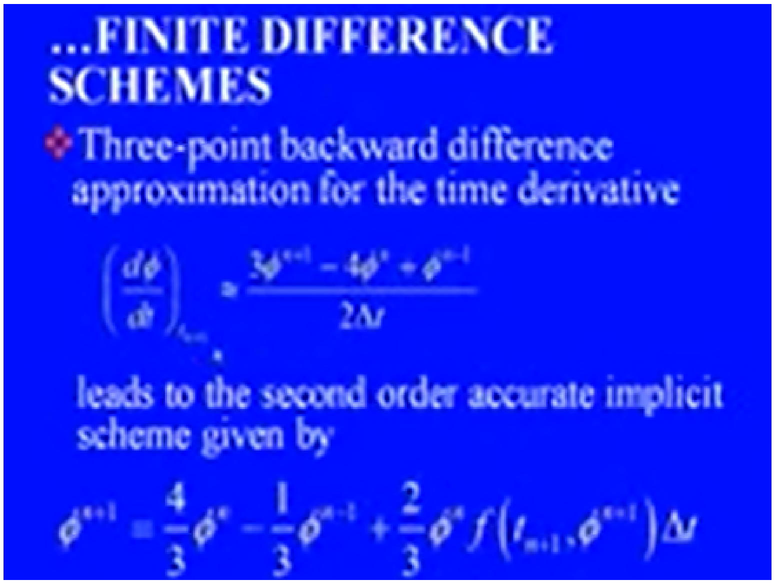

So we can use finite difference approximation of the time derivative to construct a time marching scheme similar to a 2 level and multi-level methods for instance this forward difference approximation which gave us explicit Euler method. This we have already seen earlier that is to say we replaced our time derivative, del phi/del t/phi n+1-phi n/delta t=ftn phi n and it gave us explicit Euler method phi n+1 = phi n+delta t times f (tn phi n).

Similarly, if you do 3 point backward difference approximation for time derivative that is d phi/dt at tn+1 this can be approximated as 3 phi n+1-4 phi n+phi n-1 /delta 2 delta t. So this leads to second order accurate implicit scheme which would be given by phi n +1= 4/3 phi n-1/3 phi n-1 +2/3 phi n+f(tn+1,phi n+1) into delta t. So this way we would have stopped our discussions on different time integrations schemes are concerned and if you want to have a full further details.

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You can refer to the book by Chung or Ferziger and Peric or this book by Wood practical times stepping scheme by Oxford. So all these 3 books they contain a lot more time integration schemes for any cell value problems.