Course Name: Turbulence Modelling

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Lecture – Lec52

52. Reynold's Stress Modelling (RSM): governing equations - I

Okay, so let us get started with Reynolds stress modeling. So, so far we have been seeing only this eddy viscosity modeling right. So, we will go to the next level of complexity. So, which is Reynolds stress modeling RSM. Sometimes it is also called Reynolds stress transport modeling because you are solving a transport equation for each of the Reynolds stresses.

So, Reynolds stress transport modeling. RSTM. These are also called as second moment closures because in the when we did eddy viscosity modeling we primarily closed the momentum equation right. Your $\overline{u}$, $\overline{v}$, $\overline{w}$ equations were closed by modeling the Reynolds stresses using Boussinesq.

So, that is the first moment closure. So, this particular class of models that is Reynolds stress models RSMs are called second moment closures. the name comes because here we are solving for ah Reynolds stresses itself that means we are solving the closed form of Reynolds stress equations the second moments that is of solving closed form of your Reynolds stress equations. transport equations are being solved here. So, if you recall now from eddy viscosity models you had 6 unknowns in the RANS equation right.

So, if you recall RANS equation you have 6 unknowns that is your ui prime uj prime average 6 stresses, 3 normal stresses, 3 shear stresses and in the eddy viscose 3 model this was closed using Boussinesq. Here we are not doing anything of that sort ok. So, in RSM ok no Boussinesq is used used to close the unknowns. So, instead we solve for a transport equation for each of the stresses ok. So, instead instead we solve for we solve the transport equation for each Reynolds stresses.

but it requires modeling because this equation you saw that it has far too many unknowns right. The Reynolds stress equation that we derived from first principles if you recollect right. So, the exact form of equation. So, the exact equation looks like

$\frac{\partial}{\partial t}\left(\overline{u_i u_j}\right) + \overline{u}_k \frac{\partial u_i' u_j'}{\partial x_k}$ in some literature and I think if I recollect correctly even when we derived j was the repeated index and it was i' k'. you can throughout replace the divergence term here the index k with the j.

So, in some literature it is interchanged. So, we will get $u_i' u_k'$ and divergence term will be ∂x_j ok. Yes, but we will take up this because in the when we model the turbulence kinetic energy equation and the eddy viscosity we use $\overline{u'_i u'_j}$. So, for consistency I am taking that and making k as the repeated index the divergence that is it. When you expand it, it does not change right.

We have throughout replaced i with j sorry k yeah j and k are interchanged throughout. So, on the right hand side is the trouble you have extra unknowns. First we have the diffusion rate term which is $\frac{\partial}{\partial x_k}\left(-\frac{p' u_j'}{\rho}\delta_{jk} - \frac{p' u_i'}{\rho}\delta_{jk}\right)$ ok. So, this was your pressure diffusion rate. and then I have the turbulent diffusion rate which is $-\overline{u_i' u_j' u'_k}$ the 27 unknowns and then the viscous diffusion rate which is $\nu\frac{\partial}{\partial x_k}\left(\overline{u_i' u_j'}\right)$.

So, this is your diffusion rate. symbolically let us say I write it as Dij, this can also be symbolically written as Cij here, the advection rate or the convection rate term Cij Dij term. And then I have the pressure strain rate term. this we escaped in the turbulence kinetic energy equation by continuity, but that term appears here which is good because this will take care of your redistribution rate and therefore, it takes care of anisotropic nature of turbulence. It also handles your two component limit.

So, there is lot of advantage of this term coming back to the equation not for modeling not from the modeling perspective. because pressure term we always ignored, but this term we cannot. So, this is an important term in the Reynolds stress equation. So, we have this $\overline{\frac{p'}{\rho}\left(\frac{\partial u_i'}{\partial x_j} + \frac{\partial u_j'}{\partial x_i}\right)}$ and then the strain rate fluctuations which is $\left(\frac{\partial u_i'}{\partial x_j} + \frac{\partial u_j'}{\partial x_i}\right)$ the pressure strain rate. What did we use the symbol here? Is it Φ_{ij} or ok you can use this or sometimes this is also used in the literature symbolically.

So, I have the two terms and then two more which is the production rate term So, I have $-\overline{u_i' u_k'}\frac{\partial \overline{u_j}}{\partial x_k} \quad -\overline{u_j' u_k'}\frac{\partial \overline{u_i}}{\partial x_k}$. So, this is your production rate. can call it p_{ij} and finally, the dissipation rate term which is $-2\nu\overline{\frac{\partial u_i'}{\partial k_k}\frac{\partial u_j'}{\partial x_k}}$ is your dissipation rate. We call it ε_{ij} let us say symbolically. So, we have the exact equation available.

We just need to go ahead and model the equations now. Of course, it has more unknowns than the k equations, but we will see what we can do with that. So, the left hand side may not be modeled nothing to model on the left hand side. So, I have access to the Reynolds stresses here right. So, nothing to model on LHS.

Only on the right hand side I have the terms that are unknown. We can take up with the diffusion rate term first D_{ij} term. So, D_{ij} model ok. So, before that I would like to talk about interesting advantage here. We will take up the production rate P_{ij}.

So, now the question is do we need to model this? The production rate of turbulence kinetic energy P_k was modeled right, it had Reynolds stresses working against the main strain rate terms right $-\overline{u_i'u_k'}\frac{\partial \overline{u_j}}{\partial x_k} - \overline{u_j'u_k'}\frac{\partial \overline{u_i}}{\partial x_k}$ that term was there. So, that was modeled using Boussinesq. What about here? The production rate term sorry yeah. We do not have to model this at all because you are computing for the Reynolds stresses. So, it is available to you.

