Course Name: Turbulence Modelling

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Week - 12

Lecture – Lec71

71. Large Eddy Simulations: Smagorinsky model – II

So, let us see the next type of model, which is the dynamic Smagorinsky model. This is by far a very popular model used in most of the commercial solvers and open source codes. Dynamic Smagorinsky, Smagorinsky model is also very robust, provided you know the model constant what to use. Otherwise, this is a popular dynamic Smagorinsky model. The reference for it is Germano et al 1991.

So, as I said in this model in this SGS model, the Smagorinsky constant, that is, the C_s, is not user-dependent but calculated; that is, it is dynamic. So, this particular approach, where the model constant in the SGS model is computed rather than fixed by the user, is also used in other modelling approaches. For example, you can take a note. We will not go into this models, but you can the concept is similar.

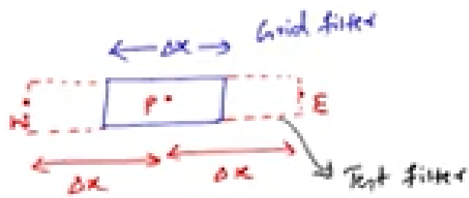
So, similar approach is also used in that is dynamically computing the model constant. Similar approach in what is called a VMS model that is variational multiscale and WALE model that is wall adapting local eddy viscosity, but we will come back and focus on the dynamic Smagorinsky model, ok? So, what we do first in a dynamic Smagorinsky, as I said, is we have to compute the model constant dynamically. So, that means we here define what is called a test filter ok. So, first, we define a test filter. So, what is this test filter and why it should be different than the grid filter is that this test filter let us call this I use a cap here.

This test filter is twice that of your grid filter ($\widehat{\Delta} = 2\Delta$). So, here, where Δ is your grid filter that is filtering operation done at the grid level if you are using implicit filtering right. So, grid filtering and this is the test filter this κ is the test filter. So that means even in finite volume method if you are using implicit filtering that means your grid size, ax size and filter size when you use a dynamic Smogorinsky model you must do an explicit filtering operation right. You actually took advantage of finite volume method where you are not doing actually a filtering operation at all the grid size was acting like a filter size right.

But if you use dynamic Smagorinsky, one filtering operation has to happen. You need to do additional calculations here. So, there is a computational cost will slightly increase compared to a Smagorinsky model ok. So, now, why do we define this is basically, we will see what advantages it would bring to compute that model constant eventually. So, schematically, I can write this as, let us say I have a grid and am writing this in 1D, say this is Δx ok.

Those who are familiar with finite volume terminologies I call it P node, east node and a west node. So, now we are going to do a. So, this what I have written delta x as let us say this is the grid filter. this is your grid filter size. So, the test filter is twice that of that.

So, this will be like, so this is $2\Delta x$ here, right? So, this particular part is your test filter. I am writing it in 1D for easy illustration. So, we define a test filter twice that of the grid filter and we apply this filtering operation to Navier Stokes ok.



So, let us start with Navier Stokes equation. So, consider the NS equation, which is dou ui by dou t plus dou by dou xj ui uj equal to minus 1 by rho dou p by dou xi plus nu dou square ui by dou xj square.

$$\frac{\partial u_i}{\partial t} + \frac{\partial u_i u_j}{\partial x_j} = -\frac{1}{\rho}\frac{\partial p}{\partial x_i} + \nu \frac{\partial^2 u_i}{\partial x_j^2}$$

Now first we filter this on the grid level that is grid filtering is applied. So, apply grid filtering. So, I get essentially if I just take the same equation here, copy it. So, I am applying the grid filtering operation. This of course, does not have the convective part that we already know that we added.

So, we can add a convective part here that is plus $\frac{\partial \overline{u_i u_j}}{\partial x_j}$ grid filtering on both sides. So, here also you can add or I will just add it here $\frac{\partial \overline{u_i u_j}}{\partial x_j}$ is added on both sides the convective part this is the convective terms added on both sides.

$$\frac{\partial \bar{u}_i}{\partial t} + \frac{\partial \overline{u_i u_j}}{\partial x_j} = -\frac{1}{\rho}\frac{\partial \bar{p}}{\partial x_i} + \nu \frac{\partial^2 \bar{u}_i}{\partial x_j^2} \longrightarrow 1$$

$$+ \frac{\partial \bar{u}_i \bar{u}_j}{\partial x_j} \qquad\qquad + \frac{\partial \bar{u}_i \bar{u}_j}{\partial x_j}$$

Grid Filtering

Now we apply test filtering. So, this, of course, up to this, you are anyway computing on your LES the test filtering operation we have to do. So, apply test filtering to let us call this equation 1 here to this part.

to equation 1 ok. So, if I do this I would get. So, I have dou ui by dou t plus dou ui uj by dou xj equal to minus 1 by rho grid filtered pressure gradient and also the grid filtered viscous term. So, now I am going to apply test filtering on all this. Again this equation does not have a convective part.

