Course Name: Turbulence Modelling

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Week - 11

Lecture – Lec66

66. Large Eddy Simulations: Filtered Navier-Stokes Equations - I

Okay, so let us get started. So, we were looking into larger dissimulations. So, we saw the difference between filtering and resolving and we understood what is ah you know the conceptually we understood what is sub grid scale when we talk about it and what is what we mean by residues. So, we look at the governing equations today. So, essentially the governing equations are we have to start from first principles Navier Stokes equations and filter it.

So, what we have is filtered Navier Stokes equations filtering is nothing, but your volume averaging or low pass filtering ok. So, we start with Navier Stokes incompressible form. So, that is the Navier-Stokes equations. We now perform the filtering operation on that ok.

So, the over bar represents filtering . So, now unlike Reynolds averagedequations Reynolds averaged Navier-Stokes where the ensemble averaging and differentiation they commute. you take the derivative of a mean quantity, ensemble mean quantity or you take mean of the derivative of that quantity both are same. So, we could move the over bar operation in RANS inside the differentiation operator. It was $\frac{\partial \bar{u}_i}{\partial t}$ ok, but that is not the case here.

Here over bar as I said over bar represents over bar represents filtering volume averaging ok. So, that means, we have two types of derivative here a spatial derivative and a temporal derivative. So, what about the first one temporal derivative and spatial filtering? So, volume averaging is nothing, but spatial filtering here right. So, the $\frac{\partial}{\partial t}$ term and the filtering they actually commute that is because the filter volume or your control volume is not a function of time right. So, note that filtering and derivative with respect to time.

it is the $\frac{\partial}{\partial t}$ commute . So, we can ah move this filtering operation inside the operator here that we can see the illustration is straight forward. So, if I have $\frac{\partial ui}{\partial t}$ filtering I can write this as the actual filtering operation which is volume averaging 1 by v, v is the volume integral v $\frac{\partial ui}{\partial t}$ dv ok, the volume averaging part here . So, this is nothing but now I can take out the time derivative outside $\frac{\partial}{\partial t}$ $\frac{1}{v}\int_{v} u_i dv$ that is because volume is not a function of time since volume is not a function of time. So, this is nothing but $\frac{\partial \bar{u}_i}{\partial t}$ filtered velocity ok.

So, this is good so far the temporal derivative part was fine, but the spatial derivative will create a problem. So, the question is what about spatial derivatives? the answer is not ah straight forward because there is an exception for it. So, in general they do not commute the filtering operation over this $\frac{\partial}{\partial xi}$ $\frac{\partial}{\partial xj}$ terms they do not commute. So, I cannot move the filtering operation over the primitive variables. So, in general filtering and the $\frac{\partial}{\partial xi}$ a special derivative do not commute.

So, in general. So, there is an exception which is a rare. So, the exception is only when you say I am using a homogeneous filter. Homogeneous filter implies a constant grid size. You have made a mesh which is equidistant everywhere only in that case the it is you can move this filtering operation inside ok.

So, exception is homogeneous filters where the control volume size is constant you are essentially performing a filter filtering operation over the mesh ok. Sometimes the filter size can be same as mesh size sometimes it may not be that is your choice, but the filter filter size cannot be smaller than the mesh size right your volume averaging the data. So, it can be same as the mesh size or larger than that right 2 times 4 times like that. So, if CV size your control mesh size is constant then this can be an exception right that is a bit rare in engineering problems we cannot afford to have equidistant mesh or having a control volume size constant everywhere right that is uniform grid spacing that is an exception. in all the engineering problems it is common to have a stretch mesh ok.

So, so in general we cannot do this. So, what is the way out? We do not even have a governing equation ready for us without even solving it right. Let us take any spatial derivative term Φ can be pressure or velocity or anything. So, $\frac{\partial \phi}{\partial x_i}$ filtering this is nothing, but $\frac{1}{v}\int_{v} \frac{\partial \phi}{\partial x_i} dv$ ok. So, now as I already said this volume However, this volume in general is a function of the spatial coordinate except that CV constant right exception in that case.

So, what do we do? There is actually a numerical way out is like a stalemate here we cannot even have a governing equation ready. One way of achieving ah the governing equation here is that have a numerical ah solution ah and this particular solution is actually provided by Gossel and Moin 1995 ok. They showed that the spatial derivative term, ∂/∂xi, can be moved out of the integral with a numerical error that is of the order of magnitude of delta squared, where delta is the mesh size. So, if you can afford to live with that, you can actually move the $\frac{\partial}{\partial xi}$ outside the interval. So, what we achieve is that if I follow this, I essentially get the average of $\partial\varphi/\partial$xi; sorry, the filtering of $\partial\varphi/\partial$xi is equal to what it is now.

I am now moving the spatial derivative outside. So, I calculate the integral of $\frac{1}{v}\int_{v} v \mathbb{d} v$, plus the error, okay? So, that is equal to $\frac{\partial\bar{\phi}}{\partial x_i}$ plus the error that is being introduced; this error is omitted in LES, okay? This numerical error is neglected in LES because we essentially want the filtered derivative of Φ with respect to xi to be equal to the filtered derivative of Φ bar with respect to xi. You want to substitute this left-hand side with this particular term, $\partial\bar{\varphi}/\partial$x; this is omitted, but this delta is now a function of your mesh. In a DNS, the mesh size will be as tiny as, let us say, the Kolmogorov microscale. The error is tiny, but it is not in LES.

