Course Name: Turbulence Modelling

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Week - 12

Lecture – Lec72

72. LES: Dynamic Smagorinsky model and Scale similarity models – I

Ok, let us get started. So, we were looking into the dynamic Smagorinsky model. So, we have learnt about test filtering, and we also understood the concept of dynamic Leonhard stresses, which is essentially the L_{ij} was equal to $T_{ij_{SGS}}$ at the test filter level minus tau ij SGS test filtered this is equal to ui filtered uj filtered both at the grid and the test level minus ui bar uj bar. Sorry, this is whole test filtering, and we learned that this is exact now this entire part is exact.

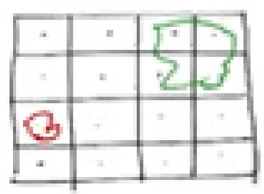
$$\mathcal{L}_{ij} = T_{ij_{SGS}} - \widehat{\tau_{ij_{SGS}}} = \underbrace{\widehat{\overline{u_i}\,\overline{u_j}} - \widehat{\overline{u_i}}\,\widehat{\overline{u_j}}}_{exact}$$

We have access to this because the ui bar is the grid filtered velocity that you are calculating in LES, and on the test filter level you can do one filtering operation. So, this is the test filter level SGS. This is the grid filter level SGS. So, essentially, what we are doing is in a Smagorinsky model we want to compute.

If you recall, in Smagorinsky, the length scale is the sub-grid length scale, right? So, this l_{SGS} let us call it this was essentially $C_s\Delta$ right. So, to compute ν_{SGS}, I am using this length scale square followed by a inverse time scale which was the square root of $2S_{ij}S_{ij}$ filtered. So, the l_{SGS} is the sub-grid scale contribution or the sub-grid scale eddy obviously, has to be smaller than the Δ that means, C_s has to be less than 1, right? It has to be a fraction right. So, since this is a subgrid, this is the subgrid contribution subgrid scale or an eddy, right? So, therefore, C_s is less than 1 and we saw that the values were like 0.25 or 0.01 or whatever right. So, a fraction value and this is what we want to compute instead of fixing in dynamic Smagorinsky we want to compute C_s right. So, to do that what we did is we let us say we took a mesh, not a mesh. Let us say this is the grid filter level. Ok, this is the grid filter level.

So, infinite volume method implicit filtering this is where you are actually calculating your data right you are computing data on this cells where mesh size is same as filter size. So, I have access to this. So, let us say here this is let us say there is a sub-grid scale eddy here and the contribution of this sub-grid scale eddy is what you want to compute in LES using dynamic Smagrinsky model, and I do not know that and to do this I want Cs delta right to compute the ν_{SGS}. So, the ν_{SGS} was your $(C_s\Delta)^2$ right, the L square followed by the inverse time scale $(\bar{S})$ right. So, this was your meter square this is $1/s$.

So, this is the sub grid scale eddy contribution that I want to find out. And since I have a mesh like this, let us say I have a resolved. Let us say I have this particular green eddy. Let us say it is captured using this mesh at the grid filter level. So, the idea of this dynamic Smagorinsky is that, ok, I have access to an eddy, which is captured at the grid filter level.



Grid Filter

I can make it a sub-grid eddy at the test filter level, right? That is the idea. So, what we just did is we just said ok. So, we make a test filtering operation. That is, after calculating at the grid level, I make a test filter right. So, this is our test filter here this particular teddy that we had this now becomes sub grid right.

So, this is resolved here using the grid filter, and this is your subgrid. Now, this becomes a subgrid here, but I know the exact data on the grid filter because you are computing your LES on a particular grid filter. I am just taking a test filter which is twice that of grid filter. So, that I know what it is from this, I can estimate, suppose if I am going to, you know, change the mesh or resolve it further, what should be the sub-grid contribution by going one level up right. We are not going let us say one level down that you can do it in LES convergent test you can reduce the mesh instead of this 4×4 you can do 8×8 or 12×12 so on.

