7. Navier Stokes: the governing equations - 2

So, now, if I expand like previously, I can see how many equations I get. So, let us say for i equal to 1, I would like to get equation for u1, equation for u1 velocity. So, I get $\frac{\partial u_1}{\partial t}$ plus, now j is repeated. So, we need to sum over j. So, I get, but i equal to should be 1, i should not be changed, i is the free index here.

I can take value 1, 2 or 3. So, we are looking into i equal to 1 means u1 velocity component, looking into the momentum along the x direction, x1 direction. And therefore, j is the summation part. So, you have to write $u_1\frac{\partial u_1}{\partial x_1} + u_2\frac{\partial u_1}{\partial x_2} + u_3\frac{\partial u_1}{\partial x_3}$, j is summed over the summation rule.

j is summed over on this side, I get $-\frac{1}{\rho}$, the pressure gradient along the x1 direction plus 1 by rho; here, j is repeated again, $\frac{\partial \tau_{ij}}{\partial x_j}$, i is 1. So, $\partial\tau_{1j}$, j is repeated. So, $\frac{\partial \tau_{11}}{\partial x_1} + \frac{\partial \tau_{12}}{\partial x_2} + \frac{\partial \tau_{13}}{\partial x_3}$. So, I have one equation here for velocity u1 and I need of course, the velocity is 2 and 3 also. but for that I can write an expression for i equal to 2, i equal to 3, I will get equation for u2 and u3 velocity.

So, the left-hand side is completely taken care of and but I have an extra unknowns on the right-hand side. So, similarly, if I write an equation for I equal to 2 the equation for u2, I will not write the whole part. so you can write. So, I am going to just look at this particular term here. So, this is $\frac{1}{\rho}\frac{\partial p}{\partial x_2}$. Here it is $\frac{\partial u_2}{\partial x_2}$ and then the other term we have to write down. So, here this term is important. So, let us look at dou now i equal to 2, $\frac{\partial \tau_{21}}{\partial x_1} + \frac{\partial \tau_{22}}{\partial x_2} + \frac{\partial \tau_{23}}{\partial x_3}$. Similarly, for i equal to 3, the equation for the third velocity component. So, it is $\frac{\partial u_3}{\partial t}$ plus you can write the convective term 1 by rho dou p by dou x3 plus 1 by rho dou 31, i equal to 3.

So, tau ij, $\frac{\partial \tau_{31}}{\partial x_1} + \frac{\partial \tau_{32}}{\partial x_2} + \frac{\partial \tau_{33}}{\partial x_3}$. So, we have three equations. So, if I look at number of equations here, I will write it here, number of equations and number of unknowns. So, I have three equations, equation for u1, u2 and u3 from conservation of momentum, correct? So, this will also help me solve the continuity equation which requires u1, u2, u3. This is taken care of.

But what are the unknowns in the three momentum equations here? So, we had unknowns u1, u2, u3. that I need to solve which is fine. In addition to that, I have pressure term and then I have the last terms here. It is totally 9 terms, it is a tensor, but it is a symmetric tensor whether it is tau ij or tau ji, you will get the same. That is the easiest way to check whether the tensor is symmetric or not.

So, when you expand it, you will get the same. So, we expand it as τ_{ij}. Now, you can do τ_{ji}

and give values, you will get the same tensor. that means it is symmetric tensor. So, we can look into only the diagonal components here $\tau_{11,}\ \tau_{22}, \tau_{33,}$and the off diagonal three components that is symmetric.

So, τ_{12} and τ_{21} are the same. So, we can look at this. So, I get six unknowns here, extra, which is $\tau_{11,}\ \tau_{22}, \tau_{33,}$, and three normal stress terms. And then, 3 shear stress terms, which is $\tau_{12,}\ \tau_{13}, \tau_{23,}$. So, I get 4 plus this 10 unknowns.

So, now, for pressure, those who have studied CFD would know that you can use continuity equation to solve for pressure. So, that is kind of taken care of. So, if I say continuity equation will help me this particular term using continuity equation. So, this can be taken care of. Either if you know the method like let us say simple or something like that you can do that or one can also derive an equation for the pressure term We would need that as we go into deeper into the turbulence modeling we will derive that equation later, right now it is not required.

So, one can also derive an equation for pressure which is called Poisson equation using continuity and the momentum equation one can derive that and solve that, that is also possible that is used extensively in CFD techniques that route. So, in any case we have 4 equations in total, right. So, I can say 3 plus 1. The other one is this continuity. So, 4 equations are there, 4 unknowns are taken care of, but the additional unknowns is this part.

This is the extra unknowns that is causing the problem, 6 extra unknowns. So, when number of equations are lesser than the number of unknowns. number of equations is less than the number of unknowns. What do we call this closure problem? Mathematicians call this a closure problem because you cannot solve the simultaneous equations when number of unknowns are more than the equation. If you have 3 unknowns, you need 3 equations to solve it as simultaneous system of equations.

So, this is the closure problem. So, what do we do? How do we proceed? We have extra unknowns. We have not even started turbulence modelling. We have not even looked at this is just Navier-Stokes equations. How do we proceed now? This you should have studied in the fluid mechanics course.

Exactly. So, we are using a model, right? This model, is it universal? It can be used for any type of fluid? Exactly. So, you have already been introduced to a model. So, a model is required to close the equations. So, what is that model is? Newton's law of viscosity for a Newtonian fluid. So, for a Newtonian fluid, using Newton's law of viscosity.

