Course Name: Turbulence Modelling

Professor Name: Dr. Vagesh D. Narasimhamurthy

Department Name: Department of Applied Mechanics

Institute Name: Indian Institute of Technology, Madras

Week - 7

Lecture – Lec38

38. New model k- ε and model constants – I

Let us get started. In the last class, we looked at the complete k epsilon model constants like all the 5 constants how it came about for a standard k epsilon model. And we also just cursorily you know I took a look of the modeling philosophy behind an RNG k epsilon model where one of the constants is computed using ratio of 2 timescales. And then we also saw one equation example of an one equation eddy viscosity model, right? The Prandtl's model. There are other models as well for this, and as I told I am not going to every eddy viscosity model, I will not discuss that will be the entire semester.

But there are some important eddy viscosity models that you will come across. So, I will only deal with that one and one such one is what is called a k omega model ok. So, we look at this today. k omega model all right.

So, first, we will see why we need a k omega model when we already have, let us say, the standard k epsilon, right? It looks like it is functioning well, and it is, you know, at least the standard k epsilon model is very popular, numerically stable. So, we will see why do we need an another model actually, ok?. So, this reference for this is Wilcox 1988, this is the article. There is also a book written by him which has lot of information on k omega based models. So, if we going to look at again a turbulent boundary layer ok.

So, the same example of a turbulent boundary layer. So, we have a wall and then a turbulent boundary layer growing over it. So, what are the boundary conditions that are required? So, we have the velocities, right? $\bar{u}, \bar{v}, \overline{w}$. So, the velocities requires boundary condition both in the this is free stream if you are going to simulate let us say a turbulent boundary layer and also on the wall boundary conditions. So, wall boundary conditions and free stream boundary conditions are required.

So, let us say this is the y direction, this is the x and out of plane is z. So, at let me use another color at y equal to 0 that is the wall, right? So, your $\bar{u}, \bar{v}, \overline{w}$ goes to 0 velocity that is known and together with it, we are computing in k epsilon model two extra parameters k and epsilon. So, what will happen to k at wall y equal to 0 goes to 0, right? k is made up of correlation of velocities. So, velocity goes to 0 its fluctuation will also goes to 0. What will happen to epsilon? Epsilon goes to maximum epsilon.

The exact expression for epsilon is the correlation of velocity gradients $\left(\frac{\partial u_i'}{\partial x_j}\right)^2$. So, the gradients are large there. So, epsilon becomes maximum. So, epsilon goes maximum epsilon becomes maximum on the wall. So, we need to know we cannot define what is maximum you do not even know right.

So, that there is an issue there it is not an issue there are solutions for it, but as of now it looks like it is not an easy boundary condition to give. something to give like maximum epsilon is maximum I do not know what to that I will come to that what can be done. But at least you know it is not giving any so far any stability issue. So, I am essentially looking into numerical stability to see whether any issue occurs here, ok? So, with this let us consider the model parameters and see their stability, right? So, consider the k epsilon model or k epsilon model components or k epsilon model behaviour, k epsilon model behaviour in the near wall zone, in the near wall zone, and close to the wall. and the focus is on stability not accuracy.

So, focus is on numerical stability not accuracy. So, let us look at some of the terms these terms are that are going to 0 is fine epsilon becoming maximum it does not cause any numerical stability issue if you figure it out how to give a boundary condition which can lead to maximum epsilon. But let us look at other components. So, we have eddy viscosity that you have to compute first in a k epsilon model right. So, we have eddy viscosity ν_t which is equal to $C_\mu \frac{k^2}{\varepsilon}$.

So, y goes to 0, that means as you approach wall, y goes to 0, your k as I said goes to 0 and epsilon goes to maximum. So, what will this approach to? 0 only, epsilon goes to maximum. So, ν_t approaches 0, stable. I am not looking into the accuracy issue. Only looking into whether it is numerically stable.

So, this is completely fine here, stable here. So, now what are the other things that I have to compute? I have to compute k and epsilon to get eddy viscosity. So, let us look at the k model equation. So, I have $\frac{\partial k}{\partial t} + \overline{u_j}\frac{\partial k}{\partial x_j}$ equal to the diffusion term $\frac{\partial}{\partial x_j}$ of, I have only two diffusions that we modelled that is viscous and turbulent. $\frac{\nu_t}{\sigma_k}\frac{\partial k}{\partial x_j}$ and then the production rate terms, so-called Pk minus epsilon.