So, we add that again here adding the convective term to this equation. So, I have dou by dou xj. ui bar uj bar, we are essentially getting a filtered Navier Stokes equation at the test level ok. The previous one was at the grid level, this is at the test level. So, again I add it here dou by dou xj ui uj at the test level.

$$\frac{\partial \hat{\bar{u}}_i}{\partial t} + \frac{\partial \widehat{\overline{u_i u_j}}}{\partial x_j} = -\frac{1}{\rho}\frac{\partial \hat{\bar{p}}}{\partial x_i} + \nu \frac{\partial^2 \hat{\bar{u}}_i}{\partial x_j^2}$$

$$+ \frac{\partial \hat{\bar{u}}_i \hat{\bar{u}}_j}{\partial x_j} \qquad\qquad + \frac{\partial \hat{\bar{u}}_i \hat{\bar{u}}_j}{\partial x_j}$$

Test Filtering

Again same arguments, convective terms added on both sides. So, now what do we do? We simply rewrite this equation here. So, I get the left-hand side part dou ui bar by dou t plus I take the convective term here ui bar uj bar by dou xj equal to the pressure term dou p bar by dou xi, the viscous term nu dou square ui bar by dou xj square, and, of course, the sub-grid stresses at the test level ok. So, that will be minus of I am moving the one term to the right-hand side. So, I get minus dou by dou xj of I have ui uj average test filtered minus ui bar uj bar test filtered ok.

$$\frac{\partial \hat{\bar{u}}_i}{\partial t} + \frac{\partial \hat{\bar{u}}_i \hat{\bar{u}}_j}{\partial x_j} = -\frac{1}{\rho}\frac{\partial \hat{\bar{p}}}{\partial x_i} + \nu \frac{\partial^2 \hat{\bar{u}}_i}{\partial x_j^2} - \frac{\partial}{\partial x_j}\left(\widehat{\overline{u_i u_j}} - \hat{\bar{u}}_i \hat{\bar{u}}_j\right)$$

So, this is the stress at the test filter level. So, I will call this not τ_{ij}, but $T_{ij_{SGS}}$ at the test filter level this particular term, but what is this particular $\overline{u_i u_j}$ if I go and look into equation 1 where we added the two convective terms here? Here in this equation, if I rearrange here, the right-hand side part $\tau_{ij_{SGS}}$, we already know $\tau_{ij_{SGS}}$. By moving one term to the right-hand side, I would get ui uj over bar minus ui bar uj bar. This is your SGS stresses, right? So, using this let us call this star. Now, using equation star in this final equation here I can see that ui uj bar is nothing but right.

$$\tau_{ij_{SGS}} = \overline{u_i u_j} - \bar{u}_i \bar{u}_j$$

So, here, or I can say $\overline{u_i u_j} = \bar{u}_i \bar{u}_j + \tau_{ij_{SGS}}$. So, I can substitute this particular term in the $T_{ij_{SGS}}$ test filtered level ok. So, I get essentially. So, the left-hand side terms remain the same, equal to the right-hand side part. The pressure minus 1 by rho dou p by dou xi test filtered the viscous term, and then we are substituting for that $\tau_{ij_{SGS}}$ here.

So, I get minus dou by dou xj of so uiuj bar is replaced by $\tau_{ij_{SGS}}$ plus ui bar uj bar. But this entire term uiuj, if you go back to the previous slide, you can see here, here you can see this uiuj bar had a test filtering operation. So, we are replacing this ui uj over bar with tau ij SGS plus ui bar uj bar and a test filtering over this here. So, I get a test filtering operation over this entire part here and then minus, I have the other two terms, which is ui bar test filtering and uj bar test filtering here. So, I get ui bar test filtering, sorry, and then uj bar test filtering ok. So, this is your entire $T_{ij_{SGS}}$ this whole thing.

