6. Navier Stokes: the governing equations - 1

So, let us get started. Welcome you all to the class. So, yesterday, we looked at some review of Cartesian tensors right. So, I think many of you are familiar some who are not let us hope that it was clear. So, we looked at only the basic part of it.

If you are interested in knowing more there are books on Cartesian tensor calculus. We do not need all those things. So, the basics what is required that you should know the summation rule and you should be able to identify the free index and if you want to change it through change the free index throughout, not for only one or two terms that is the basics. So, we look at we practice today one or two terms before we go into the governing equations.

So, we discussed one thing called, so, we are still at the theory part where we looked at Cartesian, Cartesian tensors. So, we looked at what is called vorticity. So, the omega in your vector form, which I told you, is not so useful when we go into the world of turbulence, right? So, this is written in tensor notation as omega i equal to your $\epsilon_{ijk}\frac{\partial u_k}{\partial x_j}$, ok. So, now let us see what is the vorticity component in the i equal to 1 direction. So, let i be equal to 1.

implying you are looking into the coordinate direction 1. Let us say x, y, z then x is 1 let us say or x_1. So, then omega 1 will be, i is the free index here j and k are repeated. So, it has to be summed over it 2 times. So, let us write first $\epsilon_{1jk}\frac{\partial u_k}{\partial x_j}$, ok? Now, we apply the summation rule once.

Let us sum it over j first. So, I get $\epsilon_{ijk}\frac{\partial u_k}{\partial x_j} + \epsilon_{12k}$, we are summing over j. So, it is $\frac{\partial u_k}{\partial x_2} + \epsilon_{13k}\frac{\partial u_k}{\partial x_3}$. Now it is readily seeable we do not have to sum over the k for the first term because we can readily see from the Levi-Civita epsilon, epsilon has to take cyclic or anti-cyclic values it should be 1 2 3, 2 3 1 so on or 3 1 2 and so on it is 1 1. So, it does not matter what k value takes here since 1 is repeated it will be 0 here correct.

So, I do not need to you can expand it and make it 0, but it is smart to just say this is anyway going to 0 here. So, this term I do not have to expand. I will only expand the other two terms to see. So, this gives me now, now I expand or sum over. So, here I have summed it over, summation over j.

Now, I will do the summation over k, Einstein summation. So, I do now over k for both the terms. So, I get $\epsilon_{121}\frac{\partial u_1}{\partial x_2}$. I will write it below for this plus $\epsilon_{122}\frac{\partial u_2}{\partial x_2} + \epsilon_{123}\frac{\partial u_3}{\partial x_2}$. Again, you see here this is 0, 1 to 1 is 0, 1 to 2 is also 0.

Only one term remains. Now I expand the last term here. We are only expanding these two terms. So, now I expand this third term which $\epsilon_{131}\frac{\partial u_1}{\partial x_3} + \epsilon_{132}\frac{\partial u_2}{\partial x_3} + \epsilon_{133}\frac{\partial u_3}{\partial x_3}$. So, again this is 0.

this is 0 leaving out only two terms. So, the final notation is basically your ϵ or ω_1 is essentially equal to; ω_1 is equal to your $\epsilon_{123}\frac{\partial u_3}{\partial x_2} + \epsilon_{132}\frac{\partial u_2}{\partial x_3}$. Now, what is the sign for this? 1, 2, 3 is positive, it is positive one, it is in the cyclic, 1, 3, 2 in the counter cross. So, it is a negative sign. So, therefore, we have.

Therefore, ω_1 is equal to plus $\frac{\partial u_3}{\partial x_2}$ minus; I will just say plus, sorry. So, it is positive here, minus of dou u2 by dou x3. So, we do not need to remember the sign here unless you want to work with a vector form. The Cartesian tensor rule itself will tell you it is much easier; you do not have to remember the sign, whether it should be assigned negative or positive to the gradient. Similarly, you can expand to ω_2, ω_3 and so on.

This is just a practice. So, you need to get used to this and as I said we are mostly going to use this Einstein summation rule repeated indices you have to sum it over that particular index ok. And sometimes we use this free index rule to change an index throughout that we do it fine. So, with this we can go ahead and we are ready to start with our governing equations. And now, when I talk about governing equations, that means the governing equations for turbulent flows.

Obviously, the Navier Stokes, as I said, knows about turbulence; it knows about laminar, transitional turbulent flows; it knows about everything. But the equation is not revealing to us what is the origin of turbulence, which term knows about turbulence, that we do not know. Everything is concealed inside Navier-Stokes. So, governing equations, you can take it as chapter 2 actually, this particular part. Let us call this chapter 2.

Governing equations for what? Of fluid motion right, and we are particularly interested in turbulent fluid motion, not the laminar. So, note here is that the Navier Stokes equations, Navier-Stokes system of equations constitutes three. Many times I see the students think only the momentum equation is the Navier-Stokes equation. It is a system of three conservation of mass, momentum, and energy. So, the Navier-Stokes system of equations knows about turbulence, but it is concealed.