But at any grid filter level I am going one level up to find out what is the SGS contribution and using this I estimate the C_s ok. You have a question? No ok. So, using that information so we said this dynamic Leonard stresses right. So, this was our dynamic Leonard stresses, which luckily is exact. I have access to the filtered velocity.

I make one test filtering operation and find out the L_{ij}. No modeling required here. So, the objective now is to find out C_s dynamically. For that what we do is first we use the Smagorinsky idea also for the test filter level ok. So, now recall the $\tau_{ij_{SGS}}$ model, ok? This is the $\tau_{ij_{SGS}}$ model.

Essentially, you are saying it is $-2(C_s\Delta)^2$, that is your length scale, sub-grid length scale, the inverse time scale and, of course, the strain rate filtered strain rate, right? So, this is your this is your ν_{SGS} in a Smagorinsky model. So, I am recalling here Smagorinsky and then followed by of course, you have plus one third tau k k delta ij. I am not using the the idea that is suggested by Pope where you take 2 third k SGS that requires k SGS to be computed using a let us say one equation model. Instead we go back to the original equation where tau kk is introduced. So, this is I have an equation here let us call it 1 this is at the grid filter level $\tau_{ij_{SGS}}$.

$$\tau_{ij_{SGS}} = -2\underbrace{C_s^2\Delta^2|\bar{S}|}_{\nu_{SGS}}\bar{S}_{ij} + \frac{1}{3}\tau_{kk}\delta_{ij}$$

I am using the same concept now also for the test filter level right. So, assume Smagorinsky model also for $T_{ij_{SGS}}$ that is the test filter level, right? So, what we do is $T_{ij_{SGS}}$, modeling analogous to the $\tau_{ij_{SGS}}$ here. Of course, this is minus 2. We can call it C_s^2, let it be there for some time. We do not know what it is.

And then now it is a test filter size will come square of that ok Cs delta cap square followed by test filtered inverse time scale here on the strain rates S ij grid filtered test filtered. Of course, the other part which is one third T kk delta ij let us call it equation 2 here. or we call it let us say 2 and 3 that is better this is 2 and this is 3 call this 1 right.

$$T_{ij_{SGS}} = -2C_s^2\hat{\Delta}^2|\hat{\bar{S}}|\hat{\bar{S}}_{ij} + \frac{1}{3}T_{kk}\delta_{ij}$$

So, now we would like to of course, here where this S test filtering is nothing, but square root of 2 S ij Sij grid filtered test filtered and S ij test filtering is nothing but your half of dou ui by dou xj plus dou uj by dou xi grid filtering test filtering.

$$|\hat{\bar{S}}| = \sqrt{2\hat{\bar{S}}_{ij}\hat{\bar{S}}_{ij}} \quad \text{and} \quad \hat{\bar{S}}_{ij} = \frac{1}{2}\left(\frac{\partial\bar{u}_i}{\partial x_j} + \frac{\partial\bar{u}_j}{\partial x_i}\right)$$

So now I can use this equation 2 and 3 in 1, I can substitute to find out I have exact data L_{ij}, but the objective is to find C_s, and C_s is appearing in equation 2 and 3, and therefore I would like to substitute 2 and 3 in 1 to find the nature of the constant C_s.

So using 2, 3 in 1, I get this L_{ij} is equal to I have the first part is, of course, $T_{ij_{SGS}}$ right. So, $T_{ij_{SGS}}$ I am substituting here. So, which is minus 2 Cs square delta square plus one-third T kk delta ij, and then this is the $T_{ij_{SGS}}$, and I have $\tau_{ij_{SGS}}$ test field and then test filtering operation on that. So, $\tau_{ij_{SGS}}$ is this is minus of that. So, it is minus of $\tau_{ij_{SGS}}$, which is minus 2 Cs square delta square S ij over bar plus one third tau kk delta ij and then there is a test filtering operation on the entire part this is $\tau_{ij_{SGS}}$.

