**Course Name: Turbulence Modelling**

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**Week - 12**

**Lecture – Lec73**

**73.LES: Dynamic Smagorinsky model and Scale similarity models - II**

Scale similarity models . So, the idea here is if you recall the the energy spectrum E (k), k is a wave number ok. So, in that let us say you have a cut off wave number $k_c$ is the cut off wave number let us say ok. cutoff wave number as I said before is that like you are resolving up to that particular eddy size.

This wave number is of course, representing a certain eddy ok. Wave number is related to the wavelength of a particular eddy. So, the anything from the origin to up to cutoff is the resolved part here. So, this is the resolved part and this is the modeled part The SGS is supposed to model the last tail of this contribution.

So, if you simply recollect this energy spectrum with a cutoff wave number. and ah this and then of course, you have a Kolmogorov length scales at this end and you have your either energetic length scale or a pseudo length scale whichever you want to use it to account for the other one. Here the idea is that the scale is scale should be similar on either side of the cutoff wave number. That means, the eddy which is just above or ah below this cutoff wave number ah should have similar behavior.

So, there should not be any abrupt ah change at the cutoff right. So, the Smagoronski and other models it models everything before the cutoff wave number sorry everything after the cutoff wave number and before it is resolved by the mesh. Here the idea came that the eddies or scales just below and above or just around this cutoff wave number has to be similar that is the idea of scale similarity or scale similar right. So, in this type of model. So, the model philosophy is scales just above and below the cutoff wavenumber $K_c$ ok or similar that is scales smaller than is scale smaller and larger than delta can think of as this right.

Since the cutoff wave number corresponds to your filter size or the grid size that you are capturing. So, the eddies that are just smaller and larger of the delta they should be similar ok or  similar scales just above and below the cutoff wave number or smaller and larger than your delta or similar and therefore, the name came scale similar . So, to do this  we go back to $\tau_{ij\,SGS}$ because every SGS model or in LES is about modeling or finding this 6 unknowns sub grid scale stresses $\tau_{ij\,SGS}$ and the exact form of this is $\overline{u_i u_j}$ which is the unknown we do not have access to this and you have $\bar{u}_i \bar{u}_j$ is the exact equation. Now what I can do is this ui further I can decompose ok. So, here ui itself is decomposed into the filter and its fluctuation or not the fluctuation the residue ok.

So, this is nothing, but your filtered component  and this is the residue . So, if I split this ui into these two components, I can rewrite this as I have ui filter plus residue and then uj filter plus its residue  filtering operation over it minus Ui filter Uj filter. So, you can rewrite this as now it is looking lot similar to this Reynolds decomposition even though it is not like we have simply splitting the instantaneous velocity into its filtered and residue. So, now, I have Ui bar Uj bar filtering on top of it. first component and then I have the two components which have interaction between filtered and residue which is u i bar u j filter plus I have u i residue u j filtered filtering on top of it plus u i residue u j residue filtered.

$$\tau_{ij\,SGS} = \overline{u_i u_j} - \bar{u}_i \bar{u}_j \quad ; \quad u_i = \bar{u}_i + u_i''$$

$$= \overline{(\bar{u}_i + u_i'')(\bar{u}_j + u_j'')} - \bar{u}_i \bar{u}_j = \overline{\bar{u}_i \bar{u}_j} + \overline{\bar{u}_i u_j''} + \overline{u_i'' \bar{u}_j} + \overline{u_i'' u_j''} - \bar{u}_i \bar{u}_j$$

$$\tau_{ij\,SGS} = \underbrace{\overline{\bar{u}_i \bar{u}_j} - \bar{u}_i \bar{u}_j}_{\text{Leonard Stresses (exact)}} + \underbrace{\overline{\bar{u}_i u_j''} + \overline{u_i'' \bar{u}_j}}_{\text{Cross-Stresses}} + \underbrace{\overline{u_i'' u_j''}}_{\text{Reynolds SGS Stresses}} \longrightarrow \circledast$$

minus Ui filtering Uj filtering ok. So, here I can rewrite this as $\tau$ ij Sgs is equal to the first term and the last term minus Ui Uj. and then the other two terms which is where there is interaction between filtered and residue plus ui and the last term which is somewhat looking like a reynolds stress. So I have classified this into three types of terms. Is there any term that is exact here or all terms requires modeling? The first term on the right hand side is exact right.

