Computational Fluid Dynamics
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Lecture - 25
Two Level and Multi-Level Methods for First Order IVPs

Welcome to our new module on time integration techniques, in the previous models we have learned the techniques of finite difference discretization for governing equations whereby we obtain a system of discrete algebraic equations if a problem would still state in a previous module we discuss some techniques to solve the system of algebraic equations to both linear as well as non linear time.

Now in case if we are dealing with a time dependent problem the finite difference finite volume or finite element discretization are using will lead to a system of ordinary different differential equation in time, now we need some numerical techniques to integrate that system of ordinary differential equation what we popularly call as initial value problem so this what would be the focus of our module 5 on time integration techniques.

Now remember there is plenty of literature available there are many books which are the available in maths, literature on numerical solution of initial value problems so we are going to sample it in a few lectures in this module only those techniques which are most widely used and they are very simple and easy to program in the context of temperature flow dynamics in particular and computational mechanics in general.

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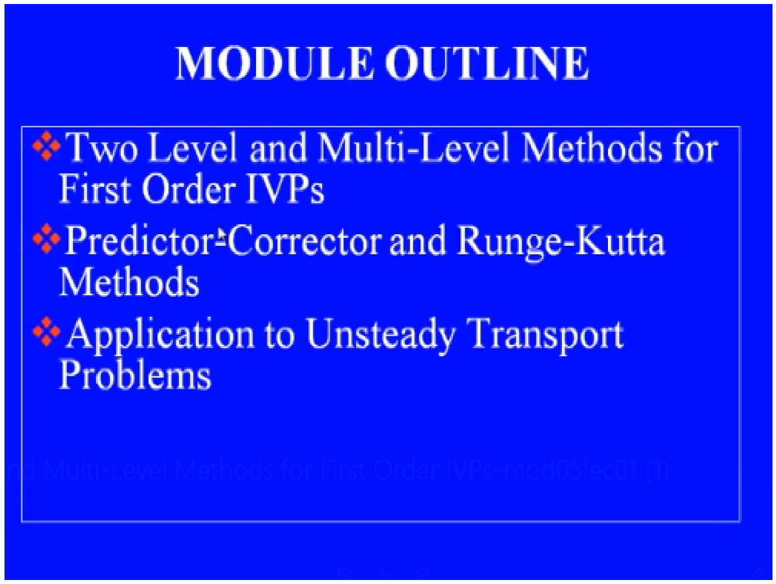

So outline of a module, we will first have a brief look at what we mean by time integration methods how do they work and we will look at 2 level methods multi-level methods, 2 level methods are the ones which are involved or try to obtain the solution in terms of the functional value of 2 time levels and multi-level methods it would involve functional values at many time instants.

In the context of first order initial value problems and then we would have a look at wider class of methods which are known as predictor corrector method and Runge K is also very popular and these methods they require only one initial condition and next we will have a look at application of some of these methods earnestly transport problems we primarily focus on the parabolic problems of heat conduction and advection,

Diffusion the later problem is an adjective of what? Nave a strokes equation as well simplified version of that so this way they are going to focus on application of explicit as well as implicit time integration techniques to these 2 class of problems and would analyze the behavior a few popular time schemes. Now we would focus exclusively on first order initial value problems, the reason is very simple a second order initial value problems.

If we encounter for certain cases they can be easily converted into a system of first for sort of initial value problems this region office would primarily be on first order initial value problems.

Now we will start with the first lecture in which you are going to focus on 2 level and multilevel methods for first order initial value problems.

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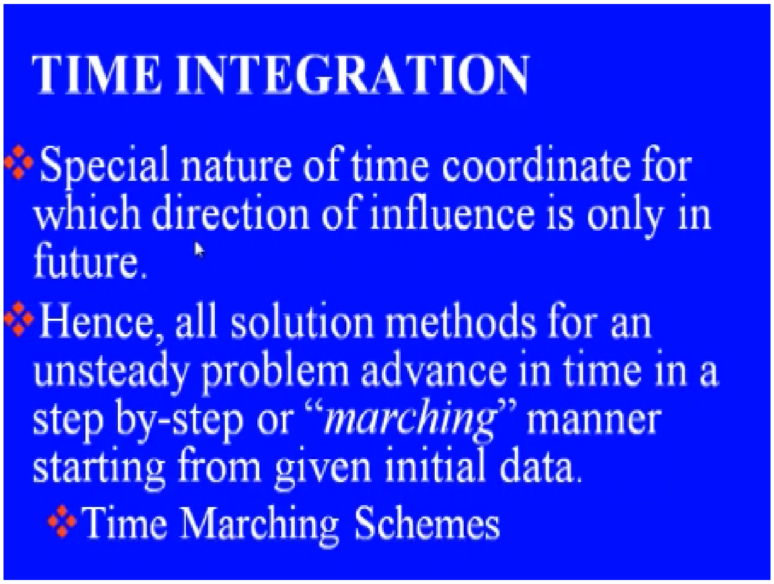

Okay now this is specific nature to time coordinate has got a special nature in contrast to our space coordinates X and Z which can extend in both directions from negative x axis to positive x axis and similar negative x axis to positive y axis and so on when we deal with time dependent problems we do not want a solution at a given instant what we call an initial instant which is called T=0 that is known to us we know the state of our system at time T=0.

Or let us call it T0 if you prefer to and we want to find out what will happen to the system in future so we are primarily interested in only one direction that is the direction of influence is only future and this specific nature of time coordinates that allows us to construct a very efficient class of schemes and in fact all the methods which we would look at all a solution methods for an un steady problem they advance in time and in a step by step.

Or what we call marching manner starting from a given initial data our approach would be very simple say suppose you got T=T0 that is where we know the solution we will take a small time increment try and frame an algorithm using which we can find out now what solution at time T=T0+delta T0 now once we know the solution T0+delta T that becomes our new initial condition and then we can use the same algorithm to obtain the solution at next time step and so on.

So we will proceed what we call a step by step manner or as if we are marching in a space in a given direction so that is the reason why the solution techniques for initial value problems are also referred to as time marching schemes.

