**Course Name: Turbulence Modelling**

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**Week - 4**

**Lecture – Lec22**

## 22. Turbulent Boundary Layer: Order of Magnitude analysis - I

Let us get started again with the turbulent boundary layers. So, to understand the nature of a turbulent boundary layer, I would like to consider a certain class of flow and then apply the equation that we have derived ok. So, first I would like to consider So, I would consider a thin boundary layer, a thin boundary layer, it could be a thin shear layer also. So, what this thin boundary layer means is that we would need two length scales. So, if I take the let us say this is a wall, an infinitely long wall.

then we have this boundary layer growth. So, let us say this is our growth of the boundary layer. Then we are saying that we need two length scales. This is L which is a function of x, x is the x coordinate here.

Let us say I am using x y coordinate here to make it simple and this is your the thin region that Prandtl talked about that is the boundary layer thickness which is a function of x. As you walk along the x direction, the boundary layer grows. So, a thin boundary layer implies that you would need two length scales here to define this flow. One is L, a horizontal length scale, $\Delta$ is a vertical length scale, both functions of x. So, when I say consider a thin boundary layer, it means two length scales required to define the flow ok.

So, that means we have here we have chosen L of x and $\Delta$ of x in the x, y directions respectively. Two scales. So, we are introducing scales now. We are already taking a problem. So far, we have done only governing equations.

Now, we have taken a problem and we are introducing scales to see how turbulent boundary layer looks like. Two scales introduced and we will also consider statistically stationary flow. The flow has come to a statistical stationary state. So, consider statistically stationary flow. So, this implies that the $\frac{\partial}{\partial t}$ of the average term is 0.

The mean is it is a statistically stationary flow, I am not talking about statistical stationarity that you achieve in any. This flow I am taking it as statistically stationary flow that is the mean is no longer changing with time in this problem ok and then I would also consider statistical homogeneity in one direction. So, I am going to take a planar boundary layer. So, turbulence is still 3D here, but from mean flow dynamics point of view, it will be a planar boundary layer. That means the out of plane z direction, I am taking it as homogeneous.

We already studied what is statistical homogeneity. So, I am considering statistically homogeneous homogeneous turbulence along z direction. This implies $\frac{\partial}{\partial z}$ of any statistical quantity is 0. So, I have this condition. So, with this condition now we will look into our RANS equations.

What happens to RANS equations? So, consider RANS equations now. Constant density isothermal flow, right. So, ρ is constant which means it is isothermal flow. So, the equation is essentially you have your $\frac{\partial \overline{u}_i}{\partial t} + \overline{u}_j \frac{\partial \overline{u}_i}{\partial x_j} = -\frac{1}{\rho} \frac{\partial \overline{p}}{\partial x_i}$ + I have the viscous term u $\frac{\partial^2 \overline{u}_i}{\partial x_j^2}$ - $\frac{\partial}{\partial x_j}$ the Reynolds stresses.

Of course, I also have the continuity equation $\frac{\partial \overline{u}_j}{\partial x_j}$ equal to 0. This is my Reynolds equation for an incompressible flow. Now, with the consideration I have a statistically stationary flow, this goes away. because of this condition here statistically stationary flow. So, bean flow dynamics is no longer changing with time right.

So, now we will expand other terms to see what it looks like and I have also the third direction gradients along the third direction, gradients of the mean quantities along the third direction is 0 ok, gradient of not the fluctuation statistically homogeneous. So, if I expand it I would get if I write in terms of x y coordinate i equal to 1. So, I would get essentially a u momentum equation, u bar equation which will be I will have $\overline{u}\frac{\partial \overline{u}}{\partial x}$. $\overline{p}\frac{\partial \overline{u}}{\partial y}$. You can write in the Cartesian tensor also.

Since it is coming to a planar and we would not need so many, it is only a scale analysis. So, I do not need a tensor so much. So, I am going back to this x y coordinate system. And then I have $-\frac{1}{\rho}$.

So, i = 1. So, pressure gradient is applied along the x direction. So, I have $\frac{\partial \overline{p}}{\partial x}$.. And so, the third one is already 0 here. So, I am not writing it here, but I can write it down. The third one which is your $\overline{w}\frac{\partial \overline{u}}{\partial z}$ , this is 0 because of the statistical homogeneity along the z direction.

So, I have $-\frac{1}{\rho}\frac{\partial \overline{p}}{\partial x}$ + again the three components one $\frac{\partial}{\partial x}$ will be 0 here. So, I have $\nu\frac{\partial^2 \overline{u}}{\partial x^2} + \nu\frac{\partial^2 \overline{u}}{\partial y^2} + \nu\frac{\partial^2 \overline{u}}{\partial z^2}$. again $\frac{\partial}{\partial z}$ term is gone and then I have the Reynolds stress terms, 3 Reynolds stress terms minus $\frac{\partial}{\partial x}$ of your, this is j is repeated, u is there, u' is always there because i equal to 1. So, u' u' $\frac{\partial}{\partial y}$ u' v' - $\frac{\partial}{\partial z}$ u' w' $u_1u_1$ $u_1u_2$ $u_1u_3$ Again $\frac{\partial}{\partial x}$ is gone because of statistical homogeneity.