Exact means I can do this. I have Ui bar access because that is what is being computed. I can do one more grid filtering operation on that. So this is exact for me here right. So, this is exact as well as it is looking lot similar to the dynamic Leonard stresses at the grid filter level. So, this is actually called Leonard stresses ok. So, this is called Leonard stresses luckily it is exact no modeling required and these are called cross  stress terms cross stresses because it is looking like there is a cross interaction between the filtered and the residue component here. This is as I said looking somewhat like a Reynolds stress. So, we call it Reynolds SGS stresses.

So, we will call this equation let us say let us call it star some number. So, among these three one is exact the other two requires modeling. If you do that the $\tau_{ij\ sgs}$ is closed and you can solve the LES calculation ok. So, symbolically this is written as this $\tau_{ij\ sgs}$ is written as $L_{ij}$ Leonards stresses + $C_{ij}$+ $R_{ij}$ . So, this is exact it means no modeling required.

This is a significant advantage over your Smagorinsky-type models, where all the SGS fluctuations are modeled. Here, at least some part of the SGS fluctuation is accurate; the $L_{ij}$ component is correct over Smagorinsky. The other two have to be modeled; these two also have to be modeled. So now there are two main classes of models here: one is called a Bardina model, and the other is a special model. So that explains how the $C_{ij}$ and $R_{ij}$ are modeled.

The first one is the Bardina model. So before that, you can say that, or let us talk about the Bardina model no issue. So, the Bardina model is from 1980. So, the idea here is that you can say this $C_{ij}$ is the cross term, and we assume that it is responsible for the interactions between the scales that are just above and below the cutoff number because it contains terms that are both residue and filtered. Here, it is assumed that $C_{ij}$ is responsible for the cross interaction between the filtered and the residue scales.

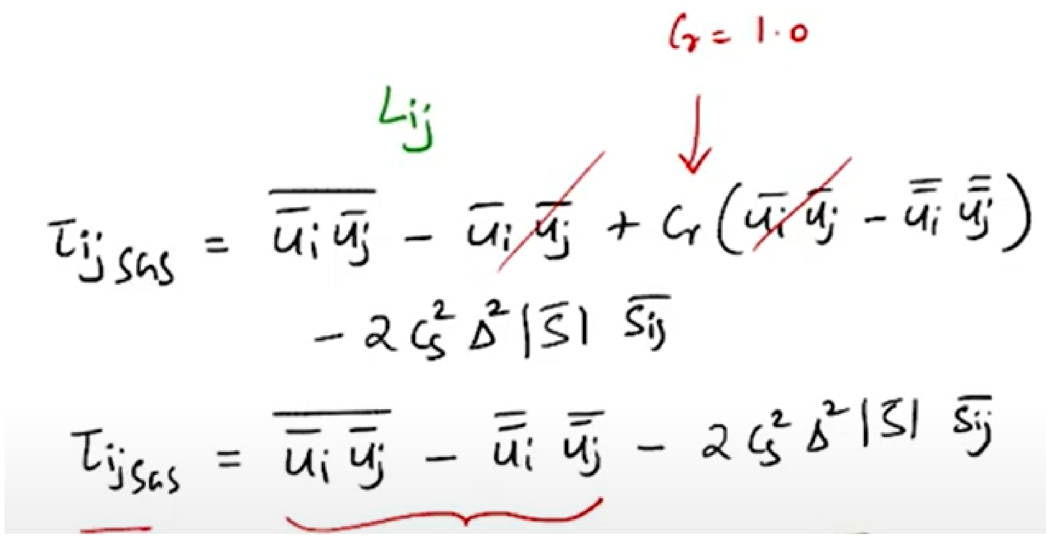
In this model, $L_{ij}$ is computed. So, the $\tau_{ij\ sgs}$ can be obtained first by modeling the $C_{ij}$ model. It is modeled as $C_r$, a model constant, followed by $\overline{u_i u_j} - \overline{\overline{u_i u_j}}$. Remember, filtering again in LES in general is not the same as the original filtered quantity for box filtering; the only exception is the cutoff filter. In the Bardina model, the model constant $C_r$ is set equal to 1.