So, here again, we look at what happens to these terms as I approach the wall. So, y goes to 0 again same thing here k must go to 0 epsilon going to maximum. So, these are gradient terms here: the first, second and third, right? All these are gradient terms. So, it will not cause any numerical stability issue, only gradient terms here. So, they are fine, gradient terms.

So, this is fine, this will not cause any numerical stability issue. So, let us look at then the Pk and epsilon. So, we have now Pk, Pk model component, that is the production rate of turbulence kinetic energy. You can write the Pk model if you want; Pk is equal to your using Boussinesq, we have $2\nu_t\overline{S_{ij}} - 2/3k\delta_{ij}$, and Boussinesq closure for Reynolds stresses followed by your mean strain rate.

Again, y tends to 0. k tends to 0; epsilon goes to maximum. So, ν_t already is going to 0 here. So, ν_t will not cause any problem ok. S_{ij} is again gradient term here. This is a gradient term that will not cause any issue.

You can compute the gradient close to the wall and k goes to 0 right. So, the k also going to 0, k goes to 0 ν_t goes to 0, no issue. This is again gradient. So, these are all fine. So, Pk model is numerically stable.

This is also stable, just like eddy viscosity. So, far so good. Now, what are the other terms we have? We have epsilon, for epsilon we compute an epsilon model equation. So, let us go back and see epsilon model equation whether that is also giving any numerical stability issues. So, I have dou epsilon by dou t plus uj bar dou epsilon by dou xj equal to dou by dou xj of the diffusion rate which is nu plus nu t by sigma epsilon dou epsilon by dou xj plus P epsilon minus let us say symbolically I am writing it as epsilon epsilon, but that is not the one.

$$\varepsilon_{model}\ eqn \Rightarrow \frac{\partial \varepsilon}{\partial t} + \bar{u}_j\frac{\partial \varepsilon}{\partial x_j} = \frac{\partial}{\partial x_j}\left(\left[\nu + \frac{\nu_t}{\sigma_\varepsilon}\right]\frac{\partial \varepsilon}{\partial x_j}\right) + P_\varepsilon - \varepsilon_\varepsilon$$

So, we know what the expression for this actually. So, here let us look at now again same arguments here these are all gradient terms, this entire part gradient terms which are all fine. I will get no numerical stability issue from gradient terms. So, we will see what happens to this P_ε, how we model this? P_ε is essentially depends on production rate followed by there was a time scale. So, P_ε model is nothing but $C_{\varepsilon 1}P_k\varepsilon/k$.

So, now, if I just look at this And of course, as I said, y tends to 0, k tends to 0, epsilon should go to maximum. Now, this appears like it is going to give a stability issue. P_ε model as y tends to 0, k tends to 0 means P_ε becomes tends to infinity, but it is not. If you implement it like this, yes, let us expand what is Pk.

because Pk is fine. So, let us see what happens to that. So, sometimes, while writing code, you have to be careful how you write it. If you are going to implement in a way where the code will blow up that is you will have you will get something like divide by a very small value or 0 then your code can blow up. So, now, if I expand it, I will get the Pk, which is $\left[2\nu_t\overline{S_{ij}} - 2/3k\delta_{ij}\right]\frac{\partial \overline{u_i}}{\partial x_j}(\varepsilon/k)$. So, I have here epsilon by k, epsilon by k.

So, now what is ν_t? ν_t is $C_\mu \frac{k^2}{\varepsilon}$. So, if I substitute that here, I get $C_\mu \frac{k^2}{\varepsilon}$. So, if I have this, then what do I get? 2 third k times, yes. So now, first term, we take it up, which is C epsilon 1 2 C mu k square by epsilon Sij bar dou ui bar by dou xj followed by epsilon by k. Now, you see epsilon epsilon will cancel out, k will cancel out now it has a k dependency.

So, the divide by k that issue which appeared here does not exist when you substitute the eddy viscosity here, right? So, if you implement this form that divide by 0 error blowing up will not occur in the code ok. So, you have to be careful when you try to implement the code, and then you have the other, which is minus, sorry, so it is minus c epsilon 1, 2 third k delta ij dou ui bar by dou xj epsilon by k. This is correct, right? Yes. So, here again, the k will go out, and again, the divide by 0 is gone. So, I have the second term as 2 third epsilon-delta ij; epsilon goes maximum, so not a problem.

$$C_{\varepsilon_1} 2 C_\mu \frac{k^{\cancel{2}}}{\cancel{\varepsilon}} \bar{S}_{ij} \frac{\partial \bar{u}_i}{\partial x_j} \frac{\cancel{\varepsilon}}{\cancel{k}} - C_{\varepsilon_1} \frac{2}{3} \cancel{k} S_{ij} \frac{\partial \bar{u}_i}{\partial x_j} \frac{\varepsilon}{\cancel{k}}$$

So, the conclusion here is that this is numerically stable. So, P_ε model essentially does not give any issues, so it will have only a k dependency part. So, this is stable. So, no issues here with the P_ε also. Even though initially it looks like epsilon by k will cause a problem upon substituting it does not.

