Course Name: Turbulence Modelling

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Week - 2

Lecture - Lec08

8. RANS equations I (I)

I welcome you all again. So, we were looking into the Reynolds averaged Navier-Stokes equations yesterday. Reynolds averaged Navier-Stokes or the so-called RANS equation. and we first took the continuity equation right. So, and then upon Reynolds decomposition and averaging we obtained two sets of equations right? So, we have for continuity, RANS equation for incompressible flows.

It is constant density. So, continuity equation, we had three sets, or I used j right? $\frac{\partial u_j}{\partial x_j}$ that is the instantaneous the instantaneous version and upon Reynolds decomposition and averaging I got $\frac{\partial \overline{u_j}}{\partial x_j}$ equal to 0 for the mean continuity equation, right? And finally, we subtracted one from the other to get the continuity equation for the fluctuation this is for the fluctuation. So, three sets of equation and they look similar. So, which is great.

So, upon averaging for the mean as well as for the fluctuation random component they look similar. So, we will see what happens to that in the momentum equation. So, we look into the mo $\frac{\partial u_i}{\partial t} + u_j \frac{\partial u_i}{\partial x_j}$ on. So, let us slide the instantaneous momentum equation first. So, we have of the form the left-hand side of the momentum equation or the acceleration terms

unsteady and convective acceleration terms, and then we have the pressure gradient term, $\frac{\partial p}{\partial x_i}$ plus. So, we had this $\frac{\partial \tau_{ij}}{\partial x_j}$ and then using the Newton's law of viscosity we had this formula τ_{ij} is set equal to $2\mu S_{ij}$ minus $2/3\mu S_{kk}$, ok.

$$\frac{\partial u_i}{\partial t} + u_j \frac{\partial u_i}{\partial x_j} = -\frac{1}{\rho}\frac{\partial p}{\partial x_i} + \frac{\partial \tau_{ij}}{\partial x_j} \quad ; \quad \tau_{ij} = 2\mu S_{ij} - \tfrac{2}{3}\mu S_{kk}$$

So, this S_{kk} is nothing but $\frac{\partial u_k}{\partial x_k}$. So, this goes to 0 in an incompressible flow here, right? So, this term goes to 0 since $\frac{\partial u_k}{\partial x_k}$ equal to 0 continuity part. So, only this term survives.

$$\frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)$$

So, if I substitute this, so this is essentially 2μ and then . . So, essentially, I get two terms, and if I substitute here, I would get this last term here, this will be $\frac{\partial}{\partial x_j}$ of the first term, which is $\mu \frac{\partial u_i}{\partial x_j} + \frac{\partial}{x_j}\left(\mu \frac{\partial u_j}{\partial x_i}\right)$. Now what will happen to the last term here? Does it survive? See the last term here, we can differentiate this term first with respect to the j and then with respect to i. So, the total when we expand it this is, this term does not change j and j are repeated indices. So, I can first differentiate with respect to j, that means I can rewrite this as, I can say this is nothing but $\frac{\partial}{x_i}\left(\mu \frac{\partial u_j}{\partial x_j}\right)$. First differentiate with respect to j, then differentiate with respect to i, no problem.

And therefore, this is 0 again due to continuity. So, only one of the term survives here, ok. So, only this term is surviving here, right? So, we have the equation, the instantaneous momentum equation. Now, we apply Reynolds decomposition and average ok. So, if I do this, Reynolds decompose and ensemble average.

So, Reynolds decomposition implies, I need to now split, let us call this equation, what was the number? The last equation was 4, right? So, we can call this equation 5. Which is missing? Oh yeah, so $\frac{1}{\rho}$ is missing, that is true. So, I will put here $\frac{\mu}{\rho}$ is anyway constant. I can move in and out of the differential operator ok. So, it is essentially becomes kinematic viscosity now. Molecular viscosity divided by the density becomes kinematic viscosity.

