Course Name: Turbulence Modelling

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Week - 10

Lecture – Lec57

57. Pressure-Strain rate modelling for RSM – II

Therefore, using Green's function, we can obtain a solution for equation 3, which is, so $\phi(a)$ is this. So, now I am essentially going to get $P'(a)$ is a solution that I want that is P' essentially, and this is Green's function is minus 1 by 4 pi. So, it is minus 1 by 4 pi integral volume integral and then I have the entire source. I have also this 1 minus b by a that is there, but before that s of b, I am going to write. So, I get a minus of this entire source term, which is 2 dou ui bar of b by dou xj.

I am writing the main strain part first here and then dou uj prime of b by dou xi. So, this is the second term on the right-hand side that you had the mean strain component, and that is at the point source location b dou uj bar by dou xj dou uj prime by dou xi that particular component plus, I have the Reynolds stress component, which is dou square by dou xi dou xj of I have ui prime of b and uj prime of b minus the Reynolds stresses, which is ui prime of b uj prime of b average. I have taken minus here minus of the bracket, and therefore, the sign changed. Earlier it was Reynolds stresses minus these two product of the fluctuation since minus is taken out it has become the other way ok.

So, I have this the entire part and then I have db cube by b minus a. So, this is my source term here. So, this entire part here is nothing but S of b the source term $G(a, b)$ is here minus 1 by 4 pi b minus a db cube that volume integral. So, this is the entire source term that I have replaced here. S of b, a complicated source term indeed where I have the rapid and slow parts.

$$p'(a) = -\frac{1}{4\pi}\int_V \underbrace{-\left[2\frac{\partial \bar{u}_i(b)}{\partial x_j}\frac{\partial u_j'(b)}{\partial x_i} + \frac{\partial^2}{\partial x_i \partial x_j}\left(u_i'(b)u_j'(b) - \overline{u_i'(b)u_j'(b)}\right)\right]}_{s(b)} \frac{db^3}{|b-a|}$$

So, this is your rapid part, this is the slow part, density. So, p prime I can say constant, so p prime I can put it as rho here, p prime by rho, I keep it here constant. Okay, so at least now we have an expression for p prime at a location a depending on point sources, which is Reynolds stresses, and mean strain at b, okay? So now we have to this is still not complete modelling part we still have the right-hand side the source term is an unknown. I have no access to this in the, when I am computing the Reynolds stresses, so I still need to make some modelling assumptions. First thing is to see in a modelling whether I can get rid of some terms.

$$\frac{p'(a)}{\rho} = -\frac{1}{4\pi}\int_V -\underbrace{\left[\underbrace{2\frac{\partial \bar{u}_i(b)}{\partial x_j}\frac{\partial u_j'(b)}{\partial x_i}}_{\text{Rapid}} + \underbrace{\frac{\partial^2}{\partial x_i \partial x_j}\left(u_i'(b)u_j'(b) - \overline{u_i'(b)u_j'(b)}\right)}_{\text{Slow}}\right]}_{s(b)}\frac{db^3}{|b-a|}$$

For example, in the slow term can I make some assumption to get rid of one of the term in that can you assume a flow where some terms vanish in the slow part, like statistically homogeneous flow if flow is statistically homogeneous in all three directions what will happen to the gradient of this average term here, this particular component, it will be 0. So, I am going to assume homogeneous turbulence. These are model assumptions. So, I am going to start with, so let us call this equation 4. So, let us make some model assumptions.

So, what is that is that first thing is I am going to consider homogeneous turbulence, statistically homogeneous turbulence, which implies I have, let us say, $\frac{\partial}{\partial x_i}$ of some averaged quantity 0 where i can be i equal to 1, 2 or 3, 1, 2, 3. Not or and 1, 2, 3, I would say, because I need a fully statistically homogeneous flow to get rid of that particular component. So, the consequence of this is one term will go away. This is the first model assumption, statistically homogeneous. I can make another assumption is that on the rapid part if I am going to assume that this mean strain $\frac{\partial \overline{u_i}}{\partial x_j}$ if I am going to say that as I walk along, this is exhibiting small changes.

What does that imply? It implies that the mean strain is almost constant. When it is constant, I can take it outside the integral. So, I would like to assume that it is constant or almost constant. So, I am going to say that let spatial variation of $\frac{\partial \overline{u_i}}{\partial x_j}$ be small that $\frac{\partial \overline{u_i}}{\partial x_j}$ is almost constant. I would like to do that so that I can take it outside the integral.

