Course Name: Turbulence Modelling

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Week - 6

Lecture – Lec34

34. Standard k-ε model and the model constants - I

Okay so let us get started again. So in the last class we looked at modeling the k and ε equation and then one phenomena that we used is that there is some kind of a local equilibrium with respect to the production rate and dissipation rate okay. So I am showing here my own DNS data here. So we are essentially looking into the ratio of this production rate to dissipation rate. This production rate here, this means this is actually P_k over ε .

Capital P here indicates the production rate of turbulence kinetic energy. This is not the model data. This is the exact data coming from DNS calculations. So when I see here this is of course for plotted for two different types of roughness in a plane Couette flow as well as the smooth.

So regardless of whether the wall is smooth or rough you can see that there is a zone where this ratio is nearly 1 right. So the P_k is balancing ε let us say in the middle of the channel here and this is of course much more dominant when we take only a smooth turbulent plain Couette flow that is this black dotted lines. You can see the ratio is more or less 1 in the core of the channel. So there is some kind of a local zone within your turbulent flow where such a equilibrium exists between the production rate and the dissipation rate. So you can say turbulence is in some kind of a local equilibrium.

in that zone and this concept is what we are using when we model the k and ε equation that the production rate of ε it relied on the production rate of turbulence kinetic energy itself right. So, because of the this ratio becoming equal to 1, but if you notice closer to the wall that is where major activity is occurring there this is not true the production rates are two three times larger than the dissipation rates. So, there is no local equilibrium in the zones where there is lot of turbulence generation is occurring and close to the wall and this is true even when you have there is no roughness also even if you look at a smooth wall you see here the peak generation rate occurs in the so called buffer layer as I said that is where you have this peaks located which is about twice that of ε okay. So we make use of this concept in modeling that the flow is locally in equilibrium. The turbulence is in local equilibrium that means locally in some zone within a flow.

This is not true for all types of turbulent flows, but essentially we are using turbulent boundary layers as the starting point for all the modeling considerations and we make use of this to come to some assumption ok. So, we made the so called k-ε model right. this is what we did in the last class. So, we had two transport equations for the k and ε ok. So, in total now we have if I take the three momentum equations.

So, I have three equations here and I also have the continuity equation to help me that gives me one more equation and to close the Reynolds stresses in the mean momentum equation we had Boussinesq and eddy viscosity and to get all the terms there we essentially needed two more transport equations which is our k and ε equation. So an expression any transport equation for k and ε so this gives me two more transport equations to solve. The eddy viscosity and business these are all comes inside the equations here. So, essentially you have this many transport equations to solve here ok. And the only parameters that were not known so far is the 5 model constants.

So, we had C_μ and then sigma k sigma ε and then $c_{\varepsilon 1}$, $c_{\varepsilon 2}$. So, if I know 5 of this model constants then we can go ahead and solve your equation. So, we will see how to get these 5 constants here today. So, for that I will start with the first one which is C_μ to get C_μ. to get C_μ this occurs in the eddy viscosity equation right.

So, ν_t equal to C_μ square by ε . So, C_μ is required in the eddy viscosity which is of course present in many different terms in the other transport equations. So, what I do is to get this we use a particular flow problem ok. So, we consider here consider the log law region of a turbulent boundary layer, smooth turbulent boundary layer. Log law region of a smooth turbulent boundary layer.

So what is so special about this log law region of a smooth turbulent boundary layer when we look into the budget of the turbulence kinetic energy. So we are essentially modeling the k and ε . So if I look at the exact k equation and look at its budget or even if you look at the three normal stresses which are the constituents of the turbulence kinetic energy right, k is half of $u_i^{'}u_j^{'}$. So that budget I have already shown to you in the previous classes. So, if you look into the budget of that you see that there are two dominant terms in the log law region that is the production rate and the dissipation rate all others are somewhat smaller compared to these two.

So, that is why we have taken this space specifically the log raw region ok, but the model is of course, being applied everywhere even when there is no turbulent boundary layer you have to remember this these are model assumptions here I am taking a very small zone of a particular flow. but the model the k ε model will be used for all kinds of flows jets, wakes, combustion wherever you may not even see a turbulent boundary layer or a log law region ok. But these are model assumptions you have to pay attention here. So, if I take a log law region of a smooth turbulent boundary layer ok. So, from the budget from the budget of turbulence kinetic energy we can see we can see that this P_k and ε that is I just showed you in the previous graph where P_k by ε is more or less 1 in the core right.

So, we can see that P_k and ε the production rate of turbulence kinetic energy and the destruction rate of turbulence kinetic energy are much much larger than other terms ok. So, P_k by ε are the dominant dominant terms in the log law region. So, we make use of this. So, if I have this then we have the k model equation right. So, if I take the k model equation So, we already have that equation that we derived which is $\frac{\partial k}{\partial t} + \overline{u}_j \frac{\partial k}{\partial x_j} = \frac{\partial}{\partial x_j}$.