So, Reynolds decomposition implies here for both u_i and the pressure, both has to be decomposed into mean and fluctuation. So, if I write now equation 5 becomes

$$\frac{\partial (\bar{u}_i + u_i')}{\partial t} + (\bar{u}_j + u_j')\frac{\partial (\bar{u}_i + u_i')}{\partial x_j} = -\frac{1}{\rho}\frac{\partial (\bar{P} + p')}{\partial x_i} + \nu \frac{\partial^2 (\bar{u}_i + u_i')}{\partial x_j \partial x_j}$$

. So, now we be essentially have to average the entire equation now right and averaging and addition they commute. So, it is separated out as well as the differential operation here. So, I need to essentially average the entire equation.

$$\overline{\frac{\partial \bar{u}_i}{\partial t}} + \overline{\frac{\partial u_i'}{\partial t}} + \overline{\bar{u}_j \frac{\partial \bar{u}_i}{\partial x_j}} + \overline{\bar{u}_j \frac{\partial u_i'}{\partial x_j}} + \overline{u_j' \frac{\partial \bar{u}_i}{\partial x_j}} + \overline{u_j' \frac{\partial u_i'}{\partial x_j}} = -\frac{1}{\rho}\overline{\frac{\partial \bar{P}}{\partial x_i}} - \frac{1}{\rho}\overline{\frac{\partial p'}{\partial x_i}} + \nu \overline{\frac{\partial^2 \bar{u}_i}{\partial x_j^2}} + \nu \overline{\frac{\partial^2 u_i'}{\partial x_j^2}}$$

Now, we have to average. Ensemble average. So, now the first term does not change anything. Ensemble averaging the mean does not change anything to it. It has achieved its true mean characteristics. So, ensemble averaging has no meaning here.

So, this will remain as it is. So, $\frac{\partial \overline{u_i}}{\partial t}$. This operation is unchanged. What happens to the second one? this is average of the fluctuation, right? So, this goes to 0. So, this term remains because this is already mean advection term.

So, I get $\overline{u_j}\frac{\partial \overline{u_i}}{\partial x_j}$. Now we have few terms here. what will happen to the second term or the second term of this advection uj bar is the true mean. So, it is like a constant right it can be taken outside the averaging operation and therefore, ensemble averaging of the fluctuation here is 0. So, this term goes to 0.

Same thing here there is a mean strain $\frac{\partial \overline{u_i}}{\partial x_j}$ is the mean strain or mean velocity gradient which is a constant operator here. It has achieved its true mean therefore, it is outside the operation you can take it out and therefore, its ensemble mean of u_j' is 0. What about this term? Yes, if you recall Chapter 1, Statistical Analysis, that is the reason I started with statistical analysis to introduce what correlation is. Correlation can go to 0, it can take a positive or a negative value that is why I had this. So, I have u_j' and then the gradient of u_i'.

We do not know what it is now, ok. At least we cannot say it is 0, we do not know. So, we have to retain this term. So, we retain this term ok. I will move it to the other side of the equation later because I want to write it in terms of equation 5 right.

Equation 5 has the way we write momentum equation is an unsteady, unsteady acceleration, and then there is a convective term, and then right-hand side, we have some forces. So, I would like to write in terms of that, and therefore, I keep it like this. So, on the right-hand side, I have this mean pressure gradient. which of course survives and then this particular term average of a pressure fluctuation is 0 that is why we needed the true mean concept if you are working with arithmetic mean we cannot achieve this equation right we are not doing any experiments we are not doing any simulations. just by assuming that true mean exist we are able to achieve an equation for mean continuity, an equation for mean momentum, right? We are trying to get to some place without doing any experiments or simulation right.

For that of course, we must believe that the true mean exists. So, this is fine. So, this term survives. So, the viscous stress term which is $\nu\frac{\partial^2 \overline{u_i}}{\partial x_j^2}$. What about the last term? The average of the fluctuation again goes to 0, ok.