So, with these two model assumptions let us see what would happen. So, what would happen for the first one is that this particular term in equation 4, the $\frac{\partial^2 u_i^{'}(b)u_j^{'}(b)}{\partial x_j \partial x_j}$, this average term will be 0, the consequence of the first assumption in equation 4. And the consequence of the second one is already I said this will become almost constant that means this term $\frac{\partial \overline{u_i}}{\partial x_j}$ can be taken outside the outside the integral. These are the consequences of two model assumptions here, ok. So, I am going to do these two and get an expression for pressure strain rate.

Now, since I have $p^{'}/\rho$, I can multiply the remaining part and average to get the equation for pressure strain rate. So, multiply equation 4. So, I am going to let us copy this particular term here or it is ok. Let us go to, so multiply equation 4 by $\frac{\partial u_i^{'}}{\partial x_j} + \frac{\partial u_j^{'}}{\partial x_i}$ and Reynolds average or ensemble average. So, what will this be? So, I am going to get $p^{'}/\rho$, of course, the constant value multiplied by $\frac{\partial u_i^{'}}{\partial x_j} + \frac{\partial u_j^{'}}{\partial x_i}$average.

So, now I have got an expression for it now which is now equal to the right hand side part which is. So, it is if you go back and see it is minus of minus 1 by 4 pi. So, I can essentially get plus 1 by 4 pi integral volume integral here and then I have the source term here which is 2. Now, if I just go back to the previous slide you can see here. These are point sources at b.

I am now multiplying this with the solution at point a here. So, I cannot multiply on the right side and still call it ij index because right now it is one is a function of b the other one is a function of a. So, I need to change the indices on the right hand side. Since on the left-hand side, I have multiplied by the velocity fluctuation gradient $\frac{\partial u_i^{'}}{\partial x_j}$ on the right-hand side, I cannot use i and j, so I am going to change to something else anyway i and j are repeated. It is eventually sum of nine terms, so I am going to use some other index here for consistency okay; so, I am going to do $2\frac{\partial \overline{u_k}}{\partial x_l}$, so i and j replaced by k and l of b this is the point source and dou ul prime of b by dou xk plus I have the dou square dou xl dou xk of uk prime of b and then ul prime of b.

This is my entire source multiplied by this particular term on the left-hand side. So, I have dou ui prime of a by dou xj plus dou uj prime of a by dou xi and then the average term, and then I have this db cube by b minus a, the delta function. So, note that this particular part here is a function of a, not b, and therefore, I can push this inside the integral. The integral is valid for the point source b, but the what I have multiplied is a function of a, so it can go inside the integral. So, you can take note here.

So, here i is set i comma j or replace with k comma l, since this is a function of b and here this part. So, here since this is a function of a, it can be moved inside the integral. So this is my source term right S of b, the entire source. So the left hand side, I have what I wanted a pressure strain rate term. So now the pressure strain rate, at least I know now that what it will depend on the right hand side if I want a solution for that one.

$$\overline{\frac{p'(a)}{\rho}\left(\frac{\partial u_i'(a)}{\partial x_j}+\frac{\partial u_j'(a)}{\partial x_i}\right)} = +\frac{1}{4\pi}\int_V \overline{\overbrace{\left[2\frac{\partial \bar{u}_k(b)}{\partial x_l}\frac{\partial u_l'(b)}{\partial x_k}+\frac{\partial^2 (u_k'(b)u_l'(b))}{\partial x_l \partial x_k}\right]}^{S(b)}\underbrace{\left(\frac{\partial u_i'(a)}{\partial x_j}+\frac{\partial u_j'(a)}{\partial x_i}\right)}_{\text{Since this is a } f(a)\text{, it can be moved inside the integral}}}\frac{db^3}{|b-a|}$$

pressure-strain-rate

[i,j are replaced with k,l since this is a function of (b)]

So, I can split this again into the two parts that is the rapid and the slow parts. So, left hand side is equal to. So, I get the first rapid part, which is 1 by 2 pi, and then I am moving the integral here the mean dou uk bar the model assumption dou xl of b integral V dou ul prime of b by dou xk multiplied by dou ui prime of a by dou xj plus dou uj prime of a by dou xi averaging db cube by b minus a. So, that is my rapid part here rapid term.

$$LHS = \frac{1}{2\pi}\frac{\partial \bar{u}_k(b)}{\partial x_l}\int_V \overline{\frac{\partial u_l'(b)}{\partial x_k}\left[\frac{\partial u_i'(a)}{\partial x_j}+\frac{\partial u_j'(a)}{\partial x_i}\right]}\frac{db^3}{|b-a|}$$

Rapid term

And then the slow term which is plus 1 by 4 pi of integral V dou square by dou xl dou xk of uk prime of b ul prime of b multiplied by dou ui prime of a by dou xj plus dou uj prime of a by dou xi average db cube by b minus a. This is your slow term. Let us call this equation 5 here.