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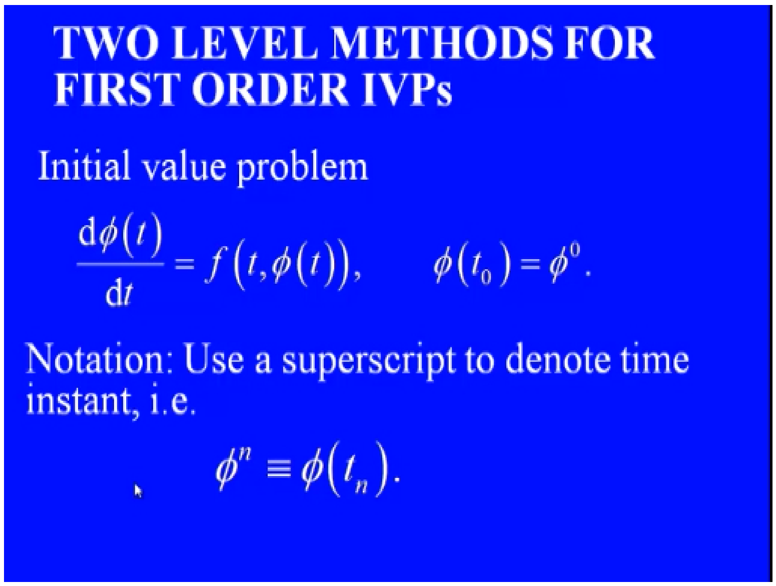

Now let us focus on 2 level methods for first order initial value started initial value problems so a general initial value problem what we call generic initial value problem for a scale of phi in these small t here denotes server time coordinate so d phi t/dt = functional, now his functional f it is depends on t as we well as phi t t so this is our initial value d phi/dt=ft phi and we know what does the initial state of a system that is or phi at time t=0 is specified as phi 0.

Now in finite differences to denote these spatial grid locations we used subscripts, so there are no subscripts available there so what we would normally prefer to use for time levels is or time instant to indicate discrete time instants we would use superscripts. So a generic superscript will say n which indicates time tn. So we will use phi sorry phi superscript n this would be used to indicate the value of the functional phi at time instant tn.

Now as I mentioned earlier if you want to get a solution to this problem.

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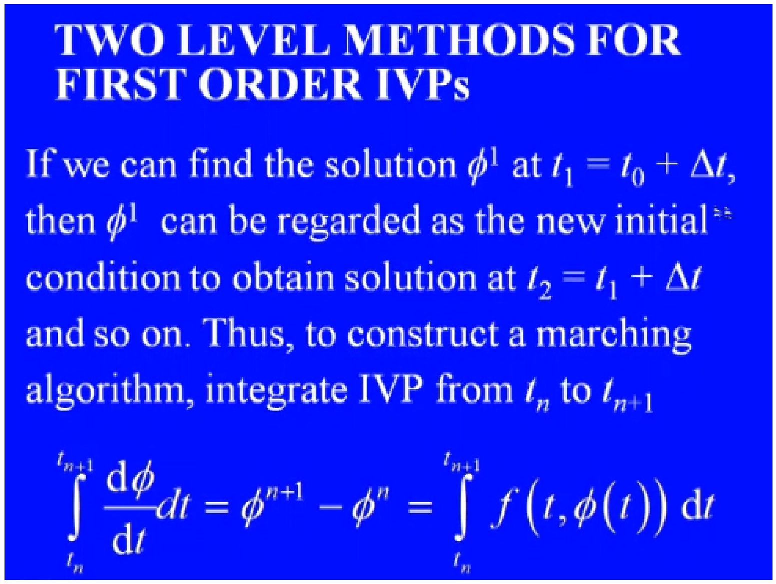

Suppose we want to find a solution at a time instant t1=t0+delta t where delta t is a very small time increment value so if we can find this early solution phi 1 at time instant t1 then phi1 can be regarded as the new initial condition to obtain the solution at t2=t1+delta t, similarly once we know the solution value phi1 at time instant that can be used as an initial guess to obtain the solution s at t2+delta t that is sin instant t3and so on.

So that is all we need to do is if you want to construct an algorithm for time integration we can start off from a generic time instant tn to tn+1, So if we can somehow obtain an algorithm to obtain our solution that would do our job to solve our initial value problem so easiest option would be less what, let us do that at less integrate our initial value problem on left hand side we have got d phi/dt integrated from time tn to tn+1, tn is the time instant.

When we know the solution and tn+1 is time instant at which we want to find out our solution so this integral is very simple it evaluates it as phi n+1-phi n and what we get on right hand side integral tn to tn+1 of f t, phi t dt now please note here this integral on right hand side it involves functional values which are still unknown. Phi as a functional of time we do not know this what we want to find out from our numerical scheme.

So if you want to make use of this particular formula what do we need to do we have got to integrate, we have got to approximate this functional.

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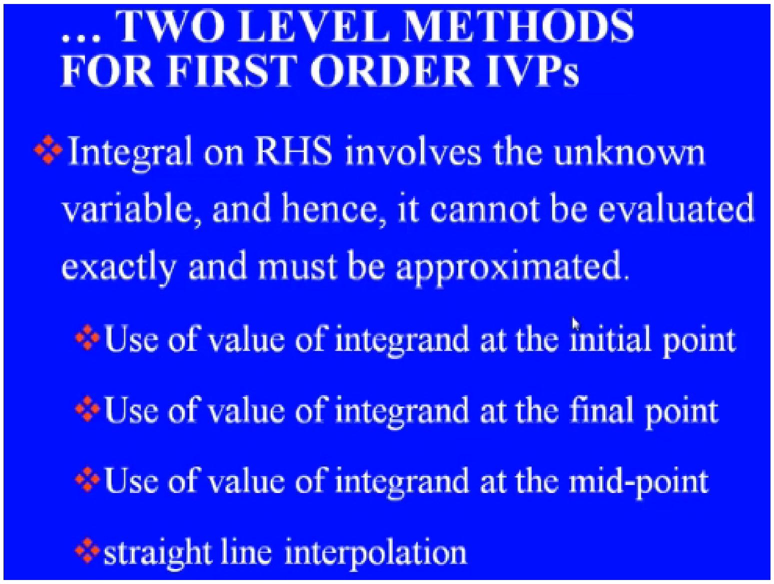

So integral on the right side there is tx2tn+1 f of t gamma phi dt it involves unknown variable phi and hence it cannot be evaluated exactly and it must be approximated and there are various ways in which we can approximate and we will learn few of the ways which will lead to some different solution algorithms. Few simple things would be like we can use the value of integrand at initial point we can use the value of integrand at the final point.