I am just rewriting what has been already done $\left(\nu + \frac{\nu_t}{\sigma_k}\right)\frac{\partial k}{\partial x_j}$ plus your production rate term which is $2\nu_t s_{ij} - - \frac{2}{3}k\delta_{ij}$ times your mean strain rate minus ε . This is your k model equation. So, if I take the k model equation here and we apply this log law region where P_k and ε are dominant terms ok. So, if I consider that then what would happen to the entire as I said in the entire budget for such a problem P_k and ε are the dominant terms. So, I would like these two to balance each other.

So, the equation essentially reduces to. So, in the log law region that is considering turbulence in local equilibrium. Equation 1 reduces to essentially 0 on the left hand side equal to P_k - ε to first order other terms are smaller. So, that means entire left hand side is 0, the diffusion terms are also smaller. So, in such a zone we will have this kind of a consideration.

So, what do I get for P_k - ε = 0? That means equation 1 becomes $2\nu_t s_{ij}$ which is 2 by 2 $\frac{\partial \overline{u}_i}{\partial x_j} + \frac{\partial \overline{u}_j}{\partial x_i}$ alright $- \frac{2}{3}k\delta_{ij}$ multiplied by ok I already have this term which is $\frac{\partial \overline{u}_i}{\partial x_j}$ $- \frac{2}{3}k\delta_{ij}\frac{\partial \overline{u}_i}{\partial x_j}$ = ε . So this is the equation as reduced to this simple form in the log law region. So now I am going to consider the same flow as before right. We have already seen the data that is we have used a statistically stationary flow right.

The data has already been shown to such a problem where the flow has come to a statistical stationary state and then that means this is gone because it is statistically stationary in a fully developed turbulent plane quiet flow or a turbulent pipe flow and so on. And it was statistically homogeneous also right. So, statistically homogeneous to that is you consider the same consider the fully developed turbulent plane Couette flow or any smooth turbulent boundary layer it is going to be the same equation. So, that means, you had $\frac{\partial}{\partial x_1}$ of your average quantity here was 0 statistically homogeneous. statistical homogeneity and then you had $\frac{\partial}{\partial x_3}$ is 0, $\frac{\partial}{\partial t}$ is anyway 0 statistical stationarity and then when it is fully developed we also know from data that this two velocity components are 0, this we have already seen and discussed.

So, I am going to use this considerations in this particular reduced form of the modeled k equation. So, if I do all this, so what do I get? Of course, this is gone. So, essentially I have now $\nu_t \frac{\partial \bar{u}_i}{\partial x_j} \frac{\partial \bar{u}_i}{\partial x_j} + \nu_t \frac{\partial \bar{u}_j}{\partial x_i} \frac{\partial \bar{u}_i}{\partial x_j}$ ok. And then I have $-\frac{2}{3} k\delta_{ij}$.

So, this particular term exists only when δ_{ij} is 1 ok. When δ_{ij} is 1 what will happen to $\frac{\partial \bar{u}_i}{\partial x_j}$ 0 incompressible flow right. So, this term should go away incompressible flow right δ_{ij} equal to 1 incompressible flow right. So, this term goes away. So, I get only this consideration here on the left hand side.

So, if I now expand these two terms this is repeated indices on the right hand side I have a scalar. So, both of this two terms has to be scalar, i is repeated twice, j is repeated twice. So, I need to expand this Einstein summation that you all know. So, I get ν_t this if I expand I get dou let us set i = 1 now $\frac{\partial \bar{u}_i}{\partial x_j} \frac{\partial \bar{u}_i}{\partial x_j}$, ok.

So u_2 I do not even have to look into that part but anyway I am now expanding first in I sum of three terms here which is $\frac{\partial \bar{u}_2}{\partial x_j} \frac{\partial \bar{u}_2}{\partial x_j}$ plus $\frac{\partial \bar{u}_3}{\partial x_j} \frac{\partial \bar{u}_3}{\partial x_j}$. I can readily see that this term is gone because $\bar{u}_2$ is 0 in such a flow and $\bar{u}_3$ is 0 ok here. So, I am considering a very specific flow problem and a local zone there also. So, I have only one particular term on the left hand side remaining.

So, we will see the other term which is plus ν_t this particular term again I said let us say i = 1 now. So, I get $\frac{\partial \bar{u}_j}{\partial x_1} \frac{\partial \bar{u}_1}{\partial x_j}$ ok. So, now what is $\frac{\partial}{\partial x}$ of any statistical quantity? 0 here statistical homogeneity. So, $\frac{\partial}{\partial x_1}$ of this is 0 right homogeneous right this is fully developed here this is because it is fully developed. So this entire term goes away here $\frac{\partial}{\partial x_1}$ of this is gone even if we expand it you will see that eventually this entire term goes away the cross term that is ji ij term goes away equal to your ε .

