Course Name: Turbulence Modelling

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Week - 10

Lecture – Lec56

56. Pressure-Strain rate modelling for RSM – I

So, let us get started again. So, in the last class, we looked at taking Laplacian or, getting a Laplacian equation for the pressure both for instantaneous as well as the mean. So, for that, we took a divergence of your Navier Stokes. You got this equation. Of course, you also did Reynolds decomposition here.

So, you took divergence and Reynolds decomposition. And then we also took divergence of the RANS equation to get equation 2. So, equation 1 and 2 is by doing this taking divergence or divergence of this Navier-Stokes and RANS equations. So now what I do is I simply subtract one from each other.

$$\frac{\partial}{\partial x_i}(NS) \Rightarrow \frac{\partial \bar{u}_i \bar{u}_j}{\partial x_i \partial x_j} + \frac{\partial \bar{u}_i u_j'}{\partial x_i \partial x_j} + \frac{\partial u_i' \bar{u}_j}{\partial x_i \partial x_j} + \frac{\partial u_i' u_j'}{\partial x_i \partial x_j} = -\frac{1}{\rho}\frac{\partial^2 \bar{p}}{\partial x_i^2} - \frac{1}{\rho}\frac{\partial^2 p'}{\partial x_i^2} \longrightarrow (1)$$

$$\frac{\partial}{\partial x_i}(RANS) \Rightarrow \frac{\partial}{\partial x_i}\left(\frac{\partial \bar{u}_i \bar{u}_j}{\partial x_j}\right) = \frac{\partial}{\partial x_i}\left(-\frac{1}{\rho}\frac{\partial \bar{p}}{\partial x_i}\right) - \frac{\partial}{\partial x_i}\left(\frac{\partial \overline{u_i' u_j'}}{\partial x_j}\right) \longrightarrow (2)$$

So, equation 1 minus equation 2 will give me an equation for Laplacian for pressure fluctuation or an equation for pressure fluctuation. So, we already have in Navier-Stokes an equation from that we can derive a Poisson equation for instantaneous pressure. We can also get an Poisson equation for mean pressure. We are trying to get a Poisson equation for fluctuating pressure. So, by doing 1 minus 2 I can easily see that this particular term goes away.

So we have this one, this one, and this one will go away, right? They cancel each other, and then the mean term this and this cancel out 1 minus 2 minus of this minus of this minus of minus this becomes plus. So, basically, these two terms cancel away the mean pressure gradient.term as well as this ui bar uj bar term. So, they cancel out. So, what I get here is basically dou ui bar uj prime by dou xi dou xj plus dou by dou xi dou xj ui prime uj bar plus dou by dou xi dou xj ui prime uj prime this minus of this so is equal to I have minus 1 by rho dou square p prime by dou xi square minus of this so this becomes plus dou by dou xi of dou or I can say dou square dou xj ui prime uj prime average. So, that means essentially, if I rearrange, I am going to get a Laplacian for pressure. This implies, dou square p prime by dou xi dou xi is equal to, I get here also there is a missing dou square term here, also here, here, here, all these places. So, then I get this as dou square by dou xi dou xj of your ui prime uj prime average and then I have another term which is looking like this. So, I have minus dou square ui prime uj prime by dou xi dou xj minus dou square ui bar uj prime by dou xi dou xj minus dou square ui prime uj bar by dou xi dou xj, alright? So I can group this together to make it like.

$$\text{Eq}(1) - \text{Eq}(2) \Rightarrow \frac{\partial^2 \bar{u}_i u_j'}{\partial x_i \partial x_j} + \frac{\partial^2 u_i' \bar{u}_j}{\partial x_i \partial x_j} + \frac{\partial^2 u_i' u_j'}{\partial x_i \partial x_j} = -\frac{1}{\rho}\frac{\partial^2 p'}{\partial x_i^2} + \frac{\partial^2 \overline{u_i' u_j'}}{\partial x_i \partial x_j}$$

$$\Rightarrow \frac{1}{\rho}\frac{\partial^2 p'}{\partial x_i \partial x_i} = \frac{\partial^2 \overline{u_i' u_j'}}{\partial x_i \partial x_j} - \frac{\partial^2 u_i' u_j'}{\partial x_i \partial x_j} - \frac{\partial^2 \bar{u}_i u_j'}{\partial x_i \partial x_j} - \frac{\partial^2 u_i' \bar{u}_j}{\partial x_i \partial x_j}$$

So this particular term, you can see this is nothing but dou square by dou xi dou xj of ui prime uj prime average minus, wait this particular term minus of this, correct minus I have ui prime uj prime here, correct? Density. The density term yes 1 by rho term correct, right? So I am clubbing the two terms which is looking like Reynolds stresses. One is Reynolds stress the other one is just a product of two fluctuating velocities the way the reason I am clubbing this is because there are other two terms here which has mean velocity components ok? So these two terms has mean velocity ui bar uj bar So, therefore I am moving them separately and keeping this separately here. So, I get basically this particular term here, and then I have minus of I can write this. Yeah, I can write this as dou by dou xi of this and this.