$$LHS = -\frac{1}{\rho}\frac{\partial \hat{\bar{p}}}{\partial x_i} + \nu \frac{\partial^2 \hat{\bar{u}}_i}{\partial x_j^2} - \frac{\partial}{\partial x_j}\Big(\underbrace{\widehat{\tau_{ij_{SGS}} + \bar{u}_i \bar{u}_j} - \hat{\bar{u}}_i \hat{\bar{u}}_j}_{T_{ij\,SGS}}\Big)$$

So, if I go back to the previous slide. So, see this that this $\tau_{ij_{SGS}}$ what I have written here this is applicable at the grid level right $\tau_{ij_{SGS}}$ is at the grid filter level. So, now, what do I have here? This is applicable at the test filter level Tij SGS at the test filter level. So, this at test filter level that means, so if I go back here and then see $\tau_{ij_{SGS}}$ here for all length scales less than the Δ only this $\tau_{ij_{SGS}}$ is applicable right up to Δ you are resolving. All the turbulent scales up to Δ it is being resolved.

Scales turbulent fluctuations or scale smaller than delta is not being resolved and that is modeled. So, this $\tau_{ij_{SGS}}$ accounts for all the sub grid scale stresses for L less than Δ correct. Grid filter level that is where L is less than Δ. for all the SGS length scales smaller than the filter size. So, if this is L less than Δ at the test filter level this has to be L less than 2Δ at the test filter level.

So, that means here this is applicable for L less than the test filter. So, then obviously, that means for all L for all scales in between the grid filter level and the test filter level and intermediate level that L will be in between this and the sub-grid scale fluctuations or sub-grid scale stresses in between the test filter level and the grid filter level is called dynamic Leohard stresses and intermediate stresses ok. So, what we do now is we define what is called a dynamic Leonard stresses which is nothing but let us call it L_{ij}, which is equal to this $T_{ij_{SGS}}$. So, this $T_{ij_{SGS}}$ is equal to these three terms. So, if I subtract $\tau_{ij_{SGS}}$ filter part from this.

So, before I introduce this, I can simplify this saying that see the $\tau_{ij_{SGS}}$ what we have got is nothing but $\tau_{ij_{SGS}}$ plus ui filter uj filter, and then there is a test filtering operation. Now, test filtering or any filtering and addition commute. So, I can do test filtering of this plus test filtering of this, and then I have minus ui grid filtered, uj grid filtered and then, of course, a test filtering operation. So, this is again at the test filter level. this particular term is at the grid filter level.

$$\underbrace{T_{ij_{SGS}}}_{\text{(test filter level)}} = \underbrace{\widehat{\tau_{ij_{SGS}}}}_{\text{(grid filter level)}} + \widehat{\bar{u}_i \bar{u}_j} - \hat{\bar{u}}_i \hat{\bar{u}}_j$$

So, if I subtract one from the other I will get the intermediate part that is what is called a dynamic Leonard stresses. So, Lij is nothing, but Tij SGS minus tau ij SGS test filtered. Ok, which is equal to ui grid filtered uj grid filtered test filtering minus ui bar uj bar test filter. So, this of course, here this is at the intermediate level at intermediate level this. So, that means, it is for all delta less than L less than test filter size.

$$\mathcal{L}_{ij} = T_{ij_{SGS}} - \widehat{\tau_{ij_{SGS}}} = \widehat{\bar{u}_i \bar{u}_j} - \hat{\bar{u}}_i \hat{\bar{u}}_j$$

Now, you may wonder why are we doing all this test filtering Navier-Stokes ones, grid filtering, are we heading somewhere? Can you see this L_{ij} equation? Do you see something useful here in the dynamic Leonard stress? Look at this term. Do you, is it an unknown or is it known for you? It is known, right? See, eventually, in the LES, you are computing the velocities at the grid filter level, right? So, at and then you are making one test filtering operation let us say at every time step ok. You are stopping the simulation doing test filtering and then time matching again. So, ui bar is available to you you just need to do one test filtering operation at a filter size twice that of your grid filter size. So, what I have here is the Lij is available to you, and this is exact.

You do not need to model this right. You have an exact term, and you can compute this easily, right? So, this dynamic Leonard stress or an intermediate SGS stress is used to actually get an equation for the sub-grid scale constant. The ultimate aim is to find that constant $C_s$instead of giving a value right. For that, we first this is a one step is to get this dynamic Leonard stresses ok. So, we will see later how to actually get this constant C dynamically.