We do not know the origin of turbulence, right? So, the question is the origin of turbulence; which term knows about turbulence? For that, we use Chapter 1 basics that we studied, that is the we looked at the statistical analysis. We know now if a quantity is given, we know how to split into its statistical component and the random component. The random is a turbulent. So, we know the mean and the fluctuation. So, we can now write equations for the mean fluid motion as well as the fluctuating fluid motion.

That is what we are going to do now. So, Navier-Stokes system of equations now, so first we look into conservation of mass here, conservation of mass. I will proceed only in terms of Cartesian tensor notations, leave out the vector form. So, we write this as $\frac{\partial\rho}{\partial t} + \frac{\partial}{\partial x}$, let us say i or j, let us say j here, rho uj equal to 0. It could be i also, i or j or k does not matter.

So, now the first term is, of course, similar to you in vector form. The second one is what you would write it as in a divergence form. That I have not written. This is much more nice because the entire Cartesian tensor comes in the tensor calculus. And therefore, it is readily appliable.

The rules that you know in calculus, it is readily appliable here. So, now I can easily apply the chain rule for products or product rule here for the second equation, so second term here. So, I can apply, I can say applying product rule, or you may call it chain rule, whatever you are familiar with. If I do this, I get $\frac{\partial\rho}{\partial t} + u_j\frac{\partial\rho}{\partial x_j} + \rho\frac{\partial u_j}{\partial x_j}$ = 0. Now, here, to proceed further in this entire course, I am going to take constant density, right? So, this entire turbulence modeling course I am going to go with constant density flows, ok.

So, now consider ρ equal to constant or you can say density which is ρ equal to constant. If I do that, I get the first term will go away ρ is constant right. So, obviously this term goes away for ρ equal to constant. The second term also goes away for ρ equal to constant. Only I get the last term which is your $\frac{\partial u_j}{\partial x_j} = 0$ which is what you call a continuity equation in compressible flow.

Continuity equation in incompressible flow. So, whenever I mention constant density it obviously implies incompressible flow. So, I do not have to explicitly say that again constant density will imply incompressible flow. So, this is our equation for continuity that you already know, even now we have not looked into turbulence and mean motion, it is still Navier-Stokes system of equations. So, now we go to the momentum equation, conservation of momentum.

So, before that we can expand this particular term $\frac{\partial u_j}{\partial x_j}$. So, I have $\frac{\partial u_j}{\partial x_j} = 0$. So, what should I do now when I see two repeated indices, Einstein summation, right? Einstein's summation over j is repeated twice. So, what do I get? It is $\frac{\partial u_1}{\partial x_1} + \frac{\partial u_2}{\partial x_2} + \frac{\partial u_3}{\partial x_3}$ = 0. So, I have only one equation here, but I have three unknowns.

So, equation unknowns, I have one equation for continuity, I have three unknowns here, u1, u2, u3 are three unknowns, u1, u2 and u3. for that I can use conservation of momentum which will give me three equations for the velocities right. So, I do not have to worry here; otherwise, it is a problem. I have one equation with three unknowns, which means you cannot solve the equation. Luckily we have conservation of momentum for three more equations.

So, we take conservation of momentum. So, we have $\frac{\partial \rho u_i}{\partial t} + \frac{\partial \rho u_i u_j}{\partial x_j} = -\frac{1}{\rho}\frac{\partial p}{\partial x_i} + \frac{\partial \tau_{ij}}{\partial x_j}$. We already discussed the gradient term, how it has to be written in Cartesian tensor notation, how the divergence term is written, how the Laplacian, all these terms we discussed in the previous slide. So, I have written the conservation of momentum in Cartesian tensor notation. And I already said ρ is constant density which implies incompressible flows. So, I have $\frac{\partial u_i}{\partial t}$plus, now I can use product rule again here.

I use the product tool again. I will get $u_i \frac{\partial u_j}{\partial x_j} + u_j \frac{\partial u_i}{\partial x_j}$ equal to I am taking the row to the right-hand side, I get $-\frac{1}{\rho}\frac{\partial p}{\partial x_i} + \frac{1}{\rho}\frac{\partial \tau_{ij}}{\partial x_j}$. I am taking the non-conservative form of the equation. Those who are familiar with fluid mechanics would know that this particular second term, the nonlinear or advection term, I have written in the non-conservative form. If you are using the finite volume method, retain the conservative form while solving this. If you are using finite difference methods, this is fine.

You can do non-conservative method. So, now I have the equation. So, it is readily seen that, any term here which will go away to 0? for constant density, this particular term, the third term on the left-hand side, it is $\frac{\partial u_j}{\partial x_j}$? So, this is, this particular term. So, this goes to 0. Therefore I have a much smaller form of the equation here.

So, let us call this equation 1 here the continuity equation and then the momentum equation, equation 2 here.