5. Statistical Analysis and Cartesian tensors - II

A correlation in time. So, there I can say I have u’ at x,y,z at time t and let us say I am looking into u’ itself at x,y,z ,t + Δ t, Δ t is the some time interval separation. So, let us say the first signal started at time t = 0, I am checking every milliseconds, okay ,2 milliseconds, 3 milliseconds and so on till I see that after some time it is not correlated anymore. So, I can take a sample. So, I collect all this data to do statistics.

So, I can do this now, average over this. and you can normalize this. If you want this is correlation in time, if you want to get the correlation coefficient then you use the reference data at time t ok. This is at time t, this is at t + Δ t here after certain time interval ok, Δ t is the time interval.

This is yes. We are generally working with ensemble average here. So, in practice only as I said you can do that. You can do I mean many of these things we do it in as an arithmetic mean. So, over bar in the course represents it is ensemble mean whenever I use it.

Ensemble mean is always a function of three dimensional space and time. It is not time average. So, if you want a correlation coefficient then you take the reference as at t and Δt keeps changing, but the reference is t. So, you can do a correlation coefficient like this. So, the coefficient, correlation coefficient is u’ of x, y, z.

t u’ of x, y, z, t + Δ t over u’ of x, y, z ,t you take the square and then average to get the correlation coefficient in time or the temporal coefficient or an autocorrelation. Now there is also, this question has come now, this is like a vortex or an eddy is coming in and I am waiting after a certain time so that I know that this has passed over it and so that I can collect another sample, that is fine. Now sometimes you are interested to know spatially how large a turbulent structure is sitting, its spatial dimensions of a turbulent eddy. For that one can do what is called a two-point correlation or a multipoint correlation.

We have looked into only single point statistics, we can do multipoint statistics. In this one interesting component is two point correlation. This is to see how large this eddy is. I mean you cannot see everything using your naked eye. You need to quantify it.

You cannot approximately say this is like looking like 10 millimeter. I need to exactly quantify the size of a turbulent eddy in three dimensions. Its length in x, y, z. Then you can do this two point correlation which is basically then I can look at u’ of let us say the two point correlation in x direction, then it is x, y, z, t u’ of x + Δ x. So, Δ x is the spatial interval that I am giving y, z, t.

This will give this information gives me like what is the spatial extent of a turbulent eddy in x direction. So, I can say that this will be useful for example, when you simulate it you need to have these eddies sitting inside your computational box right. So, if your box let us say the length in x direction is let us say 1 meter, but the turbulent eddy is 10 meters precise then obviously you are not simulating it right. So, your box has to be bigger than the turbulent eddy, the largest turbulent eddy you have. And therefore, two-point correlation helps in finding out how large computational domain you should have.

ok So, this is useful, this multi-point statistics. So, the statistical part I will stop here. I will not go into much more in depth. This is sufficient to know basically how to split the raw signal or the instantaneous data into a statistical component and the random component. Now engineers are interested in the statistical value but you need the information of the randomness to compute the statistical part and physicists are interested to learn turbulence so they are interested in the random component.

So, now we know how to split if you have the data I know how to split, but the application of this now is to we will use it in a governing equations. We want to split the terms inside the governing equation the statistical component and the random component. So, we go towards that part. This multipoint statistics is as I said it could be U' and V' but you need to know why you want to do that. right the question you need to ask the question which component I am using and which interval I am taking it could be a general correlation where you are looking into u' and v' correlation and it could be $x + \Delta x$,$y + \Delta y$,$z + \Delta z$,$t + \Delta t$ if you want to do that go ahead and do you need to ask the question why do I need this.

Now I want to study the correlation in each direction first in x, y, z separately so that I know its length. ok fine So, with this now we have to proceed to what is called the governing equations part, but before that all this we have done here is to we need to apply the statistical analysis part into the governing equation. So, the objective is is to apply what we learnt in the statistical analysis to the governing equations. to see what does the equations tell us right just by doing this decomposition the equations can tell lot of information. So, that is the objective here to apply the statistical analysis to the governing equation.

So, that we can get an equation for the mean fluid motion an equation for the random fluid motion separately ok and then we see how to model them. alright So, before that, how many of you are familiar with Cartesian tensors? At least those who have taken CFD course will be okay. Not everybody. So, let us quickly do this so that everybody is on the same page. We need Cartesian tensors throughout this course.