Yes. So, then what is the problem? There is one more term. Let us see that whether that gives an issue, which is the dissipation rate of turbulence kinetic energy or the destruction rate of epsilon. So, if I look at that, which is what we are calling epsilon epsilon, which is not, it is just a dissipation rate term of epsilon. So, this is model is C epsilon 2 epsilon epsilon by k. So, again y tends to 0 near wall behavior k goes to 0.

$$\varepsilon_{\varepsilon_{\,model}} = C_{\varepsilon_2}\, \varepsilon \frac{\varepsilon}{k} \longrightarrow \infty \quad \nabla_0 \; \nabla_0 \quad \text{Numerically unstable}$$

$y \to 0$

$k \to 0$

Here there is no help, there is no eddy viscosity, you do not have your friendly neighbor eddy viscosity to come and help you here. Here this will block. So, this tends to infinity. Numerical issue is there, numerically unstable.

you do not want it. I am not even looking into accuracy as I said. We are only looking into whether any term will cause problem as you approach wall. So, this particular term can cause an issue. There are of course solutions for that as well. I am just telling you what is the modeling argument for the genesis of k omega.

I mean somebody has to look at and that is a disadvantage in that model, therefore I am proposing this model. I am proposing no model, I am just explaining to you what is the philosophy. So, now one way of having a solution for this is what is called the k omega model. This is clear so far. You understand that this sinc term in the epsilon model blows up as y goes to 0, you will get this issue here.

So, what we do in the so called k omega model. The first thing is in the k omega model, we must define what is omega. So, an inverse time scale is defined. So, omega is nothing but let there are many ways of doing this omega model. I am just taking on a more like a modeling philosophy idea not go with the exact derivation. The exact derivation for this exists in the textbook, Wilcox textbook or some literature.

I will go ahead with more a modeling philosophy similar to the epsilon transport equation we arrived for the model. So, let omega be equal to $\frac{\varepsilon}{\beta^* k}$. So, $\beta^*$is a constant. So, $\frac{\varepsilon}{k}$ is essentially giving an inverse time scale. That is an inverse time scale turbulent time scale only epsilon by k.

So, with this term, what will happen to all the other parts? So, first of all, we have to see that this when we define this the last term in the k epsilon will not cause an issue that that is what we have to figure it out here. So, let us go back and see first how to get the k and omega equations. So, the k omega equation I will come to later, k more or less, remains the same in any type of k-based model. We saw it in Prandtl's one equation model also, more or less remains the same. Whether it is one equation k or RNG k epsilon also, the k remains the same, just the constants are different, or you take standard k epsilon or k omega, k is more or less remains unchanged.

It is the second equation which is slightly different than the one. So, now let us consider the epsilon model equation that we already looked at, right? So, which is dou epsilon by dou t? I am just rewriting it here plus uj bar dou epsilon by dou xj equal to dou by dou xj; of you do not have to write this again because I have just wrote it in the previous slide, dou epsilon by dou xj plus c epsilon 1 pk epsilon by k minus c epsilon 2 epsilon square by k. Let us call this equation 1 here now; beta is a constant; I will come to that value. So, here beta star is a model constant, all model constants I will give you at the end. So, let us so as I said omega equation can be derived very differently I will go ahead in the direction of the way we obtained epsilon model equation.

$$\frac{\partial \varepsilon}{\partial t} + \bar{u}_j \frac{\partial \varepsilon}{\partial x_j} = \frac{\partial}{\partial x_j}\left[\left(\nu + \frac{\nu_t}{\sigma_\varepsilon}\right)\frac{\partial \varepsilon}{\partial x_j}\right] + C_{\varepsilon_1} P_k \frac{\varepsilon}{k} - C_{\varepsilon_2} \frac{\varepsilon \varepsilon}{k}$$

So, you recall epsilon model equation we did not derive anything from first principles there we just made a numerical argument that epsilon model equation should look appear similar to a k model equation transport equation. Right, epsilon model equation, the way we obtained epsilon model equation. Because only numerical, we placed only numerical arguments for our own convenience. I am going to do something similar here. So, we already have defined what is omega here, right? So, this is our omega epsilon by beta star k.