$$\mathcal{L}_{ij} = -2C_s^2 \hat{\Delta}^2 |\hat{\bar{S}}| \hat{\bar{S}}_{ij} + \tfrac{1}{3} T_{kk}\delta_{ij} - \widehat{\left\{-2C_s^2 \Delta^2 |\bar{S}| \bar{S}_{ij} + \tfrac{1}{3}\tau_{kk}\delta_{ij}\right\}}$$

Now, let me just rearrange it here. Here I would like to call this C_s^2 as simply C because that is the dynamic constant or coefficient we would like to compute ok. So, here let us call this let C be equal to Cs square. your dynamic coefficient that you want to find out, dynamic coefficient. So, I can rearrange it L ij minus 1 third T kk delta ij.

and then this is minus of minus. So, this is minus, but it becomes plus on this side. So, I get plus one third tau kk test filtered delta ij equal to the remaining two terms here, which is minus $2C$. C square is replaced by C simply, and that is the common in both. So, I have minus $2C$, I have delta cap square ok, I have missed this one here and then the remaining term minus $2C$ of this particular term and then here this is plus.

So, if I take it out it becomes minus. So, I get minus delta square S Sij over bar and a test filtering operation on this particular thing. This is the test filtering operation here. I have simply rearranged all the terms ok. So, now I can set this particular term in the square bracket as M_{ij} this thing.

$$\mathcal{L}_{ij} - \tfrac{1}{3} T_{kk}\delta_{ij} + \tfrac{1}{3}\widehat{\tau_{kk}}\delta_{ij} = -2C\Big[\underbrace{\hat{\Delta}^2 |\hat{\bar{S}}| \hat{\bar{S}}_{ij} - \Delta^2 \widehat{|\bar{S}| \bar{S}_{ij}}}_{M_{ij}}\Big]$$

So, this particular thing I would like to call it as M_{ij}. So, I get I can rewrite this as essentially L ij minus one-third delta ij of so; I have T kk minus tau kk test filtered which is equal to minus 2 C Mij. So, notice this T kk minus tau kk test filtering. Let us go to equation 1. What is Tkk SGS minus tau kk test filtering? Lkk right.

$$L_{ij} - \frac{1}{3}\delta_{ij}\left(T_{kk} - \widehat{\tau}_{kk}\right) = -2CM_{ij}$$

So, I can write this as L ij minus one-third delta ij Lkk equal to minus 2 C Mij. If I rewrite this to get C, the objective was only to get the equation C right. So, I can get C is equal to now minus of L ij minus L kk or, sorry, one-third L kk delta ij minus of this divided by 2 Mij. So, if you look closely now L ij is exact right equation 1 this particular part here. So, on any grid level I have access to resolved velocity I do one test filtering operation to get L_{ij} ok. So, this you have to do it at every time step dynamically at runtime you have to calculate this coefficient C right. So, this is essentially a Smagorinsky model where C is computed every time step, and to do that, I pause every time step to compute this dynamic Leonard stresses ok get L_{ij}.

$$L_{ij} - \frac{1}{3}\delta_{ij}L_{kk} = -2CM_{ij}$$

$$C = -\frac{(L_{ij} - 1/3L_{kk}\delta_{i}j\}}{2M_{ij}}$$

Once I get L_{ij}, then I have to get the M$_{ij}$ part here. The M$_{ij}$ is essentially, you know your test filter size, you know the grid filter size, and you have strain rates at the grid level. You use that to make one test filtering operation.

You get M$_{ij}$. Then you get your C at every time step. But is C now constant or is it a function of time or a function of three-dimensional space? It is a function of three-dimensional space and time right. So, this particular equation now here. So, here C is now a function of three-dimensional space and time, your dynamic model coefficient, runtime you have to calculate this. So, but later and what you generally see is not this equation.

