Course Name: Turbulence Modelling

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Lecture – Lec20

20. Production rate of TKE and Mean TKE - I

Let us get started. So, in the last class, we looked at turbulence kinetic energy equation, turbulence kinetic energy equation. Or you can also say the kinetic energy associated with the turbulent fluid motion or fluctuating fluid motion. Kinetic energy associated with fluctuating fluid motion. That is a random component here, right? Random or turbulent, right? And we have a transport equation, which looks like we have $\frac{\partial k}{\partial t}$.

I am just rewriting what we derived yesterday $\overline{u}_j \frac{\partial k}{\partial x_j}$. equal to I have the right hand side terms which is the many different rates. So, I have the diffusion rate first which is dou by dou xj of minus 1 by rho p prime ui prime delta ij which is the pressure diffusion rate minus ui prime ui prime uj prime, the turbulent diffusion rate plus nu dou k by dou xj, the viscous diffusion rate. And then I have the production rate term, which is minus ui prime uj prime dou ui bar by dou xj, and then the dissipation rate minus nu dou ui prime by dou xj squared averaged.

$$\frac{\partial k}{\partial t} + \bar{u}_j \frac{\partial k}{\partial x_j} = \frac{\partial}{\partial x_j}\left\{ -\frac{1}{\rho}\overline{p' u_i'}\,\delta_{ij} - \frac{1}{2}\overline{u_i' u_i' u_j'} + \nu \frac{\partial k}{\partial x_j}\right\} - \overline{u_i' u_j'}\,\frac{\partial \bar{u}_i}{\partial x_j} - \underbrace{\overbrace{\nu\,\overline{\left(\frac{\partial u_i'}{\partial x_j}\right)^2}}^{\varepsilon}}_{\geq 0}$$

This particular term is what we also define as epsilon, and this is always greater than or equal to 0; it is positive, and there is a negative sign, and therefore, this is a negative rate of change of turbulence kinetic energy in the system. Therefore, the name came dissipation rate and we also discussed the diffusion rate. This is the only term which is, this is the ∇. term. This is the only divergence term that is present in the equation and therefore, we could do a volume integral of that one use Gauss divergence theorem, convert it into a surface integral, apply boundary condition and we see that the net contribution of this term is 0. And only for this particular term we can apply the boundary condition.

So, some ask the question why ui prime is going to 0 on the wall then that is present everywhere. Why is it not being applied? Because we did not, that is not a divergence term. So, we cannot do a volume integral and apply the Gauss divergence theorem there, right? So, now the only question remaining is why this particular term is called production rate. Without doing any simulation or experiments, we have kind of come to an agreement that the epsilon is the dissipation rate and, the divergence term is the diffusion rate, the transport term and also, we found that there is a pressure strain rate term in the Reynolds stress equation which is doing redistributing. So, we already agree that turbulence is a random process; it dissipates energy, transports energy or any scalar quantity that you have, and diffuses.

So, turbulence is diffusive, dissipative and it is random and it is redistribution also, it redistributes. Only thing is the production we have to see. For that I need to derive one more transport equation. So, for that let us look at the, there was a question from a student, what about the total kinetic energy? So, let us look at, if I say, the total kinetic energy, the mean total kinetic energy, if I define for this if I have to define this as $\frac{1}{2}u_i u_i$, where u_i represents the instantaneous velocity here.

So, I can do a Reynolds decomposition here, right? So, this I can write this as using a Reynolds decomposition, half of I have ui bar plus ui prime ui bar plus ui prime average throughout. I am simply doing a Reynolds decomposition here. Averaging is anyway applied, the ensemble averaging. So, I get half ui bar ui bar ok, and then the other term which survives is half of ui prime ui prime. And the other one which does not survive is this term ui bar ui prime average ok.

$$\text{Mean total kinetic energy} \Rightarrow \frac{1}{2}\overline{u_i u_i} = \frac{1}{2}\overline{(\bar{u}_i + u_i')(\bar{u}_i + u_i')} = \frac{1}{2}\bar{u}_i\bar{u}_i + \frac{1}{2}\overline{u_i' u_i'} + \frac{2}{2}\cancelto{0}{\overline{\bar{u}_i u_i'}}$$

So, this term does not survive because its average of a fluctuation is 0, ui bar is the ensemble mean. So, average of a random fluctuation is 0. So, I get only 2 terms that is remaining. So, the mean total kinetic energy can be split into the kinetic energy associated with the mean fluid motion, kinetic energy associated with the turbulent fluid motion. So, this is the or I can write it like this two parts, this is the kinetic energy of the mean fluid motion and this is the turbulent kinetic energy that we have already seen.