So, I get only a reduced form of u momentum equation here. Similarly, you can get v also equation. Let us say if I take i = 2. which is the V momentum equation. I will get $\overline{u}\frac{\partial \overline{v}}{\partial x} + \overline{v}\frac{\partial \overline{v}}{\partial y}$ .

Here also one of the term goes away which will be equal to rate minus $-\frac{1}{\rho}\frac{\partial \overline{p}}{\partial y}$ and then I have + $\nu\frac{\partial^2 \overline{v}}{\partial x^2} + \nu\frac{\partial^2 \overline{v}}{\partial y^2}$ -, here - - $\frac{\partial}{\partial x}$ (u'v') - $\frac{\partial}{\partial y}$ (v'v'). another term $\frac{\partial}{\partial z}$ terms are gone. So, there are three terms here again is going away. And of course, I have the continuity equation also. So, this equation here gets out to be for $\frac{\partial \overline{u}}{\partial x} + \frac{\partial \overline{v}}{\partial y} + \frac{\partial \overline{w}}{\partial z} = 0$.

again statistical homogeneity alongside. So, these three equations here constitute say this equation let us call this 1, 2 and this 3. This is typically called shear layer equations ok, planar shear layer equation. for a thin shear layer or a thin boundary layer.

So, I have the equation now. So, now I am going to introduce some scales to see what is called an order of magnitude estimate. So, the objective now is that in the I will consider the U momentum only now. For the U momentum equation I would like to see which is the largest term on the left hand side and which is the largest term on the right hand side which is going to balance this. It need not be a largest term on the left hand side also. Basically, I would like to see which are the two dominant terms that are balancing each other in a turbulent boundary layer.

For that I would like to do what is called order of magnitude estimates. Some of you would have done this order of magnitude estimates ok. For that I would like to choose a scale. So, we do what is called order of magnitude estimates. So, here I am going to choose a scale.

So, before that if I mention let us say if I say $\sim$ here, this symbol I am going to use for what is called order of magnitude. If I am going to say this, if I said two terms left and right side of this symbol that means they are of the same order of magnitude. If I am saying $\bar{u} \simeq \bar{v}$ means $\bar{u}$ and $\bar{v}$ i am considering of the same order of magnitude not same value same order of magnitude let's say $\bar{u}$ is one meters per second let's say $\bar{u}$ is like five meters per second then $\bar{v}$ is let's say two meters per second obviously they are different but same order of magnitude okay it's not 10 times smaller 100 times smaller but only two times smaller same order of magnitude so this means order of magnitude symbol order of magnitude. Then I am going to consider as I already said I needed two scales here, a longitudinal scale and a vertical scale L and Δ. So, we will use L as a generic streamwise length scale ok.

So, I am going to use L as a, so I will take this as a generic, this is basically a generic streamwise length scale You all understand what is a length scale, velocity scale, time scale right. I mean we are using scales here right to determine whether it is should be in feet or should be in meters. So, we use a particular scale to analyze and turbulence has turbulence is characterized by eddies, vertices of different length. and some can be as big as a kilometer and some can be like a micron. So, we are introducing scales here right that is the idea of a length scale the size of an eddy you can think of and then we have the Δ which is also a generic lateral length scale.

And then I need a velocity scale here for the two directions for the x direction and the y direction. So, I am going to choose a generic velocity scale. I will call this let $u_s$ say $u_s$. This is my generic stream wise velocity scale. And I will have $V_s$ as the generic lateral velocity scale, like your uv component.

These are the scales that I have chosen. And then I will have, if I say $\frac{\partial u}{\partial x}$. I would take this particular thing as L or sorry I will have used this as when I say $\partial x$ I am going to use L here, the scale that I am going to use for $\partial x$ and for $\partial Y$ it is Δ. the two scales that I am going to use here in the equations. And for the velocities I have if I now look into the equation I have $\partial x$ and $\partial y$ terms $\frac{\partial}{\partial x}$ $\frac{\partial}{\partial y}$ that I am using two different length scales here.

And I also have velocity $\overline{uv}$ and also changes in the velocity $\partial \bar{u}\, \bar{v}$. $\partial \bar{v}$and so on. So, for $\bar{u}$ it is $u_s$ as before. So, I am going to use $\bar{u}$ is same as for $\bar{u}$ I am using $u_s$ velocity scale, but then I have changes in u right $\partial \bar{u}$ $\partial^2 \bar{u}$ the second order terms changing. So, here I am going to consider that this changes in u either first order or second order terms they are of order of magnitude different than u bar ok.