So, now I have to do one trick here. So, since this particular last two terms has the mean velocity terms here. So, this has these two terms they have mean velocity terms. so what I do with these two is I am going to apply product rule to change the nature of it little bit. So, what I will do is I am simply going to take minus I will take the first term ui bar uj prime.

So, if I take dou by dou xi of now I have dou by dou xj of this particular term I apply product rule ok. So, I get ui bar dou uj prime by dou xj plus uj prime dou ui bar by dou xj. I have just applied product rule here. I am going to apply product rule also to the other term, which is minus dou by dou xi of again dou by dou xj of this particular term apply product rule, I get uj bar dou ui prime by dou xj plus ui prime dou uj bar by dou xj ok, again product rule. So, I readily see that some terms vanish correct dou uj prime by dou xj.

by continuity it will go away. So, this is vanishing by continuity also this last term here dou ej bar by dou xj continuity.

$$= \frac{\partial^2}{\partial x_i \partial x_j}\left(\overline{u_i' u_j'} - u_i' u_j'\right) - \frac{\partial}{\partial x_i}\underbrace{\left(\bar{u_i}\,\overset{\text{continuity}}{\cancel{\frac{\partial u_j'}{\partial x_j}}} + u_j' \frac{\partial \bar{u_i}}{\partial x_j}\right)}_{\text{product rule}} - \frac{\partial}{\partial x_i}\underbrace{\left(\bar{u_j}\frac{\partial u_i'}{\partial x_j} + u_i'\,\overset{\text{continuity}}{\cancel{\frac{\partial \bar{u_j}}{\partial x_j}}}\right)}_{\text{product rule}}$$

So I am doing all this as you recall from last class modelling is all about relating unknowns to known, and the known are Reynolds stresses, mean strain correct and dissipation rate of turbulence kinetic energy we somehow has to establish a relationship between the unknown pressure strain rate to Reynolds stresses and mean strain so you will see that slowly we will come towards that so if I work with this further so I am going to get this particular term keep it as it is no changes here. So I have this as, yeah, so I get the product rule. I can apply product rule once again here.

So I am going to get minus dou by dou xi of this particular term. So I can get minus dou by dou xi of if I push this further I get uj prime, so I have uj prime dou ui bar by dou xj. So, I am going to get, let us take this out. So, I am going to get uj prime dou square ui bar by dou xi dou xj and then minus dou ui bar by dou xj dou uj prime by dou xi using product rule again for this particular term here once more. So, I can do the same thing here. So, then I get uj bar dou square ui prime by dou xi dou xj minus I have dou uj bar by dou xi dou ui prime sorry dou ui prime by dou xj.

$$= \quad '' \quad - u_j' \frac{\partial^2 \bar{u_i}}{\partial x_i \partial x_j} - \frac{\partial \bar{u_i}}{\partial x_j}\frac{\partial u_j'}{\partial x_i} - \bar{u_j}\frac{\partial^2 u_i'}{\partial x_i \partial x_j} - \frac{\partial \bar{u_j}}{\partial x_i}\frac{\partial u_i'}{\partial x_j}$$

Again is any other term goes away here. So here if you can see I can first differentiate with respect to xi and then with respect to j here same with this one also. So I can rewrite this entire term here as so this is equal to the same thing here minus I can write this one I can say dou by dou xj of dou ui bar by dou xi minus I have I can take the other one here I am bringing this minus uj bar dou by dou xj of dou ui prime by dou xi and then minus these two terms dou i bar by dou xj dou uj prime by dou xi. So, now the last term the last two terms here.

If you see it is ij, ji; if I interchange, I am going to get the same term. So I am talking about these two terms here, this and this. If I take ij as free index and then replace i with j, j with i, I am going to get this term, correct? dou uj bar by dou xi dou uj prime by dou xi I am going to get and this is both i and j are repeated. So, it is sum of 9 terms. So, when I expand this I get 9 terms which is exactly equal to this.

So, if I apply Einstein's summation, these two are same that means this is minus 2 times correct minus 2 of this. Because this i, j are free indices, the free index rule, upon Einstein's summation, each term is a sum of 9 terms. So, I can replace ij to get the same. So, I get minus 2 of the last 2 terms, and here I get rid of this dou ui prime by dou xj dou ui this 2 goes away because of continuity again. So, I have the left-hand side I have minus sorry I have 1 by rho dou square p prime by dou xi dou xi is equal to I have this particular term dou square by dou xi dou xj of ui prime uj prime average minus ui prime uj prime and then I have a single term here which is minus 2 dou ui bar by dou xj dou uj prime by dou xi.

$$\frac{1}{\rho}\frac{\partial^2 p'}{\partial x_i \partial x_i} = \frac{\partial^2}{\partial x_i \partial x_j}\left(\overline{u_i' u_j'} - u_i' u_j'\right) - 2\frac{\partial \overline{u_i}}{\partial x_j}\frac{\partial u_j'}{\partial x_i}$$

So, I have an equation here which is looking like this. So, this particular equation is now the consequence here is first, we have an equation for Laplacian for p' here, and what we need is an equation for p' because we need to know correlation of p' with the gradient of velocity fluctuation it is a pressure strain rate correlation $\overline{p' \frac{\partial u_i'}{\partial x_j}}$, correct? So I do not have that right now. I have Laplacian for p'. At least we started somewhere, and I have Laplacian for p', and on the right-hand side, this has complicated source terms here. So a Laplacian equation so if I want to find out pressure fluctuation I need to know some complicated terms on the right hand side.