Equations get very complicated once you start to decompose the governing equations into statistical and random components. So we will quickly review what is called Cartesian tensors, Cartesian tensor notations. will not go into the depth of the entire this tensor calculus we will only look into the extreme basics what is required to write equations and decompose them all right. So, for that part so Cartesian tensor notations are useful to write equations in a compact way. So, we can write equations in a compact way in a compact fashion ok.

And it will also help with ease of programming, it is based on indices which means ease of programming as against using vector notations ok. And it is also easy to represent high rank high ranked matrices So, easy to represent or write high rank matrices. So, those who are familiar would know this otherwise it is not a problem we will quickly get into the basics. So, in the vector notation this I believe everybody will be familiar with you would write let us say velocity vector like this u with a arrow on top this basically implies you are looking into the three velocity components here u, v ,w in Cartesian tensor notation we use an index as I already said is based on indices ok. So, we write this as simply ui.

So, this will be u1, u2 , u3. It will go into an indexed based notation here and there is something called rank of a tensor, rank of a tensor. So, for this we can say let us say this is the rank, let us say we have a zeroth rank. So, a zeroth rank tensor is basically a scalar ok. So, the name for this is a scalar and the notation let us say I am using it as some scalar quantity b and number of components is basically 3 raise to n, n is 0 here.

So, this is the rank. So, this is basically a single component it is a scalar. So, an example could be your temperature, pressure and so on. So, then we have first rank tensor which is a vector instead of this arrow mark we write as I said indice we use an index bi. So, I can take value 1 or 2 or 3 b1 ,b2 ,b3. So, number of components is 3 ok.

So, example is your velocity vector right. velocity vector then you can get a second ranked tensor which is called tensor only this some of you have done even in continuum mechanics would know this people use in solid mechanics also they use the term tensor. So, I represent this as bij because this will give 9 components here a tensor. So, this is 3 raise to So, I get 9 components. For example, your shear stress τ_{ij} and then you have a third rank tensor.

This we will encounter in the turbulence class. So, we have this also called tensor. There is no special name for it. Let us say I call b_{ijk}.

this will give 27 components ok. So, there are some special tensors I will see what for example, there is something called a Levi-civitas tensor that we will use in the course ϵ_{ijk}. So, now we will simply look at the components, corresponding components of Cartesian terms with respect to the vector equivalents. So, before that let us see there are three rules of tensors, three simple rules. One is called a free index rule. So, what this essentially a free index rule means that, if I take an example it will be much more clear.

So, let us say I take an example as $a_i + sb_i = 0$. So, now when I an index i occurs here, if i in the first term is taking a value, if i takes 1 here, then the second one also it should take the same term. So, the index occurring only once is valid for the values 1, 2 or 3. So, this will be for i = 1, i is a free index here. So, this becomes a1 + s is scalar b1 = 0.

It cannot be 1 here and 2 there. And you can write this as, since it is a free index, you can change it into $a_j + sb_j = 0$ or k or anything. But you have to, if you are replacing the index, a free index, it has to be consistent throughout. This will give the same thing, j = 1 will give me the same answer. ok j = 1 will give me the same thing $a_1 + sb_1 = 0$. So, this is a free index we need to if it comes be careful that it can take value 1, 2 or 3.

And the most important one that we keep on using is called an Einstein summation rule, Einstein summation rule. So, this means that if indices are occurring twice in the same term, we have to sum it up. If an index occurs twice, index occurs twice, sum over this. Index occurs twice in the same term.

It is important. in the same term sum over it. I will give an example. Let us take an example of let us say $a_i\, b_i = 0$. What this means is? i is repeated twice, so it has to be summed over. So, this means $a_1\, b_1 + a_2\, b_2 + a_3\, b_3 = 0$ summing over all the three components.

So, obviously you see that the summation rule reduces the rank. So, a_i is a vector, b_i is a vector, but since they occur together in the same term by summing up it has become scalar. So, this has become a scalar here. So, the summation rule reduces the rank of a tensor and there is something called a maximum rule. So, this essentially implies that an index cannot occur more than twice in the same term.

An index cannot occur more than twice in the same term. For example, this a_i, b_i, c_i this is not allowed ok not allowed index cannot repeat more than twice and if it repeats twice you know you have to do the summation rule, just sum it over. These three are the simple rules that we need to know and then we look at two special tensors that we will use in the turbulence equations, special tensors one is called a Kronecker delta. So, it is Δ_{ij}, it is also called an identity tensor, also called identity tensor.