$$\frac{1}{4\pi}\int_V \overline{\frac{\partial^2 (u_k'(b)u_l'(b))}{\partial x_l \partial x_k}\left[\frac{\partial u_i'(a)}{\partial x_j}+\frac{\partial u_j'(a)}{\partial x_i}\right]}\frac{db^3}{|b-a|}$$

Slow term

You are all with me so far? Fine. So now we would like to make another approximation or consideration, so we will take term by term to model now from up to this so we have arrived at equation 5 by making two assumptions: statistically homogeneous flow homogeneous turbulence as well as considering mean strain is nearly constant on the left-hand side I have pressure strain rate term that is equal to these two what i call rapid and slow term so if I am going to consider only the rapid term So, consider only the rapid term now, or I can say modelling the rapid term. So, term by term we will model this. So, what I do is let us consider only one component there are this is actually 2 here, this is multiplied plus so sum of 2 different terms.

So, let us consider one term and see what will happen to it. So, consider one term in the rapid part of equation 5. So, let us say I take dou uj prime of a by dou xi and dou ul prime of b by dou xk. So, now if I use product rule here, going to use product rule. What will this become if I use product rule? I get dou by dou xk of ul prime of b dou uj prime of a by dou xi.

So, if I use product rule for this I am going to get the left hand side term and one more term. So, this will be minus then, correct? So, minus of I am going to get essentially, ul prime of b dou square uj prime of a by dou xk dou xi. I have just used the product rule here. So, if you take product rule only for this particular term, it will be sum of this and this.

$$\frac{\partial u_j'(a)}{\partial x_i}\frac{\partial u_l'(b)}{\partial x_k} \xRightarrow[\text{rule}]{\text{product}} \frac{\partial}{\partial x_k}\left(u_l'(b)\frac{\partial u_j'(a)}{\partial x_i}\right) - u_l'(b)\frac{\partial^2 u_j'(a)}{\partial x_k \partial x_i}$$

That is it, nothing new here. So, if you look carefully on to the right-hand side, these are two terms and the extreme right part here is uj prime of a, which is a function of a. So, for a, I have indices i and j, but there are k and l which is the index for the source b. So, dou uj prime of a by dou xk should be 0. because one is a function of a the other one is not.

So, I can write here. So, this particular term goes away or I can rewrite this as. So, I can call this dou by dou xi of dou uj prime of a by dou xk. So, this term will be 0 that is because since uj prime of a is not a function of b. And dou by dou xk is applicable to only functions of b, the gradient. So, dou by dou xk of this particular thing is 0, this goes away.

$$\frac{\partial u_j'(a)}{\partial x_i}\frac{\partial u_l'(b)}{\partial x_k} \xRightarrow[\text{rule}]{\text{product}} \frac{\partial}{\partial x_k}\left(u_l'(b)\frac{\partial u_j'(a)}{\partial x_i}\right) - u_l'(b)\frac{\partial}{\partial x_i}\cancel{\left(\frac{\partial u_j'(a)}{\partial x_k}\right)}$$

Since $u_j'(a) \neq f(b)$ and $\frac{\partial}{\partial x_k}$ is applicable to only $f(b)$

So, I have this particular term. I can simplify this further. So, this again I am going to use product rule, using product rule again, I would get dou by dou xk of I am going to push this ul prime of b inside the derivative using product rule. So, I would get dou by dou xi of ul prime of b uj prime of a minus uj prime of a dou ul prime of b by dou xi. So, dou by dou xi of this is i dou by dou xi of ul prime of b uj prime of a minus this particular term. So again, the same argument this ul prime of b is a function of b, but xi or index i is used for solution a, same argument this goes to 0 because similarly, ul prime of b is not a function of a. dou by dou xi is applicable functions of a. So, that goes away.

$$\xRightarrow[\text{rule}]{\text{product}} \frac{\partial}{\partial x_k}\left[\frac{\partial}{\partial x_i}\left(u_l'(b)\,u_j'(a)\right) - u_j'(a)\cancel{\frac{\partial u_l'(b)}{\partial x_i}}\right]$$

|||ly $u_l'(b) \neq f(a)$ and $\frac{\partial}{\partial x_i}$ is applicable to $f(a)$

So, essentially, that means I have now have this particular term is nothing but the left hand side here, which is dou uj prime of a by dou xi. dou ul prime of b by dou xk is equal to dou square by dou xi dou xk of ul prime of b uj prime of a. So, if you see, we have somehow brought a relationship between the rapid term, which essentially has a mean strain in that particular term we have said the rapid term is now somehow related to a mean strain that green particular thing that I have written dou uk bar by dou xl.

$$\frac{\partial u_j'(a)}{\partial x_i} \frac{\partial u_l'(b)}{\partial x_k} = \frac{\partial^2}{\partial x_i \partial x_k} \left(u_l'(b) u_j'(a) \right)$$

And then this particular term that is in the rapid term here, if you see I have a mean strain plus I have another term the nature of that term was not visible now if I just applying product rule twice I reveal that each of this term if you take one of this or the other one you can test it is going to be the same you will get a relationship like this. So, and there is an average here. So, this is essentially nothing but this particular term. If I average this, it will be this. So, that means the pressure strain rate is related to the mean strain, which I have access and Reynolds stresses.