This is your turbulent kinetic energy or you can say kinetic energy of the turbulent fluid motion. Kinetic energy of the turbulent fluid motion. or random or fluctuating fluid motion. Symbolically, I can write this equation as is equal to. I can write this as a $K + k$, mean kinetic energy and the fluctuating kinetic energy, two parts.

Now, I have a transport equation for the kinetic energy or the turbulent kinetic energy is available here, this particular equation on the top. Now, I will derive the mean kinetic energy equation to see what happens. So, I have mean kinetic energy, mean kinetic energy equation. So, for this we simply follow the same procedure right. So, essentially what I want is $K = \frac{1}{2}\bar{u}_i\bar{u}_i$.

So, an equation for this K, I start with the RANS equation. Start from RANS equation. The way we derived the kinetic energy, turbulent kinetic energy equation is we took the equation for fluctuating momentum right u_i' and then multiplied by another fluctuation average and so on. Since we are starting with the equation for mean kinetic energy, we start with mean fluid motion and mean momentum, right? So, RANS is nothing but your mean momentum equation or mean momentum equation. So, I have dou ui bar by dou t plus uj bar dou ui bar by dou xj equal to minus 1 by rho dou p bar by dou xi, this is the RANS equation that I am writing plus I have nu dou square ui bar by dou xj dou xj, or you can write dou xj square and then minus I have dou by dou xj of your Reynolds stresses ui prime, uj prime over bar.

$$\frac{\partial \bar{u}_i}{\partial t} + \bar{u}_j \frac{\partial \bar{u}_i}{\partial x_j} = -\frac{1}{\rho}\frac{\partial \bar{p}}{\partial x_i} + \nu \frac{\partial^2 \bar{u}_i}{\partial x_j \partial x_j} - \frac{\partial \overline{u_i' u_j'}}{\partial x_j}$$

This is my equation here. Let us call this turbulence kinetic energy equation 1 here. So, I have the RANS equation. So, all I am going to do is I will multiply this throughout by $\bar{u}_i$ and divide it throughout by 2. So, essentially, I will multiply throughout by $\bar{u}_i$.

I will start with this. So, I get, I get ui bar dou ui bar by dou t plus uj bar ui bar dou ui bar by dou xj equal to minus 1 by rho ui bar dou p bar by dou xi plus ui bar dou ui bar by dou xj. nu ui bar dou square ui bar by dou xj square minus ui bar dou ui prime uj prime bar by dou xj.

$$\bar{u}_i \frac{\partial \bar{u}_i}{\partial t} + \bar{u}_j \bar{u}_i \frac{\partial \bar{u}_i}{\partial x_j} = -\frac{1}{\rho}\bar{u}_i \frac{\partial \bar{p}}{\partial x_i} + \nu \bar{u}_i \frac{\partial^2 \bar{u}_i}{\partial x_j^2} - \bar{u}_i \frac{\partial \overline{u_i' u_j'}}{\partial x_j}$$

Now if I look at each of this term, I get for example if I write it on the on this side if I just take $\frac{\partial}{\partial t}\bar{u}_i\bar{u}_i$. If I take this particular term let us say I can use the chain rule for products and essentially it will be sum of two different terms because it is $\bar{u}_i\bar{u}_i$.