We can use the value of integrand at the midpoint and we can have any straight line interpolation and any of these approximations each one of this will lead us to work separate time integration scheme

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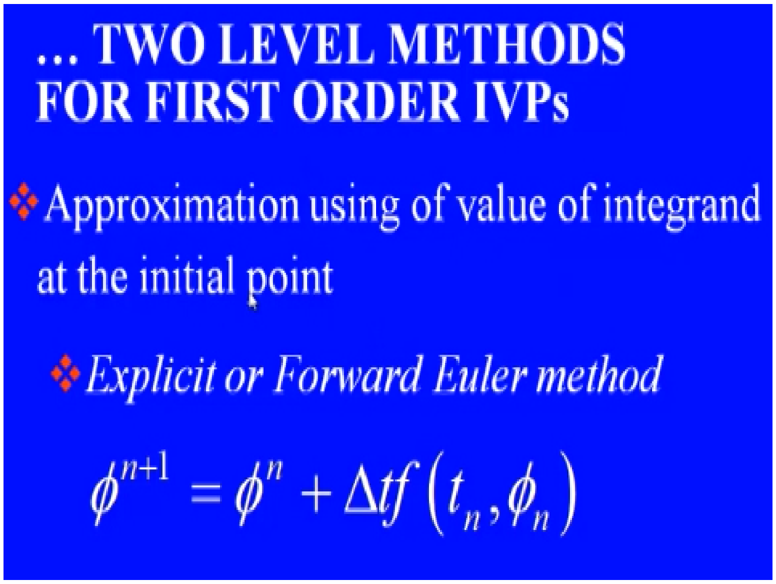

Now if you want to approximate or approximation using value of integrand initial point it leads to an algorithm what we call explicit or forward Euler method. There are straight this approximation of our integral let us move to board.

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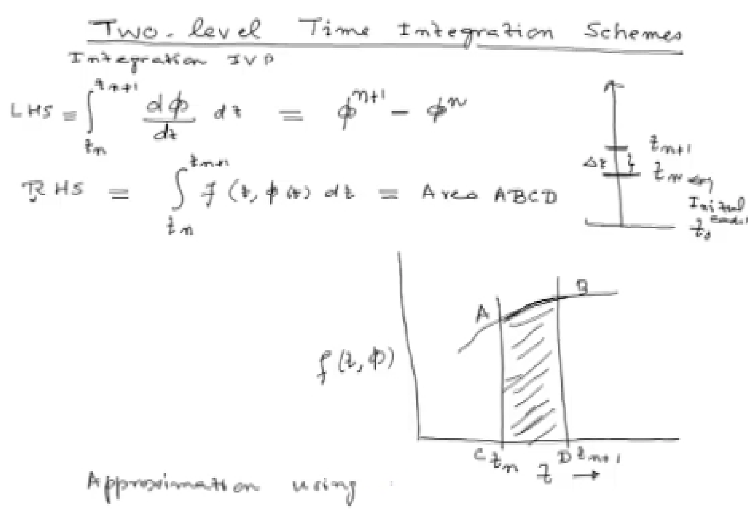

So we are interested in finding out 2 level time integration schemes for our initial value problem and we said that look we could obtain it by integrating our initial value problem in a small time interval. So if this is our time axis we will start at time t=0 suppose you have already advanced up to time instant tn we want to move on to the next time instant tn+1 so this would be the initial condition at time tn.

Which we had obtained by using our approximation or numerical induction scheme let us call this time interval as delta t. So our integration of initial value problem tn to t of n+1 d phi/dt, this was our left hand side. And this results simply x phi phi subscript sorry superscript +1 which is value with a functional phi with time instant tn+1-phi and what we had on our right hand side was the integral tn to tn+1 f of t phi d dt.

Now let us draw the graph of this functional so time axis is a functional f of t phi and suppose this was a generic functional, okay we want to integrate it between time instants tn and tn+1 that is whatever right hand side integral is stands for. So basically we are looking at the area under this curve from A to B. So let us call it ABCD, so the integral called RHS this actually area of this shape or the shaded part represented by ABCD.

So this is area ABCD, so what are different possible approximations for this area ABCD one approximation could be this approximation using value at A, So this also possible way.

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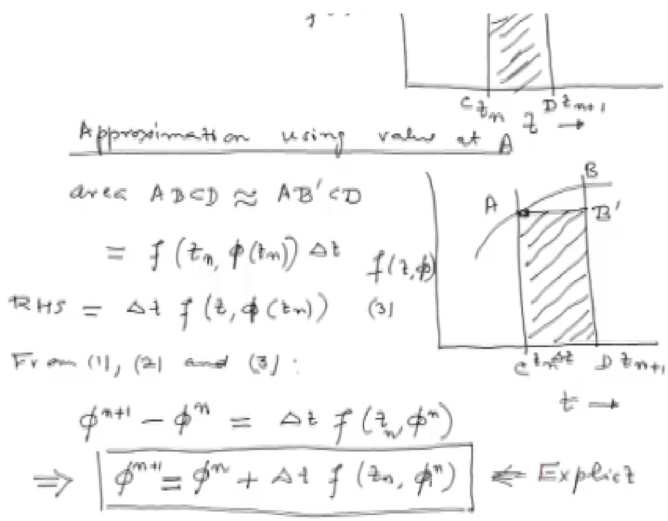

Let us redraw same diagram again, so what we mean now is that we would approximate our integral or this area ABCD by let us call this as B prime So that is area ABCD This is being approximated right area A B prime CD where it is rectangle and what are the area of the rectangle, the height is the value the functional at tm=F tn phi tn x the integral delta because what we see in this approximation results in a known quantity.

We know the time value tn and phi is also known at time instant tn. So thus our RHS become delta t times f t n 5 tn, this number of equations is 1 this was 2 and this is 3, So from 1 2 and 3, what do we get phi f n+1 and phi at –phi at instant n=delta t times f of t n phi at time instant n or we can rearrange it and+1-phi at n sorry phi n-1=phi f+delta t times f tn. So now this method is what we call this an explicit scheme.

Why we call it explicit because each term on the right hand side is known phi at n that known delta t is known and our functional f can be evaluated now. Since phi n is known both the arguments of this functional unknown. So f is known at t and phi n. So if we can evaluate this file though right hand side can be evaluated away and that will give us some number and that can be assigned to phi n+1 which becomes the value functional phi at time instant tn+1.

Now this method is called forward RS scheme and why do we call it forward Euler that we can become clear if you go back to over differential equation, let us have a look at the initial value problem.