So, $\partial\overline{u}$ $\partial^2\overline{u}$ I am going to consider it as an at least an order of magnitude it can be more than that. So, let us say it is at least 10 times smaller and it could be 100 times or 1000 times smaller also, but at least an order of magnitude different. So, I cannot use the same scale u s, I need a different scale. So, for that I will use that is the changes in u. So, I am going to use for this $\partial\overline{u}$ and $\partial^2\overline{u}$.

For both of this I am going to use a scale called $\Delta$ $u_s$. not $u_s$ but $\Delta$ us that means $\Delta$ $u_s$ is at least an order of magnitude different than $u_s$. So, that means here $\overline{u}$ and the changes $\partial\overline{u}$ or $\partial^2\overline{u}$ are of different orders of magnitude. So, what else I have? I have $\overline{u}$ , $\partial\overline{u}$ sorted out left hand side is fine.

I have $\overline{v}$ , I will come to that. I also have yeah, so I have the lateral velocity component $\overline{v}$ and also its changes. So, since it is a thin boundary layer, the $\overline{v}$ itself is tiny compared to $\overline{u}$ right. the x momentum velocity is large compared to the lateral. So, $\overline{v}$ and the changes in $\overline{v}$ I am going to take the same scale. So, there is no you know like a golden rule here that I should choose this particular scale or that you have to make an argument and choose a scale.

if you are if you do not have a thin boundary layer right if your $\overline{v}$ has significant changes then you choose a different scale for $\overline{v}$ r and changes in that so for the flow that i am considering i am making an argument that $\overline{v}$ and $\partial\overline{v}$ $\partial^2\overline{v}$ all will have same orders of magnitude okay that does not mean that they have same values So, let us say $\overline{v}$ is 0.1 meters per second, $\partial\overline{v}$ maybe let us say you know 0.2 or so it is the same order of magnitude that is what we are talking about. So, I would take for $\overline{v}$ $\partial\overline{v}$ $\partial^2\overline{v}$ for all these components I am taking a single velocity scale called Vs.

That's the lateral velocity scale that we have. So that means it's $\overline{v}$ is small here, right? $\overline{v}$ is small. And V∞ bar, if you go to the free stream, it is zero in a thin boundary layer. The same thing applies also to if you are doing a thin shear flow that is for a jet wake or other flows also if your growth of the your shear layer the growth of the jet is as I said it is a thin jet that you are taking the $\Delta$ is an order of magnitude smaller than L that kind of a jet if you have you can use the same scale. But if $\Delta$ is growing the $\Delta$ there is of course, the let us say the diameter of the jet if that is growing very fast comparable to L then you can use the same scale or a different scale you choose depending on the flow.

For the flow I have taken this seems reasonable ok. for this and then I have the Reynolds stress component right. I have this $\overline{u'u'}$ $\overline{u'v'}$ and other components right. So, here I am going to take the same turbulence velocity scale. I cannot use $u_s$ and $v_s$ because this is the momentum mean momentum. So, turbulence velocity I want it to be different not use the same, but If you recall the data that I have shown to you once about the rms data right I showed you a graph with u v w rms in a plane turbulent couttee flow.

There the values are different showing anisotropy, but they are same orders of magnitude. If you recall that figure they were like the u rms let us say 0.

5 then v rms is 0.3 and w rms is 0.2 or something. they were of the same order of magnitude, but exhibiting anisotropy. So, we are looking into only order of magnitude. So, for that reason I am going to take the for the turbulent components $\partial^2$ sorry $\partial \overline{u}\,\overline{u}$ and then I have $\partial \overline{v}\,\overline{v}$ and then the shear term $\partial \overline{u'v'}$. for all this I am taking a single turbulence velocity scale. So, this is here the small u is nothing but a generic turbulent velocity scale.

So, this implies that they are of the same order of magnitude, the u rms v rms and this u prime u' we can think it off as u rms v rms multiplied together. So, we are thinking of that kind of a quantity right. So, same order of magnitude here that is you have your u rms is of the same order of magnitude as v rms, same order of magnitude. So, now I think we have all the scales ready except one that is v bar and also p bar for the pressure and the velocity.

So, what do I do for the v bar is essentially I use the continuity equation 1. So, if I take here if I say using continuity here using equation 1, I have $\frac{\partial \overline{u}}{\partial x} = -\frac{\partial \overline{v}}{\partial y}$. So, I can simply say here that dou u bar I have a scale $\Delta$ $u_s$ by L is of the same order of magnitude as. though $\overline{v}$ that is Vs a generic velocity scale. I am just putting the scale that I have already taken here and $\Delta$.

So, this means that this gives me basically $V_s$ is nothing but I have $\Delta$ $U_s$ $\Delta$ by L. So, $\Delta$ by L indicating the boundary layer growth. right the both are functions of x, $\Delta$ is the boundary layer thickness. So, $V_s$ I have got.