So, I can write it like this $\frac{\partial}{\partial t}\bar{u}_i\bar{u}_i$ will be $\bar{u}_i\frac{\partial \bar{u}_i}{\partial t} + \bar{u}_i\frac{\partial \bar{u}_i}{\partial t}$. So, nothing but you get two of this, it is the same. So, instead, I can write this is nothing but half dou ui bar, sorry. ui bar by dou t is equal to ui bar dou ui bar by dou t. I just used the product rule here.

$$\frac{1}{2}\frac{\partial \bar{u}_i \bar{u}_i}{\partial t} = \bar{u}_i \frac{\partial \bar{u}_i}{\partial t}.$$

So, I have the first term is $\bar{u}_i \frac{\partial \bar{u}_i}{\partial t}$. I can simply replace it with half of dou by dou t of ui bar ui bar. So, I can write that part here, which is nothing but half of dou by dou t of ui bar ui bar plus the second term let uj bar be there the advective velocity. Again, similarly I have ui bar dou ui bar by dou xj, it is a spatial derivative instead of the temporal derivative. So, similarly I get the same thing.

So, similarly, half dou ui bar ui bar by dou xj is nothing, but I get ui bar dou ui bar by dou xj. So, I should be getting the same thing here. So, I have half of this half of dou ui bar ui bar by dou xj.

$$\frac{1}{2}\frac{\partial \bar{u}_i \bar{u}_i}{\partial t} + \bar{u}_j \frac{1}{2}\frac{\partial \bar{u}_i \bar{u}_i}{\partial x_j}$$

So left hand side you can already see I have dou by dou t of a K plus uj bar dou by dou xj of K. I am already getting the equation for the K which is the mean kinetic energy.

The right-hand side, we split this, so again, using product rule, I get minus 1 by rho, just a moment let me just move this to get some space. So, this is equal to the right hand side, so I have minus 1 by rho. So, I can push the ui bar inside the derivative using product rule that will give out to be, so I get dou by dou xi p bar ui bar. plus I get sorry minus, minus I get p bar dou ui bar by dou xi. Does this term survive in an incompressible flow, constant density? No, continuity equation. So, this term is going away due to continuity equation. So, this is done, sorted.

$$= -\frac{1}{\rho}\left(\frac{\partial \bar{p}\bar{u}_i}{\partial x_i} - \cancel{\bar{p}\frac{\partial \bar{u}_i}{\partial x_i}}\right)$$

Continuity

So, I have the viscous term, and the viscous term is I have plus ui; I would like to push this ui bar inside. So, I get basically dou by dou xj of nu ui bar dou ui bar by dou xj. And then I get dou by dou xj of this term, so I get minus of, I get minus nu, sorry, yeah, let nu, let us keep the nu outside for a moment, we keep this nu here, so I get minus dou ui bar by dou xj dou ui bar by dou xj, is there something missing? A bracket, this is for, the bracket holds good for this, minus of this, correct? I push the ui bar inside the first derivative, the nu should go here, nu should come back here.

again using the product rule. So, I have the another last term here that will be I can write this let me move this again need little bit more space ok. So, I have another last term here. So, it is minus dou by dou xj of, I push this inside the derivative using product rule, I get ui prime uj prime over bar minus of this right. So, I get minus the Reynolds stress comes out I get dou ui bar by dou xj . So, now let us rearrange as we already defined the K here as $\frac{1}{2}\overline{u_l}\,\overline{u_l}$.

$$\frac{1}{2}\frac{\partial \bar u_i \bar u_i}{\partial t} + \bar u_j \frac{1}{2}\frac{\partial \bar u_i \bar u_i}{\partial x_j} = -\frac{1}{\rho}\left(\frac{\partial \bar p \bar u_i}{\partial x_i} - \bar p \underbrace{\cancel{\frac{\partial \bar u_i}{\partial x_i}}}_{\text{continuity}}\right) + \nu\frac{\partial}{\partial x_j}\left(\bar u_i \frac{\partial \bar u_i}{\partial x_j}\right) - \nu\frac{\partial \bar u_i}{\partial x_j}\frac{\partial \bar u_i}{\partial x_j} - \left(\frac{\partial}{\partial x_j}\bar u_i\,\overline{u_i' u_j'} - \overline{u_i' u_j'}\,\frac{\partial \bar u_i}{\partial x_j}\right)$$

Therefore, I get this as dou K by dou t plus uj bar dou capital k by dou xj equal to the right-hand side. So, now I do the same trick with the chronicle delta. I want dou by dou xj here, I put a delta ij. So, I get dou p bar ui bar by dou xj for that it has to be delta ij, i equal to j because it was originally delta xi. So, the pressure diffusion rate is ok.