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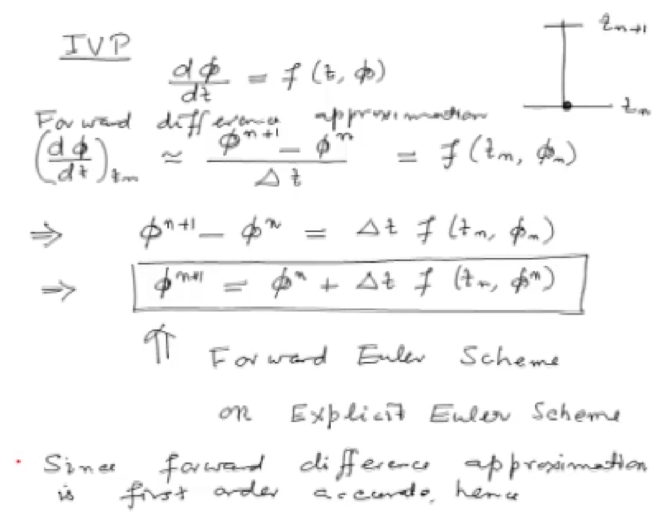

Our initial value problem there was given as d phi/dt f d phi. Now let use it forward differences scheme direct tn and the next gird point is tn+1, so can be obtained now the value of the derivative at time instant tn. So d phi/dt at tn this can be approximated as phi f at time instant n+1-phi at time instant n ÷delta t and this is equivalent to know, what is our RHS, RHS would simply be value of the functional f of time instant tn and of course we know.

So this level forward difference approximation, now what we hear in the terms what did we get phi f n+1-phi f n=delta t times f tn phi n that is phi if n+1=phi n+delta t times f tn phi. So now you can say that we obtained earlier using approximation of the integral on the right hand side by using functional value at the initial point of the time interval this exactly same algorithm or same formula can be obtained using forward difference approximation in time.

For our time derivative d phi/dt, so this is the reason this scheme is called forward Euler scheme this forward differencing was proposed by Euler and in fact the context was the solution of initial value problem so this is the reason the scheme is popularly known as forward Euler scheme and since everything on the right hand side is explicitly known this is also referred to as explicit Euler scheme.

And since we have used our forward differencing in time we can go back to what we learnt in finite difference discretization aspects since first order difference using this functional value at 2 points that is what we used in this particular formula this forward difference approximation is first order accurate.

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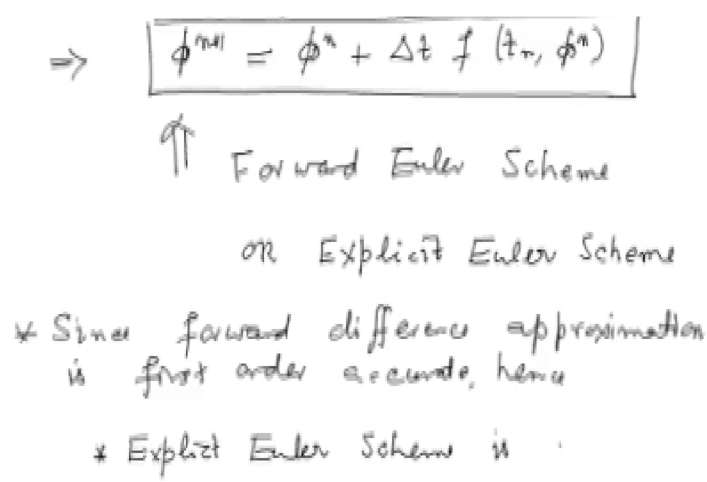

Hence this explicit Euler scheme is also what we call first order accurate. So we also got an estimate for the accuracy by switching back to the alternative derivation using finite difference discretization okay. So this is our explicit or forward Euler method, now next we can approximate our integral using the value the functional and the final point and that gives us what we call backward Euler or implicit Euler scheme.

Let us get back to our board and see why do we call it a backward Euler scheme and why we use the name implicit scheme.

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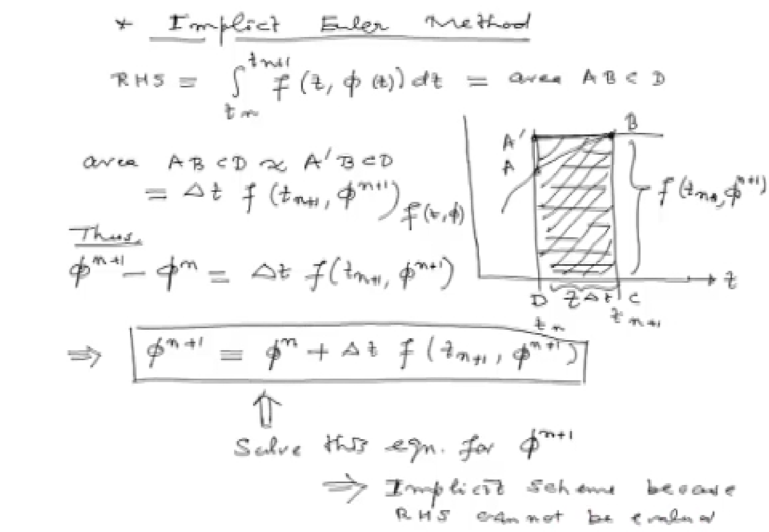

Let us focus on our right hand side integral tn to tn+1 f t phi t dt in the integral tn to tn+1 in a graphical representation so actually area is a horizontal shaded part that is what this integral is area A B C D. We want to approximate it to using the value at point B so let us call this as A prime. So now this slant shaded area is about approximations area a ABCD it has been approximated by the area A.

What is the value of A prime BCD at height BC, Now the side is the value of the functional at the tn+1 phi n+1 Tend +1 and +1 and horizontal axis delta T, we say delta t times f tn+1A phi n+1) so if you substitute this approximation into our expression for the left hand side on the left hand side we had phi f n+1-phi n this was = RHS integral and which has been approximated by delta t f tn+1 phi of n+1.

Let us rearrange the terms phi n+1=value of phi at time instant n+delta t times f tn+1 and this is 1. So now you can clearly see if you look at the right hand side, right hand side cannot be evaluated because we have got this phi n+1 which is unknown. So essentially what we have got is an equation in terms of phi n+1 which must be solved this could be a linear equation for a linear problem.

It would be a non-linear equation for a nonlinear problem so we have to solve this equation for phi n+1 and that is the reason why we call it as an implicit scheme.