You have access to Reynolds stresses here right. So, this is the biggest advantage of RSM no production rate to be modeled ok. So, this is the best part of the RSM here right because Reynolds stresses are available to that is because Reynolds stresses your $u_i' u_k'$ average is being computed and therefore, available therefore, it is readily available. So, no Boussinesq assumptions here required which is a good part the production rates are not modelled ok. So, the P_{ij} is not modelled in RSM biggest advantage.

this is the biggest advantage in terms of RSM since P_{ij} is not modelled. So, P_{ij} is already readily taken care of I do not have to do anything with it that means, I have only diffusion pressure strain rate and dissipation rates to be modeled. But you have to since you have already been looking into a eddy viscosity model and its solutions you can see that the coupling is very intense here now for the production rate term P_{ij}. So, earlier you had in the eddy viscosity model you have the RANS equation where it is coupled to the you need to compute Reynolds stresses you need the eddy viscosity and eddy viscosity required k ε or k ω and so on. So, you are solving for essentially velocity equations which requires eddy viscosity which requires k or ε and so on and they also require velocities so it is all coupled this is much more coupled than that one right.

So, you can easily see. So, however strong coupling ok. So, you need to be aware of this.

So, you need to have some CFD techniques to make sure that the model converges. So, we can see this what kind of coupling that we are talking about. So, if I have let us say I take up example say i equal to j equal to 1.

We are computing the one of the normal stresses which is u that means I am looking into the p_{11} term here. i equal to j equal to 1, p_{11}term is essentially minus 2, minus 2 of because i and j equal to 1. So, -2 of u_i', not u_i, u_1', u_k' and then $\frac{\partial \overline{u_1}}{\partial x_k}$. i equal to j equal to 1. So, this gives me sum of three terms to compute production rate for 11 for one of the Reynolds stresses that is this is the equation is $u_1'u_1'$ equation.

p_{11} appears in $u_1'u_1'$ transport equation. For that now I get $-\ 2\overline{u_i'u_k'}\frac{\partial \overline{u_1}}{\partial x_1}\ -\ 2\overline{u_1'u_2'}\frac{\partial \overline{u_1}}{\partial x_2}$ $-\ 2\overline{u_1'u_3'}\frac{\partial \overline{u_1}}{\partial x_3}$. So, that means to compute for this particular stress here $\overline{u_1'u_1'}$ you would need access to $u_1'u_1'$ itself and the two shear stress terms $\overline{u_1'u_2'}$ and $\overline{u_1'u_3'}$. So, this equation is coupled to itself as well as the two extra equations. the transport equation for $\overline{u_1'u_2'}$ and transport equation for $\overline{u_1'u_3'}$.

So, we can see what does $\overline{u_1'u_2'}$ depend on ok. So, this here we can say $\overline{u_1'u_1'}$equation is coupled to $\overline{u_1'u_2'}$ and $\overline{u_1'u_3'}$equation. So, what does this i equal to 1 and then I said j equal to 2. So, I get the $\overline{u_1'u_2'}$ equation. So, I get P_{12} the production rate for the one shear stress term that will be - $\overline{u_1'u_k'}\frac{\partial \overline{u_2}}{\partial x_k}$ - $u_2'u_k'$.

$\frac{\partial \overline{u_1}}{\partial x_k}$, i is 1 j is 2. So, if I expand this you will see what is the coupling that will happen here. So, I get $-\ \overline{u_1'u_1'}\frac{\partial \overline{u_2}}{\partial x_1}\ -\ \overline{u_1'u_2'}\frac{\partial \overline{u_2}}{\partial x_2}\ -\ \overline{u_1'u_3'}\frac{\partial \overline{u_2}}{\partial x_3}$. And then the next term which is $u_1'u_1'\frac{\partial \overline{u_1}}{\partial x_1} - u_2'u_2'\ \frac{\partial \overline{u_1}}{\partial x_2}$ - $u_2'u_3'\ \frac{\partial \overline{u_1}}{\partial x_3}$. So, the production rate for $\overline{u_1'u_2'}$ shear stress depends on 6 stress terms which is $u_1'u_1'$.

So, to compute $\overline{u_1'u_1'}$ you needed $\overline{u_1'u_2'}$ and to compute $\overline{u_1'u_2'}$ you would need $\overline{u_1'u_1'}$. They are strongly coupled to each other. In addition to that they are of course also coupled to the other stress terms and other normal stress terms also right. So, $u_1^{'}u_2^{'}$ equation is strongly or can just say coupled to $u_1^{'}u_1^{'}$, $u_1^{'}u_2^{'}$, $u_1^{'}u_3^{'}$ and then $u_2^{'}u_2^{'}$ and $u_2^{'}u_3^{'}$, 5 components. thats why I mentioned strong coupling strong coupled system of equations.

So, make generous use of under relaxation here to get any convert solution ok. generous use of under relaxation is recommended in CFD calculations to get convergence. advantage is good no need to model production rate, but strongly coupled system of equations. We are solving 6 extra transport equations here which is closely linked to each other and with the momentum equations 3 momentum equations which is also of course, linked to the pressure equation ok.

So, production rate is sorted. So, now I have the diffusion rate term.