So, that means the one term from the left-hand side. This is very important. I am moving the left-hand side term to the right-hand side. So, this is $-u_j'\frac{\partial u_i'}{\partial x_j}$. Now if you compare, let us call this equation 6.

If I now compare equations 5 and 6, this, of course, equation 5, this is gone, right? So, if I compare Equation 5 and Equation 6. we see that this is not like what we expected coming from the continuity equation the way we were being, you know, fooled by the continuity equation. Continuity equation everything looks similar right the instantaneous the continuity and the fluctuation all look similar but not for the momentum. Momentum you see the left-hand side is fine. It is looking similar to equation 5, 5 and 6 the on the right-hand side the first two terms are also looking similar right, the pressure gradient and the viscous stress term, but there is an additional term.

So, this is the sole additional term here a new term has come up this is a new term. So, upon ensemble averaging the Navier-Stokes or the momentum equation you get an additional term. So what is special about this additional term? If you look into the mean we were interested in the mean flow like any engineer but now you see if you want mean momentum you must know about randomness also. You can compute for all these four terms which is mean but to get there we must have the knowledge of this last term and only this last term knows about turbulence rest of the terms are all mean statistical mean it knows nothing about randomness. So, this is the only term which knows about turbulence ok, only term that knows about turbulence in equation 6.

So we must have the knowledge of it before we can compute even the most basic information that engineers need. You know, they would need pressure drag. They would need viscous drag, right? This is what engineers look for, even to get there, we need this information, right? And another important aspect what is the origin of this, so now we know that this is the turbulence right? So, this particular term it knows about turbulence as. So, what is the origin of this term? So, which term in the Navier-Stokes equation knows about turbulence? Exactly, the non-linear term, right? So, if you want to get rid of this term and then solve something, you know now that you cannot solve turbulence. The only term in the Navier Stokes which knows about transition, turbulence any other hydrodynamic thermal instabilities everything is known by the nonlinear term.

You cannot get rid of it assuming it is small, like if you follow some analytical methods right? So, this is the origin is from the left-hand side. So, this term originated from left-hand side that is nonlinear term or the advective or advection or convection term. So, turbulence is a non-linear process. So, that term is very important. If you say this term is not important that means your flow is laminar.

You do not have any turbulence anymore right. The equation falls back to a laminar flow. if this fluctuations are infinitesimally small then you are essentially going back to a laminar flow right. So, if you are interested in turbulence you must retain the non-linear term ok, but retaining is easier solving them is not easy. we will see we will get there what can be done to that one.

So, now I am going to do little bit of a trick here because I would like to write this this equation is fine the way it is, but I would like to write it in a different way so that you will appreciate the chapter 1 statistical analysis better ok. So, for that I will make use of the continuity equations. So, here so this was equation number what was the? This was 4. this was 4, right, and this was 3, I suppose, maybe yeah, yes, this was 3, and this was 4, ok. So, I will make use of this fluctuating continuity equation because it is 0, right? I can add or remove 0 from any equation correct? without changing anything.

So, I would like to use equation 4 because it is essentially 0. So, let us use equation 4. If I use equation 4, I have $\frac{\partial u_j'}{\partial x_j}$ =0. Can I multiply anything to this term, and it is still 0, right? I can multiply anything.

So I would like to multiply this. So we can multiply this with anything and still it will be 0, any other term and still it will be 0. So, I would like to multiply this with u_i' ok. So, multiply with u_i', $u_i' \frac{\partial u_j'}{\partial x_j}$ =0. It does not change; it is already 0; I can do whatever with it, right? So, now it is already 0; if I take an ensemble average of 0, what is it? 0.

0 right. So, ensemble, ensemble mean of 0 is 0. So, I am going to take the ensemble mean of this term. That is, I have $u_i' \frac{\partial u_j'}{\partial x_j}$. I take the ensemble mean here, it is still 0.