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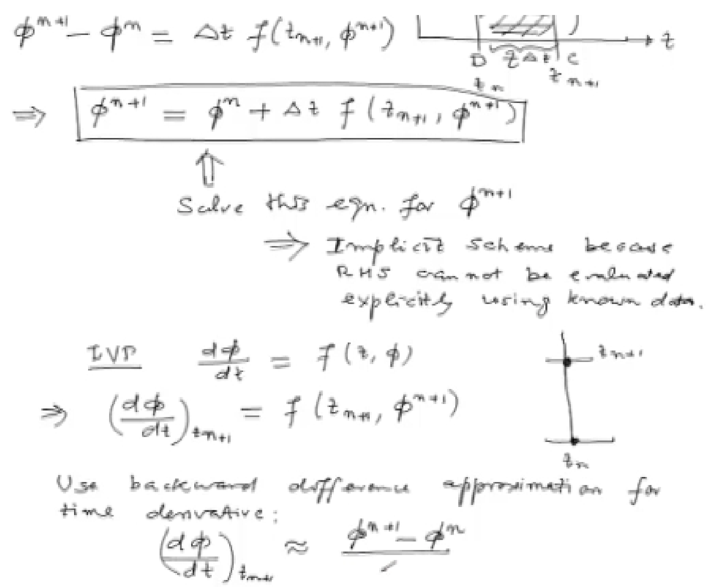

It is called implicit scheme because this RHS cannot be evaluated explicitly using known data, so whenever we mention an implicit scheme for time integration that will always lead to a equation if you say dealing with a single unknown. Or in finite difference approximation context of a partial differential equation it will lead us to us a system of equations which must be solved to get our unknown values.

The next thing is let us find out if it is it has got any connection with the backward differencing or backward difference scheme which is attributed to RS. Now let us get back to our time grid tn tn+1 and let us get back to out IVP d phi/dt=f t phi. So if you want to evaluate it at time instant at tn+1. So d phi/dt at tn+1=f tn+1 and phi n+1, Now let us use backward difference approximation forward derivative d phi/dt.

That is we would obtain the derivative in terms of the functional value at tn+1 and tm, So let us use backward difference scheme for time derivative, if we do that what do we get d phi /dt at tn+1 this can be approximated as phi at n+1-phi at time instant n ÷ delta t. Now let us combine 3 and 4, so we get, Okay if you compare equations 2 and 5 they are identical so the scheme which we have derived earlier by using approximation of the integral.

That could have been obtained using backward difference approximation in time and this backward difference approximation is also referred to as backward Euler method. So that is the reason why this method is called Backward Euler method. And it is called implicit Euler because the right hand side only available implicitly in terms of the unknown functional value.

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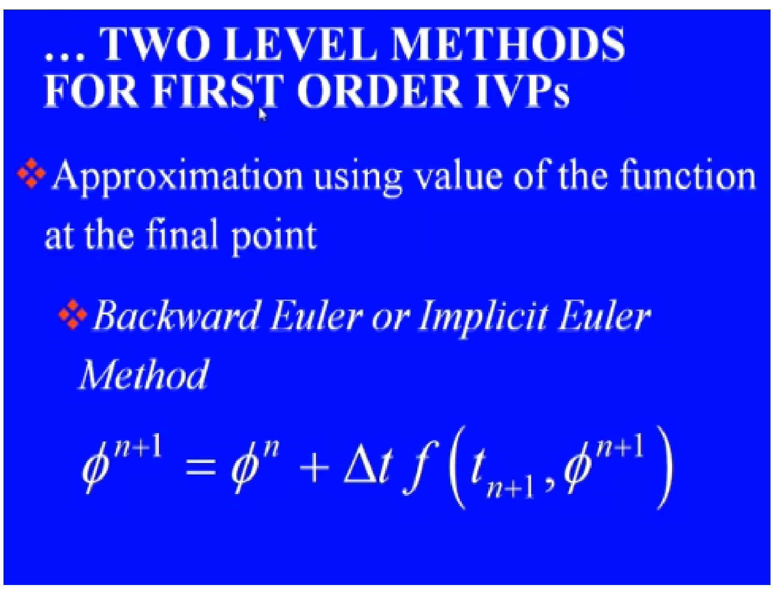

And now let us summarize the scheme so this is what we get approximation using value of the functional at the final point to get backward Euler or implicit Euler method phi n+1=phi n+delta t xf tn+1 phi m this particular equation has to be solved for an unknown value phi (n+1) the accuracy of the scheme again should be o be very clear because we have used a backward difference approximation that is 2.5 finite difference schemes whose accuracy is the first order. So this backward Euler are implicit all our methods again first order accurate.

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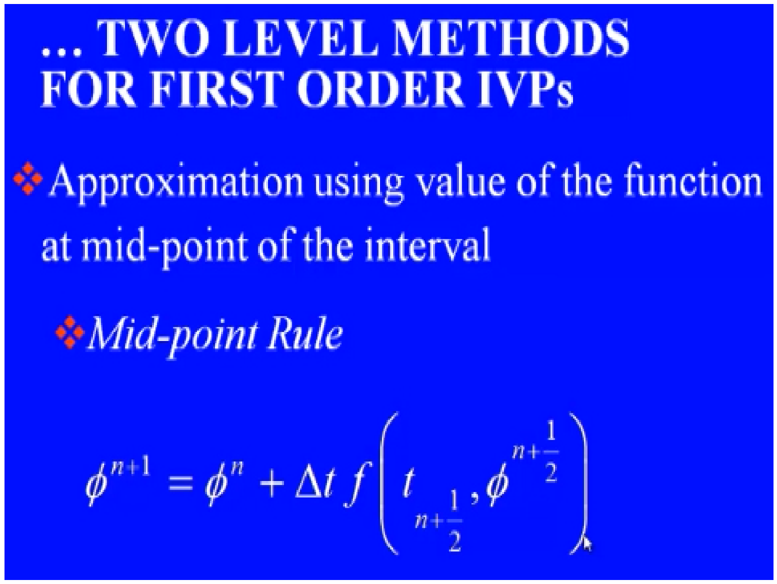

Next thing is we can approximate of our integral by using functional value at the midpoint, Now let us have a look at what do we do what do we gain by this approximation.