So, what does this model tell us? This model tells us that we can relate stress to strain; stress is proportional to the strain rate. This is the model here. So, if I apply that Newton's law of viscosity. So, I get Newton's law of viscosity, which is $\tau_{ij} = 2\mu S_{ij} - \frac{2}{3}\mu \frac{\partial u_k}{\partial x_k}$, or I can simply say Skk, so, where Sij is the strain rate tensor. Sij is your strain rate tensor, which is equal to half of your dou ui by dou xj plus uj by dou xi strain rate tensor, and skk implies i is equal to j.

So, Skk will give me $\frac{\partial u_k}{\partial x_k}$ here that component, or we can simply say it is can simply use $\frac{\partial u_k}{\partial x_k}$ also and by continuity it would go away. So, whenever we use incompressible flows that is constant density, this term will vanish because it is $\frac{\partial u_k}{\partial x_k} = 0$ by continuity equation, we would have only this part. So, we have related the stress to the strain here, and this is applicable only for Newtonian fluids, of course. So, we already have a model. So, now, this was 6 unknowns, right? This is 6 unknowns is now modeled as, here, μ is, of course, an unknown, your molecular viscosity which you can know for a particular Newtonian fluid.

So, that is taken care of and then what do you need here other than that you would need the strain rate or the velocities basically you need the velocity. So, which is known to you. So, this is known. So, 6 unknowns have been made or related to 3 known factors, and therefore, the equations are closed.

equations are closed. So, you already used a model before starting what is called turbulence modeling. So, this kind of approach is also used as we go further. So, we will start with one particular aspect of it, or the beginning of what I say turbulence modelling, is to look at the equations for mean fluid motion and equations for fluctuating fluid motion. So, this particular topic would be called Reynolds averaged, Reynolds averaged Navier Stokes or called RANS equations. So, in the Reynolds average Navier-Stokes equations, first of course, the chapter 1 part, the statistical analysis, we decompose the flow into mean and fluctuations.

So, first part is Reynolds decomposition, Reynolds decomposition, which basically when I talk about Reynolds, this is Osborne Reynolds, O Reynolds. So, we can split let us say velocity component u into its mean and fluctuation. So, this is the instantaneous velocity into the mean and the fluctuation. If I do this decomposition to the equation, I am going to get an equation for the mean fluid motion and also an equation for fluctuating fluid motion, ok? So, of course, this can also be, everything can be split. So, one can also split pressure also, instantaneous pressure as a mean pressure and the corresponding fluctuating pressure.

So, now we first look into equations for mean fluid motion. This is more exciting for engineers because they are interested in mean flows, average velocities, average pressure, temperature and so on. But this is also useful for scientists because we can separate the mean out and look into what is remaining, what is the remaining fluctuations. So, we have equations 1 and 2, which are continuity and momentum equations. So, we split this equation into two parts here.

We will do the continuity equation first. Continuity equation we take. which is $\frac{\partial u_{\mathrm{j}}}{\partial x_{\mathrm{j}}}=0$. So, I am going to split this into two, that is, Reynolds decompose, Reynolds decompose and ensemble average. This gives me $\overline{\frac{\partial u_j+u_j'}{\partial x_j}}=0$.

So, the operations I can split this into two part here, I also have to average it over. So, I have $\frac{\partial \overline{u_j}}{\partial x_j}+\frac{\partial \overline{u_j'}}{\partial x_j}=0$. I also talked about not just Reynolds decomposition, but also ensemble averaging. So, let me ensemble average. So, ensemble averaging and addition commute.

So, it can be written into two different separate terms. So, this gives me $\frac{\partial u_j}{\partial x_j}+\frac{\partial \overline{u_j'}}{\partial x_j}$. Even the differentiation and the ensemble averaging commute here. So, that is why the averaging operation is over the velocity component. So, now what would happen? We already learned from chapter 1.

The second term would go to 0, right? This is the ensemble averaging of a fluctuation. That is why I thought the difference between arithmetic mean and the ensemble mean. Arithmetic mean, it will not go to 0. So, we will be struck with this term. With ensemble mean, believing that it exists, this is 0, average of a fluctuation, ensemble mean of a fluctuation is 0 that we have already seen the proof and the ensemble averaging a true mean does not change its characteristics.

So, we get our continuity equation for mean motion, mean fluid motion as $\frac{\partial \overline{u_j}}{\partial x_j}=0$. This looks fantastic, both the equation 1 which is the continuity equation itself for instantaneous fluid motion and this particular equation you can call it let us say equation 3. the continuity equation for the mean fluid motion. They look similar.

So, so far it is really amazing. And one can also look at the continuity equation for a fluctuating fluid motion now. So, how to do that? We can also look at continuity equation for a fluctuating fluid motion. continuity equation for fluctuating fluid motion. This is straightforward. I have, this was our equation 1 if you remember correctly.

I take equation 1, equation 3. I can subtract equation 3 from 1. So, I get equation 1 minus equation 3 would give me $\frac{\partial \overline{u_j}}{\partial x_j}+\frac{\partial u_j'}{\partial x_j}$. This is the Reynolds decomposed equation 1 minus $\frac{\partial \overline{u_j}}{\partial x_j}$. So, this is equation 1 here, this part is your equation 1, Reynolds decomposed, of course.

So, this yields $\frac{\partial u_j'}{\partial x_j}=0$. Again, very interesting. So, we have got a continuity equation for the fluctuating motion also looking similar to equation 1 and 2. So, we will see in the next class whether something similar happens to momentum or some surprise is going to be there. Thank you.