I have the viscous diffusion rate term, so I can move this ui bar again here if you recollect this particular term, this term here ui bar dou ui bar by dou xj is already here, this particular term here. So, that is nothing but half of dou ui bar ui bar dou xj. the advection rate term that we used. So, that same term is coming. So, this is nothing but, so I have this, let me just move this a bit.

So, I have the viscous diffusion rate, which is nu, and this will be half of this nu dou by dou xj of half of dou ui bar ui bar by dou xj, and I have this another term square. And then I have this minus dou by dou xj of ui bar ui prime uj prime minus of minus plus ui prime uj prime dou ui bar by dou xj.

$$\frac{\partial K}{\partial t} + \bar u_j \frac{\partial K}{\partial x_j} = -\frac{1}{\rho}\frac{\partial \bar p \bar u_i}{\partial x_j}\delta_{ij} + \nu\frac{\partial}{\partial x_j}\left(\frac{1}{2}\frac{\partial \bar u_i \bar u_i}{\partial x_j}\right) - \nu\left(\frac{\partial \bar u_i}{\partial x_j}\right)^2 - \frac{\partial}{\partial x_j}\bar u_i\,\overline{u_i' u_j'} + \overline{u_i' u_j'}\,\frac{\partial \bar u_i}{\partial x_j}$$

Let us rewrite so that it looks similar to our, in the same format as equation 1, that is diffusion rates first, production rate and the dissipation rate. So, if I rewrite this, simply I get dou capital t plus uj bar dou k by dou xj equal to dou by dou xj, it is the divergence term of, first the pressure term which is minus 1 by rho, p bar ui bar delta ij and then I have the, what did we write there? We had the turbulent and the, ok. So, I will bring this minus ui bar, ui prime, uj prime over bar and then the viscous term which is minus nu sorry not minus, it is this one.

nu dou capital K by dou xj. So, this is, this becomes the dou by dou xj of capital k, this particular term here, this one. So, this is the diffusion rate term, and I have the production rate, which is plus ui prime uj prime average dou ui bar by dou xj and then lastly, this minus nu dou ui bar by dou xj square. So, I have the same thing diffusion rates here, diffusion rate I have production rate, but this is not the production rate of the turbulence kinetic energy. This is the production rate of the mean kinetic energy, diffusion rate, production rate and the dissipation rate of mean kinetic energy here.

$$\frac{\partial K}{\partial t} + \bar{u}_j \frac{\partial K}{\partial x_j} = \underbrace{\frac{\partial}{\partial x_j}\left\{ -\frac{1}{\rho}\bar{p}\bar{u}_i\delta_{ij} - \bar{u}_i\,\overline{u_i' u_j'} + \nu \frac{\partial K}{\partial x_j} \right\}}_{\text{diffusion rate}} + \underbrace{\overline{u_i' u_j'}\,\frac{\partial \bar{u}_i}{\partial x_j}}_{\text{production rate}} - \underbrace{\nu\left(\frac{\partial \bar{u}_i}{\partial x_j}\right)}_{\text{dissipation rate}}$$

This is the mean kinetic energy. So now look at the equation, let us call this equation 2. Everybody agrees? This is fine. So now let us compare equation 2 and equation 1. Let me just zoom out so we can see the both the equations together. So now look closely at Equation 1 and Equation 2.

This is your turbulent kinetic energy. If you closely watch, the left hand looks similar except that it is instead of k, I have K. Production, no diffusion rate term, it has all the mean quantities here except that the random component has come here, Reynolds stresses. So, the diffusion rate of mean kinetic energy has some influence of the Reynolds stresses here, this particular term here, that is fine. And this also looks similar, except that it is mean kinetic energy, the viscous diffusion rate, and this is the turbulence diffusion rate and the pressure diffusion rate. So, the diffusion rate is sorted and the dissipation rate of course, it is depending on the square of the mean strain here.