So now I have this on the left hand side. So if I expand this further I get ν_t times, now I sum in the j direction. So, I get $\nu_t \left(\frac{\partial \bar{u}_1}{\partial x_1}\right)^2 + \nu_t \left(\frac{\partial \bar{u}_1}{\partial x_2}\right)^2 + \nu_t \left(\frac{\partial \bar{u}_1}{\partial x_3}\right)^2 = \varepsilon$ again now statistical homogeneity for $\frac{\partial}{\partial x_1}$ $\frac{\partial}{\partial x_3}$ homogeneous along these two direction turbulence is in local homogeneity in these two directions. So, essentially I get only one term. So, colloquially if you write this, this is basically dou u bar by dou y.

if y is the wall normal coordinate right. So, I can write this as mu t dou u bar by dou y square equal to ε . So, your entire equation the k model equation reduces to this for a statistically stationary, statistically homogeneous flow along the x direction and the z direction. and considering the log law region of that flow where only P_k and ε are dominant terms. If you consider other zones close to the wall viscous terms are dominant as I said then you cannot get rid of the diffusion rates they become important.

So, I am considering only the log law region of a statistically stationary statistically homogeneous in two direction that kind of a flow. and I get this simplified form of the k model equation. Now nu t knows about C mu the whole objective is to get C mu here. So, what I do now is I would like to multiply both the sides now with mu t. So, I get so multiply both sides by ν_t I get $\nu_t \left(\frac{\partial \bar{u}}{\partial y}\right)^2 = \nu_t\, \varepsilon$.

Now, this $\nu_t \frac{\partial \bar{u}}{\partial y}$ does it ring a bell? Does it remind you of a term that you already know? turbulent shear stress using Boussinesq right. So, the Boussinesq gave us so if you recall here the - $u_i' u_j'$ over bar was your $\nu_t \frac{\partial \bar{u}_i}{\partial x_j}$ particular term. So, the $-\overline{\frac{2}{3} k \delta_{ij}}$ is basically this this particular term from Boussinesq you know that. you can I can write here. So, from Boussinesq it is $-\overline{u_i' u_j'}$ is nothing but your 2 ν_t S_{ij} $- \frac{2}{3} k \delta_{ij}$ this particular term is there.

So, considering this particular flow the last term I mean for a shear stress let us say i = 1 j = 2. If I want this particular term I get - $u_1' u_2'$ the last term goes away because δ_{12}. right so that is 0. So, I essentially get ν_t of $\frac{\partial \bar{u}_1}{\partial x_2} + \frac{\partial \bar{u}_2}{\partial x_1}$. So, again here this $\frac{\partial}{\partial x_1}$ or the u_2 term both are anyway gone for this particular flow I essentially get this form here.

So, this is nothing but - u' v' $\nu_t \frac{\partial \bar{u}}{\partial y}$. So, I am going to substitute that here. The $\nu_t \frac{\partial \bar{u}}{\partial y}$ that particular term is there. So, therefore, I get here therefore, I get - u' v' term. of course, whole square is equal to $\nu_t \varepsilon$.

But what is ν_t? ν_t from the model is $C_\mu \frac{k^2}{\varepsilon}$. So, $C_\mu \frac{k^2}{\varepsilon} \varepsilon$, these two cancel out. So, I get $\left(\overline{u'v'^1}\right)^2$ is nothing but or I can write this as basically C_μ is now equal to your $\left(\frac{\overline{u'v'}}{k}\right)^2$. So, the whole idea of doing all this now is essentially that in experiments it is much more easier to measure in the logarithmic zone and the quantities that are present there especially the correlation term. $\overline{u'v'}$ is easier to measure turbulence kinetic energy you can get from the three normal stresses.

So, from experimental data we can look at the logarithmic zone or even from A DNS data we can look and see what is this ratio looks like. for many Reynolds numbers, it is not just working for one. So, we have taken a flow no doubt, but at different Reynolds numbers and also for different flows where turbulent boundary layer is present. So, we can do that and look at what this value should be. So, if one does this, this ratio essentially comes to.

So, this ratio essentially becomes C_μ from experimental data if I put this. So, this is basically square root of C_μ will be $\frac{\overline{u'v'}}{k}$ will be 0.3 from experimental data ok. This is experimental data in the log law zone that 0.3 is a ratio of the turbulent shear stress to turbulence kinetic energy ratio.

So, obviously with this I get C_μ as equal to 0.09. So, this is the model constant that you will see in the literature. C mu value those who have perhaps use this you would know that the value is 0.09 for all the k-ε models and this is how this value has arrived one of the constant and we still have 4 more constants to go. So, we have another constant that we look today.