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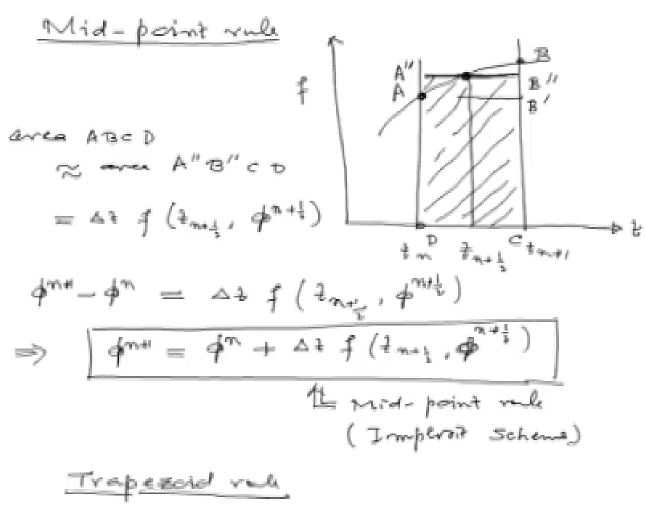

It is called midpoint rule or time axis on f. So we generic function which we want to integrate between p and tn+1, Now if we use the value at the midpoint then what happened this or what actual area would you wanted to A B C and D. Now we want to approximate it by using that area A prime B C and D, Now one thing you can clearly say that this area A prime B prime CD. It is a better approximation compared to our area which we would have obtained.

Now let us call this A double prime B double prime which we could have obtained using the functional value at the initial step or the functional value at the final step. So one thing is very clear if you use the value at the midpoint that is tn+1/2 the approximation of the integral that we get is a lot more accurate than what you would have obtained with the previous 2 schemes. So our approximation for this our area ABCD.

This has been approximated by area A double prime B double prime C and D which is= delta t times value of the function at tn+1/2 phi n+1/2subsitiute it in our original equation, so what we had on left hand side we had phi n+1-phi n and the right hand side integral becomes now delta t times f tn+1/2 phi n+1/2 phi n+1 is phi n+delta times f at tn+1/2 phi n+1/2. So this is what we call our midpoint rule.

Can we evaluate the right hand side explicitly the answer is no because we have an unknown value of the function phi at time instant tn+1/2 and in fact before we can solve this equation we have again got to make an approximation for phi n+1/2 in terms of the values at time instant tn+1 and one of the simplest approximations could be that we can substitute for phi n+1/2 as an average of phi n+1 and phi n substitute into this equation.

And solve that equation to get our solution at time instant tn+1, So this is also an implicit of scheme you can use the Taylor series approximation and some theorems from calculus to show that this scheme would be second order accurate higher order accuracy was obvious because the way we had reasoned that the approximation of the integral area using a functional value at the midpoint that is a lot more accurate approximation of the actual area.

So that explains the higher order accuracy of mid-point rule, there is a tender possibility, let us derive what we call as trapezoid rule that instead of using the values at the midpoint which actually amounts to using an average of the functional values at 2 extremes.

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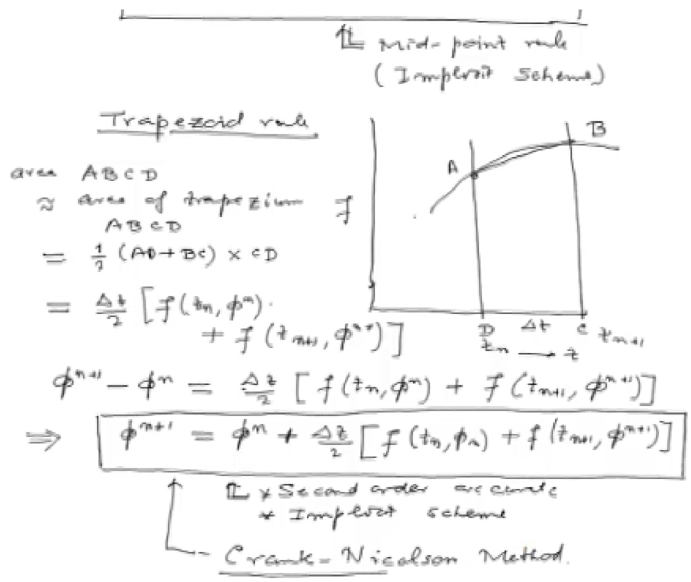

What we can do is this just interpolate the functional linearly not to say this curve is approximated by the straight line between points A and B. Now this ABCD it is now a Trapezium and we can find out what would be the area Trapezium. So area ABCD which the area beneath the curve ABCD which is = our right hand side integral, we are going to approximate by area by trapezium ABCD.

Area of Trapezium is very simple, the sum of 2 parallel sides multiply by the distance between the 2 sides which is half of AD+BC X CD, now in terms of notation function f t tn tn+1. So our AD is functional value of time emiston tn CD is delta t, so we get Delta t/2. For AD we can write F tn phi n+f at tn+1 phi n+1, substitute this approximation for the right hand side integral tool integral formula.

By integrating your initial value problem means you get phi n+1-phi n=delta t/2 F tn phi n+F tn+1, phi n+1, rearrange terms we get the formula algorithm phi n+1=phi n+delta yt/2 F tn+, phi n +f tn+1, phi n+1. By our graphical presentation due the area Trapezium, the area which we have obtained and the area which we have A prime B prime CD in a midpoint they are almost same.

So the trapezoid rule and a midpoint rule they give us the solution of almost the same order accuracy this scheme Trapezoid rule this is also formally what we call second order accurate this is also an implicit scheme. Because what you see on the right hand side of unknown value function it is phi n+1 in board a board unknown functional value this is essentially any creation which ought to be solved phi n+1.

Phi n+1 has become available directly because the right hand side cannot be evaluated explicitly so it is also an implicit scheme. Yet another popular name for trapezoid rule for this scheme is Crank Nicholson method which is derived by this gentleman and is used in solving the initial value problem and this method is also known as Crank Nicholson method.

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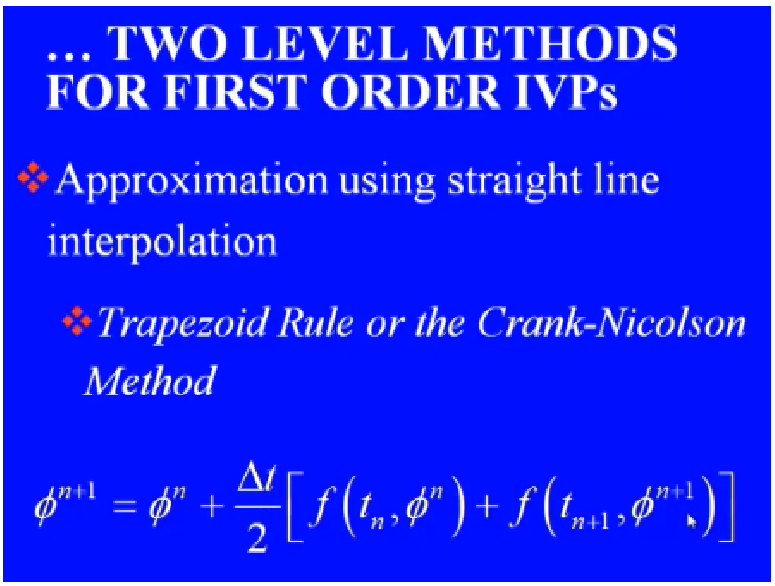

We had approximation using straight line interpolation, it gives us Trapezoid rule or the Crank Nicholson method phi n+1=phi n delta t/2 f tn, phi n+f tn+1, phi n+1, now if you see on the right hand side what do we get here in this bracket it is F t n, phi n+f tn+1,phi n+1 /2 this basically represents an average of the function or work this time interval, so it is also possible for us to generalize this scheme.