1. In the Bardina model, the $R_{ij}$ is omitted; this is not modeled, and they found that there is an issue. What is the objective of an SGS model? To dissipate the energy sent to the SGS scales and the Bardina model, it was found that it is not dissipative enough, and some energy accumulation is occurring. So, this $R_{ij}$ has to be modeled, and that is corrected in the special model, right? So, note that the main objective of an SGS model is to dissipate resolved fluctuations and resolved energy, and that is not happening here, right? The Bardina model was found to be insufficiently dissipative because this $R_{ij}$ is omitted. So, the next model by Speziale corrects the Speziale model from around 1985. So, this model also takes into account the $R_{ij}$ contribution.

So, here the $\tau_{ij\ sgs}$ is now $L_{ij}$ + $C_{ij}$ + $R_{ij}$, right? $L_{ij}$ is exact. So, there is nothing to be done similar to the Bardina model. Here, the $C_{ij\ model}$ is similar to that, except that the model constants have changed. In the C $_{ij\ model}$, $C_{ij}$ is set equal to the same as $C_r$ $(\overline{u_i u_j} - \overline{\overline{u_i u_j}})$; however, the $C_r$ model constant is 1 instead of 1.1, as in the Bardina model, and we will see why this gives an advantage later.

The R $_{ij}$ model term, based on the Smagorinsky idea, is used here. Okay, similar to Smagorinsky, it is set equal to - 2 $Cs^{2}$ $\Delta^2$ S $\overline{S}_{ij}$, which is like Smagorinsky. Here, $C_s$ is your Smagorinsky constant. Again, it is user-dependent here. Even though you are not going to define all the SGS scales, because only a fraction of the total SGS scales contributes, you will have a user dependency here.

The $C_{ij}$ and $L_{ij}$ are computed only if $R_{ij}$ has some user-dependent values going into it, right? So, if I now substitute all this into $\tau_{ij\ sgs}$ is equal to $L_{ij}$. If you recall, it is $\overline{u_i u_j}$, and then a filtering on the entire product - $\overline{u_i u_j}$ . Okay, that is the L_ij; this is your L_ij term. Now, I have this $C_{ij}$ term in the specialized model, which is $C_r$, equal to 1, followed by Ui filtered, Uj filtered, minus Ui double filtering, Uj double filtering, and finally, the Rij term, which is minus 2 Cs squared delta squared. So, it is now readily seen by setting Cr equal to 1.




This term means fewer computations for you to do, okay? This term is going away because this CR is 1 here. Therefore, this τ_ij SGS is now ū_i ū_j - u_i" u_j" - 2C²Δ²S_ij. This is the final form of your specialized model. And if you look closely, what is this? This looks exactly similar to your dynamic; Leonard stresses only that the operation is done at the grid filter level, right? This is analogous to your Lij, which is what Leonard stresses as dynamic. This provides another class of models for you: a scale similarity model, which is somewhat different from the Smogoransky model.

The advantage is that some fraction of the SGS fluctuations are computed exactly, and the $L_{ij}$ component and $C_{ij}$ are also calculated. As you mentioned, $C_{ij}$ also requires only filtering operations. There are no extra assumptions here, right? The only factor is that rij represents the modeling component, while $L_{ij}$ and $C_{ij}$ can be obtained through just one more filtering operation at the grid level.