The whole idea of LES is actually to model those small scales, considering that they are statistically isotropic. So that we can have a larger mesh, it will be economical. So, it is a technique that lies somewhere between RANS and DNS. RANS and DNS are two extremes: DNS, where everything is captured, and RANS, where every turbulent scale is modeled. LES is somewhere in between, as it partially captures some turbulent scales; the large scales are modeled, as are the tiny scales, right? In this approach, the mesh size is not very small.

So, this error, which is a function of delta squared, is not very small, okay? So, we can say that this error is not very small since delta, or the mesh size, is coarser in LES compared to the DNS mesh. So, the error is not that small here; however, that is the path we have to follow moving forward. So, now that this numerical error has been omitted, I can prepare the governing equations. So, I can use both of these; let us call this Equation 2. Using equations 1 and 2, we obtain the complete filtered governing equation.

So, I have $\frac{\partial \bar{u}}{\partial t} + \frac{\partial \overline{uu}}{\partial t} =$ filter velocity. So, it is a filtered product of velocities equal to $-1/\rho$ $(\partial\bar{p}/\partial x_i)$, which is the filtered pressure gradient of filtered pressure, or resolved pressure, plus ν $(\partial^2\bar{u}_i/\partial x_{\square}^2)$. This sentence is already grammatically correct. However, if you want to improve clarity, you might consider simplifying or breaking it into two sentences. Here's a suggestion: Thus, it is a filtered product of velocities equal to $-1/\rho$ $(\partial\bar{p}/\partial x_i)$, which represents the filtered pressure gradient of the filtered pressure, or resolved pressure.

Additionally, it includes ν $(\partial^2\bar{u}_i/\partial x_{\square}^2)$. So, you have your filtered Navier-Stokes equations. The original sentence is correct as it stands. No changes are needed. Something you have to do if you want to calculate using CFD is check if there is anything missing in this equation.

So, what does general transportation look like, and do you have all these terms in this equation? Let us refer to this equation as 3. Sorry. And this is a Navier-Stokes equation that I am not referring to. General transport equation. Do you remember the general transport equations? What does it consist of? 4 terms.

Unsteady term. Unsteady terms that are present. Convection term

Is the convection term present here? Yes or no Is that a convection term? There is a filtering; it is a single term. There is a filtering of the velocity products, and there is no convective term here. It is not $\overline{u_i u_j}$ correct? I cannot use the product rule here to bring the convective term being evaluated to prove that you must go to a non-conservative form and clarify what is being correctly evaluated.

So, an advection velocity or convection velocity is derived from the derivative. I cannot use the product rule here to do that correctly. So, there is a problem with this equation, right? So, this equation does not have a convective term, or an advective term if you prefer to call it that. We want to express it in a general transport equation where convective fluxes are present, diffusive fluxes are also included, a source term is included, and then there are source terms; however, it does not contain any of those. So, what I do is add the convective term here.

Okay, add the convective term. So, add $\frac{\partial \overline{u_i u_j}}{\partial x_j}$von both sides because I am adding the convective term. If I do this, I get $\frac{\partial u_i}{\partial t} + \frac{\partial \overline{u_i u_j}}{\partial x_j}$ plus this particular term equal to $-1/\rho$

$(\partial\bar{p}/\partial x_i) + \nu\frac{\partial^2 u_i}{\partial x_j^2}$, and I am adding the convective term. So, rewriting this in the standard form of a general transport equation, I get $\partial(\bar{u}_i\ \bar{u}_\square)/\partial x_\square$. I am moving this particular term, the second term, to the right-hand side. So, I get - 1/ρ times the partial derivative of p̄ with respect to x_i + ν times the second partial derivative of u_i bar with respect to x_j squared.

Now, I have this particular term minus the two red terms here, right? So, the convective term on the right-hand side minus this is not the two red terms, sorry. So, this particular term, is being moved to the other side. So, that means I can write this as $\frac{\partial \bar{u}_i}{\partial x_j}$ - $\frac{\partial \overline{u_i u_j}}{\partial x_j}$, or I can express it in a way where these terms remain the same. You can write this as - $\frac{\partial \overline{u_i u_j}}{\partial x_j}$ - $\bar{u}_i\ \bar{u}_\square$. Now, this particular term inside the brackets is called τ_{ij} $\tau_{ij\ sgs}$.

This is unknown here. In LES, you are computing filtered velocity and filtered pressure, not the filtered products of the velocities, okay? So, this is an unknown term, and therefore, this unknown term - $\overline{u_i u_j}$ is an unknown as well. So, note that in LES, you compute $\overline{u_i}$ and not $\overline{u_i u_j}$ okay? So, therefore, $\overline{u_i u_j}$ is an unknown. So, I have six unknowns here. So, this is again an unknown turbulent closure problem involving the $\tau_{ij\ sgs}$, right? So, this $\tau_{ij\ sgs}$ is popularly called subgrid-scale stresses. Sometimes referred to as residual stresses, the term "subgrid scale" is preferable because we understand that the residue is not exact in LES, or in terms of residual stresses.

So, this is an unknown that requires modeling. It is popularly called SGS modeling or subgrid-scale modeling. So, you are going to model only those quantities? So, again, this is not a stress, even though we are calling it stress. Just like Reynolds stresses, this particular term has units. As you can see, we are writing ∂/∂x_j (ν ∂ū_i/∂x_j), which represents the viscous stress.

So, it is written in that manner. So, the units come out to be stress. So, it is not really a stress, okay? So, not really a stressful term. You can call this sub-grid scale motion. This particular term accounts for all the residual fluid motion or sub-grid scale motion when you filter it out. So, it accounts for subgrid turbulence, which refers to turbulence that is smaller than your mesh size. That has to be included in $\tau_{ij\ sgs}$.