By using a separate weight instead of giving equal weight to the functional value like in this case we have used to give equal weight to the functional value the time tn and tn+1 if you give an unequal weight by using a separate weighting function, we get a generalized formula.

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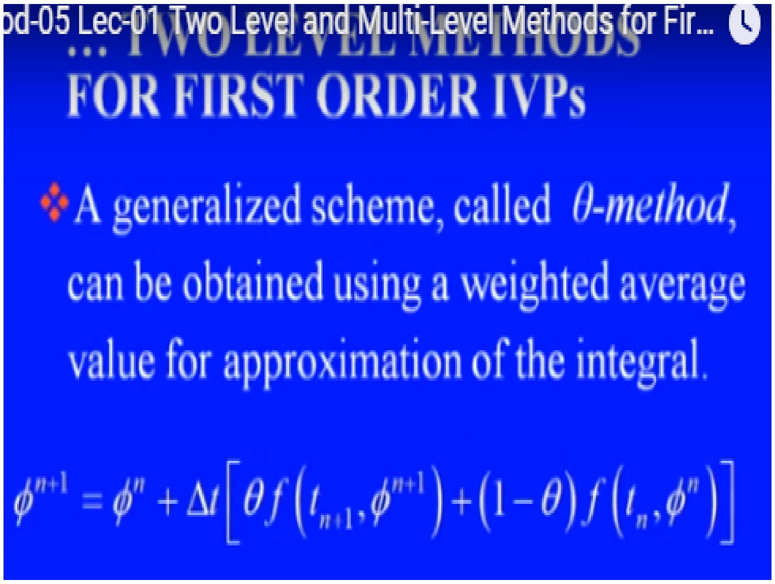

And we will call it this popularly known as theta method, so we generalize the scheme which is called theta method, theta is basically the weight which can be obtained using a weighted average value for the approximation of the integral, so what we done is this integral was approximated as delta t times within bracket theta times the functional value at time instant n+1+1-theta times functional value at time instant tm.

Okay so this gives our generalized scheme called theta method phi n+1= phi n+delta t times theta times f tn+1/2, phi n+1/2+1-theta f tn phi n for a non-0 value of theta this scheme this generic generalized method would also be an implicit method, now if you chose theta=0 that is we have given full weight to the functional value f at the old time instant tn.

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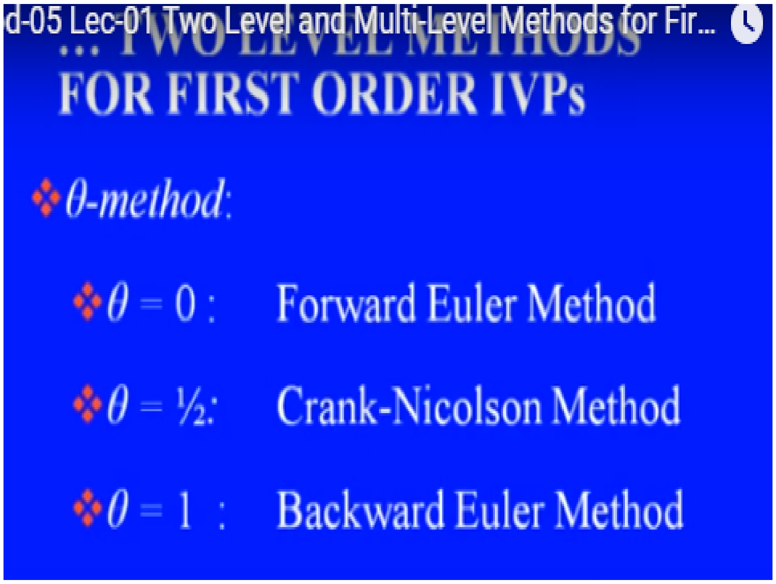

So this theta method reduces to your forward Euler method and if you use theta=1/2 that have given equal weight to the functional value at time instant tnand tn+1 so it is equivalent to our trapezoidal rule method or trapezoid rule or Crank Nicholson method, that is theta=1/2 becomes theta=1that gives us what is called backward Euler method, ok this scheme becomes identical to our backward Euler method derived earlier this a time for value theta between 1/2 and 1.

That is theta=2/3 that we call as galerkin method because that formula would correspond to what we would have obtained if you used a finite element method in time galerkin finite element method in time and use linear shape functions, so that is why that version for that particular value of theta, theta=2/3 we get what we call as Galerkin method in time, now some general comments about the accuracy of the schemes, based on our Taylor series expansion we can show these.

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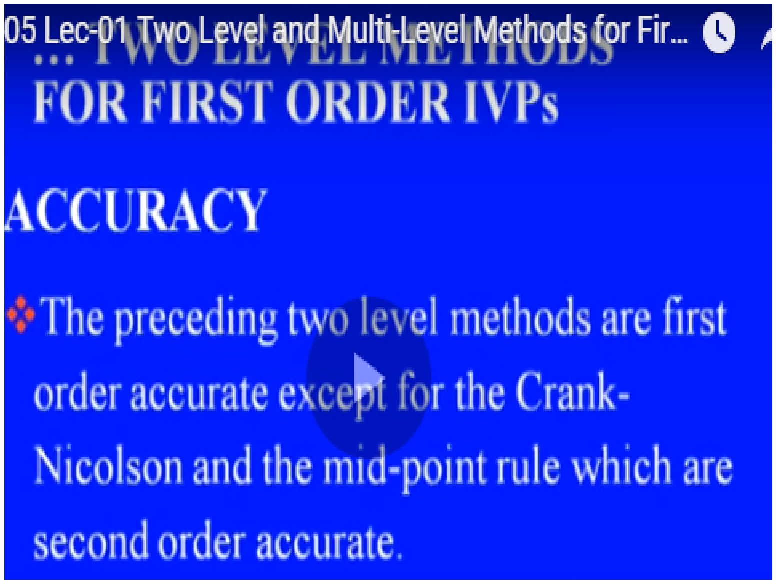

So proceeding 2 level methods we have already seen about backward Rm file forward Euler, the same holds good for anywhere other value of theta so this theta may support which are 2 level methods, first order accurate except for the Crank Nicholson method whichever Trapezoid rule or and the mid-point rule now these 2 Crank Nicholson method and midpoint rule, they are second order accurate this you can easily verify by using a Taylor series expansion.