So, Δ_{ij} can take values 1 or 0. you can take value 1 when i = j, take 0 when i not equal to j ok. And then there is another tensor that we use, another special tensor is this Levis-civitas tensor, it is a third ranked tensor. So, this is ϵ_{ijk} So, this can take value 1, 0 or -1. It can take value 1 when i, j, k values are in successively progressive. That means it is for values let us say 123, 231 and 312 cyclic, cyclically varying.

It can take 0. for anything other than the 1 and minus 1. So, minus 1 is it will go in the anti-cyclic route. So, it can take like 321, 213, 132, 0 otherwise. Cyclic, anti-cyclic, this is cyclic occurrence This is anti-cyclic. So, we have learned some tensor rules, some special tensors.

Now, we go on seeing how do we represent the terms that is in the Navier-Stokes equations using tensor notations. So, if I take one of the term is a pressure gradient term. So, certain examples, the first one is let us say the gradient of a scalar field of a scalar. So, this in vector notations let us say it is represented as If you are looking into the scalar quantity as pressure, this is nabla P.

People use this as pressure gradient term. This is represented as your $\frac{\partial P_i}{\partial x_i}$. So, gradient of a scalar, P scalar, but by performing a gradient operation, it has become scalar or vector. It is a vector now, right. So, this is P scalar here. your P is scalar let us say pressure ok, but this has become now a vector this is a vector now because it has 3 terms.

So, we have ∂P we are looking into pressure gradient along x_1 direction pressure gradient along x_2 direction pressure gradient along x_3 direction. So, it has become a vector. So, gradient operation increases the rank of a tensor and in contrast to this there is something called divergence operation. The divergence reduces the rank ok.

So, divergence of a vector field. Again this term is very familiar, if I write it you will know what it is. $\Delta.\vec{u}$, you would have seen this in fluid mechanics, right the divergence of a vector field. So, how does this represent? We represent this simply as $\frac{\partial u_i}{\partial x_i}$. This is your essentially a continuity equation, Δ.u. right So, when index is repeated, i is repeated twice, what you should do? It is a sum, sum over it.

i is repeated, so this will become as $\frac{\partial u_1}{\partial x_1}$ + as $\frac{\partial u_2}{\partial x_2}$+ as $\frac{\partial u_3}{\partial x_3}$. So, this has become a scalar here. So, divergence reduces the rank. These things you do not have to remember, which is increasing the rank, which is reducing the rank. The most important part is to write the equations in tensor notations.

Like $\frac{\partial P_i}{\partial x_i}$is your pressure gradient term, $\frac{\partial u_i}{\partial x_i}$= 0 is your continuity equation or conservation of mass in incompressible flow. Similarly, we can have rotation of a vector field. This also has a very special meaning. People call it curl of a vector or so on. So, what is this? Is there a physical meaning to this? vorticity right So, this has a special meaning called vorticity.

So, this one is we write this here we make use of what is called the third rank Levi-civitas ϵ. So, when vorticity equation is written like this and when you want to compute you get two you know differential operators for the velocity plus there is a sine component plus minus. Usually you mess up, you have to remember, but you do not have to remember if you are going to write it in tensor notations. So, I simply write this as ϵ_{ijk} ok , $\frac{\partial u_k}{\partial x_j}$. Now, you see this k and j are repeated index k , j are repeated twice.

So, we need to do summation twice summation over k summation over j when you do this then i is the only index here free index that means it is a vector. So, this is a vector and how do you write this we write this as ω_i just like a velocity vector ω_i is the vorticity vector. And ϵ_{ijk} will tell us the sign whether it is minus or plus. It can take 1, 0 or - 1. So, we will easily know what is the sign that comes here before your gradient of the velocity field whether it is - or + very easily.

So, this is represented like this and then we have the Laplacian. This you get in a viscous stress tensor term let us say the Laplacian. also comes when if you are going to solve the

Poisson equation for the pressure. So, this is Δ^2u let us say this is written as$\frac{\partial}{\partial x_j}(\frac{\partial u_i}{\partial x_j})$. So, tensor notations reveals the true form what it looks like what which differential operator you are going to use in which direction and yet compact much better than the vector notations and with vector notations you cannot proceed in turbulence. we have to proceed only in the tensor notations when we average the equations to get the statistical equations for the statistical motion and equations for the random motion of the fluid all right. So, we stop here today and I will take up any questions if you have.