So, I am going to say that I am going to replace here epsilon by beta star k. Mathematically, this is incorrect because I cannot go ahead and divide epsilon by beta star k, beta star is fine is a constant, but k is a function of time and space. So, I cannot push k inside the derivative. So, this is not, this is mathematically incorrect. I am just going to use only a numerical argument that replacing epsilon by beta star k to obtain an omega equation. So, what I am going to do here is that I am going to say divide throughout by beta star k and set it to omega that is what we want. So, this is of course purely numerical argument. So, note that we cannot really divide the equation like this since k is a function of your time and space. So, I cannot simply push turbulence kinetic energy inside the transport equation here, only a numerical convenience. if I am going to do this all I am going to do now is that I get dou by dou t.

So, instead of epsilon, epsilon by beta star k. So, I will get epsilon by beta star k here plus uj bar dou by dou xj of epsilon by beta star k. This is equal to, so essentially, we are constructing an omega equation similar to the way we obtained the epsilon equation. But an actual root exists that you can go and refer. And then nu plus nu t by sigma epsilon dou by dou xj again going to replace epsilon by epsilon by beta star k plus I have C epsilon 1 Pk by k here, epsilon I am going to replace it by epsilon by beta star k minus c epsilon 2 only one epsilon is replaced by beta star k not the two here for consistency each term we are dividing by k right that we have to remember. So, only one term which is epsilon by beta star k epsilon by k, but the other epsilon k may also want to have a beta star it you know why that is like your a sibling we have to give fruit to both otherwise the other one will cry. So, I will give a beta star also for that going to multiply and divide by beta star for that so that both get both gets to divide by beta star k.

$$\frac{\partial \varepsilon/\beta^* k}{\partial t} + \bar{u}_j \frac{\partial \varepsilon/\beta^* k}{\partial x_j} = \frac{\partial}{\partial x_j}\left[\left(\nu + \frac{\nu_t}{\sigma_\varepsilon}\right)\frac{\partial \varepsilon/\beta^* k}{\partial x_j}\right] + C_{\varepsilon_1} \frac{P_k}{k} \frac{\varepsilon}{\beta^* k} - C_{\varepsilon_2} \frac{\varepsilon}{\beta^* k} \frac{\varepsilon}{k} \frac{\beta^*}{\beta^*}$$

So, if I do this what would happen to my equation now I see that it is dou omega by dou t plus uj bar dou omega by dou xj equal to dou by dou xj of nu plus nu t by sigma omega dou omega by dou xj plus we have C epsilon 1, why C epsilon 1? We will have C omega 1, it is an omega equation, p k by k omega or omega by k basically. We have a ratio of, you can write this as omega by k now, instead of epsilon by k we had omega by k here. minus we have C epsilon 2 one of the beta star and then omega square. So, instead of having C epsilon 2 times beta star I am going to call it C omega 2 because it is two different constants.

So, I will just call it C omega 2. So, this C omega 1, 3 model constants has appeared. So, the last term if you compare equation 1 and equation let us call it 2.

$$\frac{\partial \omega}{\partial t} + \bar{u}_j \frac{\partial \omega}{\partial x_j} = \frac{\partial}{\partial x_j}\left[\left(\nu + \frac{\nu_t}{\sigma_\omega}\right)\frac{\partial \omega}{\partial x_j}\right] + C_{\omega_1} P_k \frac{\omega}{k} - C_{\omega_2} \omega^2$$

So, the last term here, this was the issue which was giving a numerical issue, as y tends to 0, this last term here blows up. That does not happen here, for us here. So, that is one argument. So, because we are essentially not looking here as epsilon or divided by k kind of a thing it is only a argument. So, if at all you want dissipation rate in your data while computing equations using k omega you can use this formula to reproduce epsilon. But in the equation we are only solving for omega which is an inverse time scale. we are never going to solve for anything epsilon and look at ratio of that with k and all this. That is only for you post processing if you want epsilon to compare with some data you can use that formula.

But together with 1 and 2, 1 is also not correct because 1 is equation for the k epsilon model. So, 2 is its counterpart or an equivalent in the omega model. So, we need a k equation in the k omega also that is also required. So, before that we need eddy viscosity. So, this forms one of the equation here. So, omega model equation we have, equation 2. So, now we need, what do we need? We need eddy viscosity, eddy viscosity ν_t. dimensionally again dimensional analysis you can do it and you get essentially k by omega instead of $C_\mu \frac{k^2}{\varepsilon}$ we get k by omega here. So, this is the formula for a the viscosity in a k omega model that gives you say meter square per second dimensionally correct. So, we have model constants that we have to look at. And then I have