And our graphical illustration also showed that the approximations which we make by using the value of the function midpoint are using linear average that is much better approximations of the area under the curve compared to what we get by using the functional values at endpoints, so that is the reason why these 2 schemes are second order accurate another thing which you worry about in time integration is what we call stability.

That is to say if you have got certain errors you know a solution how do they propagate, do they become unbounded as our time integration proceeds say they become unbounded, we call that scheme is unstable but if for disturbances remain bounded, they do not affect the solution much then we say that method is stable, now just make general some observations without putting a formal proof.

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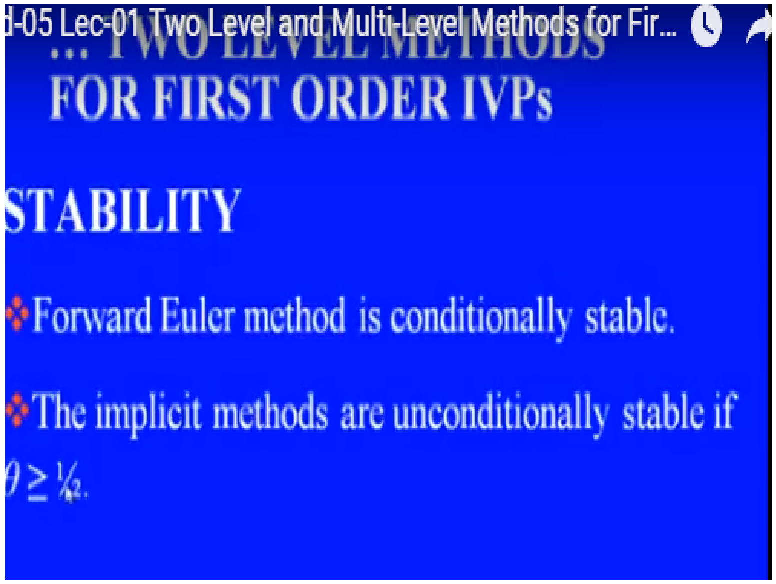

Forward Euler method is only conditionally stable, we will derive; later on when we apply to a diffusion problem that what is the stability condition, that condition is related to the value of the time step, so there is some restriction on the time step, the time step is larger than a certain value or forward Euler method can become unstable, the implicit methods let us call now the 2 level methods in terms of theta value.

So all the implicit method which correspond to theta=non 0, so if theta is greater than or=1/2 theta=1/2 will correspond to our Crank Nicholson method theta=2/3 of galerkin method, theta=1as our backward Euler method, all these implicit methods where theta=greater than 1/2, they are unconditionally stable, there is you can use any value of delta t and the method will not blow up, the solution will always remain bounded.

There is only one catch in the Crank Nicholson method with fairly large values of delta t the solution will remain bounded, but there are some oscillations which might be, numerical oscillations which might be introduced in our solution, so the implicit method which is unconditionally stable got a beautiful property though it is first order accurate it tends to produce smooth solutions even with large values of delta T.

And hence this method is preferred for obtaining steady state solutions as wells as for nonlinear problems where stability is a prime concern, now some comment about computational aspects now these explicit methods are very easy to program because we have a simple formula, right hand side can be evaluated in terms of known values.

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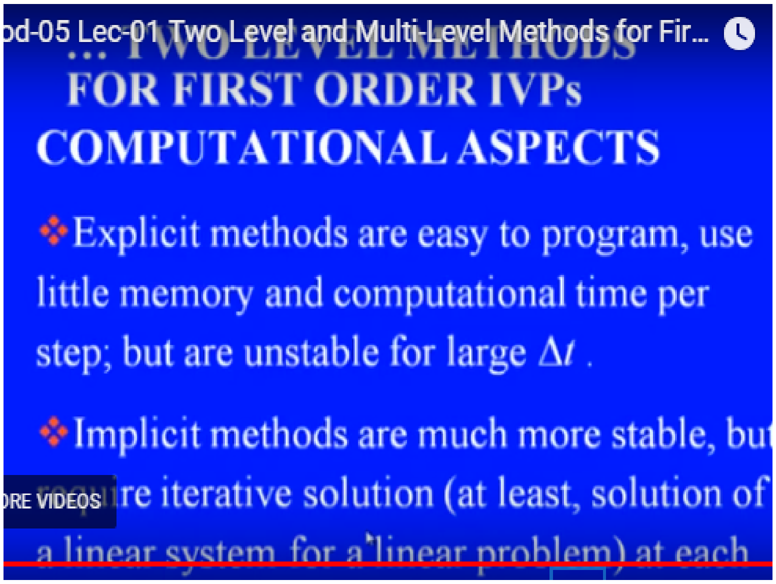

That is why they require very little memory and very little computational time per step. But remember the explicit method is unstable for large delta T. So this observation we are making in context of our forward Euler method but similar observations hold god for higher order explicit methods which we will see later on. The implicit methods are much more stable but they require an iterative solution.

Or at least solution of a linear system or a linear problem at each time step. So that is the disadvantage if implicit methods. So in CFD we will see later on when we come naval stokes equations explicit methods for used where we require very fine tolerance for accuracy, we want time accurate solution, SO their accuracy requirements will force you to use very small value of delta t and that would make that would satisfy what we call the stability conditions.

In that case we can go for explicit methods, but if you want to solve a problem, it does not vary with time, we want to find out what is steady state solutions, in that case explicit methods are too restrictive, so we will go for implicit methods used pretty large time step to quickly arrive at this steady state solutions, so now we have done with 2 level methods and the next if you want to improve the accuracy further we can go for what we call multi point method.

It is very similar to the way we saw in our finite difference approximations that using 2, we can get at the most second order accuracy with central difference approximations if you want higher order accuracy we got to use the functional values at more number of special points or more number of grid points, the same holds good time integration as well, if you want to obtain the time integrations schemes of higher order.

We should use the values of the functional multiple time instants and that what leads to what we call multi point time integration methods and this aspect we will now consider, we will consider multi point methods in the next lecture.