Interestingly, the two terms that I have here is something unique on the left hand side or the first term on the right hand side. This particular term I would call this slow term and the second term I would call it a rapid term. The reason for this is if I compare these two. Let us assume that we want to find out pressure fluctuation at a particular point right and to get this pressure p prime at some point. It is influenced by the flow field around it, and the source term here on the RHS is saying that the influence is Reynolds stresses; change in Reynolds stresses at anywhere is going to influence your pressure fluctuation and change in mean strain $\frac{\partial \overline{u_i}}{\partial x_j}$ everywhere is also going to influence your pressure fluctuation.

There are two things that are going to influence pressure fluctuation, and the mean strain is going to dominate in a flow when it wants to give a response compared to the Reynolds stresses, and therefore, the name came as slow and rapid. So, the two both are source terms here. So, the RHS terms or source terms for Laplacian for pressure fluctuation. So, this slow and rapid term this particular rapid implies that this has come from a theory called rapid distortion theory. The term rapid comes from rapid distortion theory for homogeneous turbulence.

$$\boxed{\frac{1}{\rho}\frac{\partial^2 p'}{\partial x_i \partial x_i} = \frac{\partial^2}{\partial x_i \partial x_j}\left(\overline{u_i' u_j'} - u_i' u_j'\right) - 2\frac{\partial \overline{u_i}}{\partial x_j}\frac{\partial u_j'}{\partial x_i}}$$

$$\underbrace{\qquad\qquad}_{\text{Slow term}} \qquad\qquad \underbrace{\qquad\qquad}_{\text{Rapid term}}$$

What is essentially means is that this gives rapid response over the other term because of the large mean strain. So, this essentially implies that for large take a note here. So, this last term here or the rapid term gives rapid response that is dominates over the other term for large mean strain that is $\frac{\partial \overline{u_i}}{\partial x_j}$ term. When mean strains are large this is going to dominate over the other one. Both will influence your p' at any given point changes in Reynolds stresses and changes in mean strain.

We are actually coming closer to what we want. We want to model pressure strain rate term on the known terms which is Reynolds stresses and the mean strain rate and dissipation rate of turbulence kinetic energy. At least now it is telling me that Laplacian for p prime is a function of Reynolds stresses just on the modelling argument as well as mean strain rate. We are coming closer. So, but I need an expression for p' and not Laplacian for p', right? So, how do I get this this particular equation we call it let us call this equation 3 ok let us call this equation 3.

So, this equation 3 is Laplacian for p', but what we need is an equation for p' itself. So, how to go about it? So, there is something called Green's function, which can help us get an equation for p'. So, what it does is so Green's function can help us get an expression for p'. Green's function can help us to get an equation for p'. So, what is Green's function? So, it is essentially Green's function is essentially if you have, let us say, I am going to take two points, I am going to take two points here, and I am going to call it a and b.

Let us call this a and b. So b let it be a source term. So this can be any x, y, z location. I am calling it b. So I am interested in my solution at a and the solution is basically p'. So I would like to know p prime at a location a and how it is changing by the source term at any other location b.

So, essentially, how all this Reynolds stress term, as well as the mean strain, how it is changing around, how it is influencing at p' at a particular location a. a is my solution location here, I want a solution at a, b is my point source location. So if I have these two points then I can write Green's function. So there is a reference for this. This is Chai 1945, or you can refer to some mathematics books.

So suppose a Laplacian of, let us say, some generic variable phi of a, the solution that I am looking at. So, where this Laplacian is 3D Laplace operator. So, the Laplacian of $\phi(a)$ is equal to some source at b. So, where $\phi(a)$ is a solution at point a and S is the point source. In our case, a complicated source term is here that we saw from equation 3 here.

If I have this, then the solution is essentially this $\phi(a)$. I can write it as a volume integral and a Green's function $G(a, b), S$ of b db cube where this $G(a, b)$ is the Green's function. and b is as already said it is a your point source location. So, where this $G(a, b)$ this is defined as minus 1 by 4 pi 1 by b minus a, this is a delta function here. So, this now describes this particular thing describes response of the system at the point a due to a point source at location b.

$$\phi(a) = \int_V G(a,b)\, S(b)\, db^3 \quad ; \quad G(a,b) \rightarrow \text{Green's function}$$

$$G(a, b) = \frac{-1}{4\pi} \frac{1}{|b - a|}$$

So, I have the solution here. So, now I have to apply it to equation 3 to get P(a), sorry $P'(a)$. P' at a depending on the changes of Reynolds stresses at b, this b can be anywhere as well as the mean strain changes of mean strain at location b. So, if I apply this particular I will not take this, let me just happen. So now I am going to take apply equation 3 to this Green's function to get a solution for $P'(a)$. So therefore I can say