Anything else you notice which is peculiar? Is there one term which is looking exactly same? production rate, right? This term is exactly the same. Every other term is different. Dissipation and diffusion rates are completely different compared to the equation 1. Only one term that is exactly same between the mean kinetic energy equation and the turbulent kinetic energy equation is this one, this production rate. Anything else you notice? Anything different? Sign is different.

So, what does that mean? So, I have here it has become plus here, and this is minus here, but the term is the same. So, the production rate, the production rate term, the term looks the same. but the sign is different. What does this mean? So whatever it is doing to this is a rate term, right? So whatever it is doing to the mean kinetic energy, it is doing the opposite to the turbulent kinetic energy, right? We already said there was like, okay if some. Let us say that in the turbulent kinetic energy equation 1, if this term was positive, that is

$-\overline{u_i'u_j'}\frac{\partial \overline{u_i}}{\partial x_j}$ that entire term minus of this because we do not know the sign of the gradient as well as the Reynolds stresses right.

So, the Reynolds stresses this is a sum of you know the j is repeated, i is repeated. So, it will become a scalar. It will be a sum of many terms; every term is a scalar here, just like epsilon is the sum of its i is repeated j is repeated. So, let us say the entire term is positive; if this is positive, then this is negative here in the mean kinetic energy equation. Therefore, we can say that in the mean kinetic energy equation, this term is actually removing something out of it.

So, the mean kinetic energy, there is a loss of mean kinetic energy due to this term and that term is coming back here as its production. So, whatever it is removing from the mean kinetic energy, it is giving it back to the turbulent kinetic energy. So, loss of mean kinetic energy results in generation of turbulence kinetic energy so that So there is a balance of the total energy right if there is a loss from the mean it has to go into the fluctuation and therefore the name production rate right. So this is the best explanation that we can give before having or seeing any data just by looking into the equation right no need to do experiments no need to do any simulations just equations are telling us lengthy equations but it is telling us.

some information. So, this particular therefore, we call this particular equation as production rate here. So, note here that the sign is, note that sign is opposite in equation 1. Therefore, we call this a production rate term. So, whatever it is doing here, it is doing the opposite. So, its contribution is reversed in the mean and the kinetic energy equation.

Is this clear? Any doubts on this? You can think it of like that, but it is not in a directional sense. It is just, so obviously, the total energy has to be the sum of these two, the mean kinetic energy and this. So, if there is a reduction in mean kinetic energy, it must go into the turbulence kinetic energy so that the total remains the same. So, turbulence is of course that means it is draining, I also said turbulence is dissipative. So, if you do not supply energy, turbulence will decay continuously; to sustain turbulence, there must be a supply of energy.

Yeah, so the supply happens. Come see, this Reynolds stresses itself cannot supply; the supply is coming from the mean strain. If there is an absence of a mean strain in your flow, obviously, there cannot be any turbulence. So, in shear layers, boundary layers, everywhere you get sharp velocity gradients. So, this is important.

This sharp velocity gradients or the strain is important. As I already said, this production occurs as you can think of it as an action of Reynolds stresses against the mean strain.

This is taking away turbulence coming from this mean strain. So, mean strain is important to be there in your flow. If you do not have a flow with mean strain then you will not have turbulence. Any other questions? So, now this entire chapter on governing equations is I believe is sorted.

So, we have an equation for the mean momentum, which is the RANS equation, an equation for fluctuating motion that we see that by itself is not useful unless you neglect the non-linear terms, that is, the direction of stability analysis that we do not use it. But we use that fluctuating momentum equation to derive Reynolds stress equations. And from Reynolds stress equations we contracted indices to get turbulence kinetic energy equation. And then we learnt all the terms now, four terms diffusion rate, production rate, dissipation rate and redistribution rate or pressure strain rate terms. What does it do? So, we completely understand now at least to some extent what turbulence is doing.

The definition that we gave in the first class. What does it do? And I also show from the data that turbulence is also anisotropic. From the data, I showed you that part. So now we can move to another chapter. At least we start a new chapter. We are all ready for it.