So, I am coming closer to modeling at least one term the rapid term and I have access to both Reynolds stresses and the mean strain rate. So, therefore I can say that from equation, so this previous equation let us call it this as equation 6. So, from 5, 6 I can say that pressure strain rate or the rapid part in pressure strain rate is depending on the mean strain and the Reynolds stresses. From equation 5, 6, it is clear that this pressure strain rate rapid term we can call this, let us say pi ij rapid let us say like this. This term is a function of dou uk bar by dou xl comma uj prime ui prime average. So, these are known to me, known mean strain and Reynolds stresses. ul prime right this is was that ul prime uj prime ul prime yeah ul prime these are known terms to me.

$$\underset{[\pi_{ij}\ \text{rapid}]}{\text{pressure-strain-rate (rapid) term}} = f\Big(\underbrace{\frac{\partial \bar{u}_k}{\partial x_l},\ \overline{u_j' u_l'}}_{\text{Known}}\Big)$$

So, I can make a model now this particular model for the rapid has a name it is called IP model it is a very popular model used in commercial codes also. So, the π_{ij} is called IP model. This IP model, which means isotropization of production that is IP.

There is a reference for this, which is reference article. This is a reference article. So, what this model is doing is essentially the modelling philosophy is that the rapid term partially counteracts the production rate term. So, rapid term partially counteracts production term. So if you recall pressure strain rate role is basically to redistribute ok, and we are trying to relate it to the production rate because production rate has Reynolds stress and mean strain and we have established a relationship now that The rapid term in the pressure strain rate is a function of Reynolds stresses and mean strain rate and therefore, the model itself will be like this.

So, the π_{ij}, let us call it r, or you can say rapid also. Some literature will just see r as the subscript. This model will be minus C2 rho multiplied by P ij minus 2 third Pk delta ij. So, this is the model for one part.

$$\boxed{\pi_{ij\,\text{rapid}} = -C_2 \rho \left[P_{ij} - \frac{2}{3} P_k \delta_{ij} \right]}$$

So, there is no, this is not a complete model. So, slow term also has to be model. So, the pressure strain rate term will have two models, one for rapid, one for slow. So, this is the π_{ij} model term. This is the model term, and this may look a little strange. Why is this production rate coming, but if we expand it, we will get back to the original part, which is minus C2 rho of what is P_{ij}? P_{ij} is nothing but your $-u_i^{\prime}\ u_k^{\prime}$.

You can go back and look at the equation. You will get dou uj bar by dou xk minus uj prime uk prime average dou ui bar by dou xk. This is your Pij minus 2 third Pk also, you can substitute. So, both Pk and Pij terms are functions of this is your Pij and also this Pk. So, both are functions of the mean strain rate. So, dou uk bar by dou xl comma uj prime ul prime, that is what we have accomplished here.

$$-C_2 \rho \left[\underbrace{\overbrace{\left(-\overline{u_i' u_k'} \frac{\partial \bar{u}_j}{\partial x_k} - \overline{u_j' u_k'} \frac{\partial \bar{u}_i}{\partial x_k} \right)}^{P_{ij}}}_{\text{Both are functions of } \frac{\partial \bar{u}_k}{\partial x_l},\ \overline{u_j' u_l'}} - \frac{2}{3} P_k \delta_{ij} \right]$$

So, the rapid term depends on mean strain rate and Reynolds stresses. This IP model is popular. You can use this, and the model constant is C_2 if I C_2 and density if I am going to set equal to 1 here, and if P_k is 0, then this π_{ij} rapid essentially becomes P_{ij}. So, this π_{ij} term is actually taking away what production rate is doing. Even in a canonical flow, we have seen that production rate, let us say, is in only direction along the x direction, then y and z it is redistributing.

So, this π_{ij} is taking away whatever is being produced in the absence of, let us say, if C_2 is 1 and P_k is 0, π_{ij} rapid becomes minus P_{ij}. It is essentially taking away everything that is the whole isotropization of production meaning. So, we have to still do the slow term we can take a break until and you have any questions.