There was a slight change done by Lilly. He said he introduced a slight variation of this formula. So, this is the dynamic Smagorinsky suggested by Germano et al. So, later Lilly suggested 1992 suggested C be equal to set as minus L_{ij} by 2 M$_{ij}$ was there. So, basically that one third Lkk delta by ij was dropped instead M$_{ij}$ is introduced on both sides, both the numerator and denominator, by dropping that one-third Lkk delta ij compared to if you go back and see here minus L ij is retained 2 M_{ij} is retained this one-third Lkk delta ij dropped instead it is multiplied and divided by M_{ij} ok.

$$C = -\frac{L_{ij}M_{ij}}{2M_{ij}M_{ij}}$$

Even then, even after doing this, there is an issue. Since it is a function of 3D space and time, the value fluctuates widely, causing numerical convergence issues, ok. So, this is the popular version that you will see in commercial codes or open source codes. When I say using dynamic Smagorinsky model where Smagorinsky equation is used and to compute SGS viscosity C is computed dynamically at runtime ok. So, the disadvantage here compared to the Smagorinsky model is that one disadvantage here is that is that is that the dynamic coefficient that is C fluctuates widely in space and time in x, y, z, t that means this is causing numerical stability.

because essentially the C is controlling your sub grid scale length right C_s or the $C\Delta$ and assume that if C becomes negative right for some reason then your length scale is becoming negative the sub grid scale causing stability issues. So, you must make sure that the ν_{SGS} stays positive or in the original equation if you recollect its $\nu + \nu_{SGS}$together the total viscosity is ν the kinematic viscosity plus ν_{SGS} was there. So, the total viscosity must stay positive in the equation and for ν_{SGS} to become positive the C should not take any negative value. So that happens if you do not do any other, let us say a correction here. If you just use this, it can happen because you are essentially computing at run time.

If you have 1 million mesh, you have C at all these 1 million grid points at that particular time, and then it changes every time right. So one way out is to actually use like volume averaging. okay smoothening this parameter. So solution 1 is actually average this C along, let us say if you have if you are fortunate to have a homogeneous direction, let us say in a turbulent channel flow, a plane Couette flow or in a pipe flow, we have homogeneous direction like along the flow direction, axial direction or even in the azimuthal direction in a pipe or the spanwise direction in a channel.

Homogeneous direction you can average the C. So, the C instead of becoming a 3D at a particular time step, it becomes either a planar data or a line data. So, some volume averaging is done. So, you can average C along homogeneous direction if you have them of course ok. For example, channel flow right a channel flow then upon averaging the C is now then just a function of your wall normal direction and t instead of so, the x and z are the homogeneous directions. So, this helps by averaging, but what do you do in industrial flows where you do not have such luxury of homogeneous direction.

So, you can yeah. So, you can locally average them locally around them you choose certain ah mesh around it right maybe 5 by 5, 10 by 10 or whatever locally you can average it out to smoothen. So, you do this along the homogeneous direction or average or locally average right locally average locally average C in industrial or complex problems. So, this helps this is one of the solution other solution is of course, we can come up with some kind of a limiter to make sure that the ν_{SGS} stays positive ok. So, we enforce a limiter for C so that $\nu + \nu_{SGS}$ stays positive that is stays positive, alright this is clear.

So, this completes dynamic Smagorinsky model. Exactly like Smagorinsky, except C is computed using this formulation where L_{ij} is exact M_{ij}. You can calculate using the filtering operation of your strain rates. So, now there is an another class of models. So, I am as I said in the beginning of this course I am only introducing class of models rather than discuss every type of available model. And models in eddy viscosity are can give very diverse very different results.

but in LES it is more or less since you are resolving most of the energy let us say 70, 80 percent of it. The objective of the SGS model is or the role of SGS model is limited only to account for the SGS fluctuations. So, usually regardless of the model if you have done a very good resolution if you have in LES the models work well. So, there are still many class of many types of models I am only introducing certain classes. So, we had this Smagorinsky type now there is something called scale similarity model.