18.Turbulence Kinetic Energy and it's dissipation rate - I

So, let us get started; welcome you all again. So, in the last class, we looked at the Reynolds stress equation that we applied to a plane Couette flow, and we saw the role of the pressure strain rate term, right? So, for that, we took this plane Couette flow and applied the conditions. We saw that the pressure strain rate term is responsible for this redistribution, and using the continuity equation, we showed the continuity equation for the fluctuating velocity. We showed that this is the term which is actually stealing turbulence from the direction where it is produced and giving to the directions where it is not produced. So, you may have a flow where production is dominant in all the three directions and redistribution will also be active in all the three directions you know production rates may not be equal in all the three directions, but in the simplest configuration that we have taken where production is there only in one direction, we have still seen that redistribution acts to give away turbulence to the other two directions, ok? So, now I mentioned long back that we are going to use Reynolds stress equations as the way we derived because it involves less number of unknowns and somebody asked me what is the other equation.

So, this is not useful for modeling community, but for completeness I am going to just give a note if you are interested you can do this as a homework. So, note here if you want to derive this Reynolds stress equation. And another way starting again from first principles, Reynolds stress equation, derivation from another starting point. So, what is that? Is that we take this fluctuating momentum equation.

So, we started with an equation for fluctuating velocity, we multiplied it with another velocity, another fluctuating velocity and then we averaged it to get a Reynolds stress equation. So, we start with that one, only thing is here we write it differently. So, the fluctuating, fluctuating momentum equation. This equation will be slightly different than what we have used. So, I have the left-hand side is the same i.e., $\frac{\partial u_i^{'}}{\partial t} + \overline{u_j}\frac{\partial u_i^{'}}{\partial x_j}$, equal to the pressure term is same. The fluctuating momentum equation is:

$$\frac{\partial u_i'}{\partial t} + \overline{u_j}\frac{\partial u_i'}{\partial x_j} = -\frac{1}{\rho}\frac{\partial p'}{\partial x_i} + \frac{\partial \tau_{ij}'}{\partial x_j} - u_j'\frac{\partial \overline{u_i}}{\partial x_j} - \left\{ u_j'\frac{\partial u_i'}{\partial x_j} - \overline{u_j'\frac{\partial u_i'}{\partial x_j}} \right\}$$

So, if you compare this particular equation with the equation that we have used, you will see that the only term that is different is this particular term. Every other term is the same. We just multiplied this with another fluctuation, right? $u_k^{'}$ and then averaged, we proceeded like that the starting.

This particular term was different because here we already used the incompressible consideration. So, the $\tau_{ij}^{'}$ here is nothing but your $2\nu S_{ij}^{'}$. So, we did use this condition that it is $\frac{\partial u_i^{'}}{\partial x_j}+\frac{\partial u_j^{'}}{\partial x_i}$. So, we use this and then we said by continuity one of the term will go away. So, we already use this as a simplified term and therefore, less unknowns resulted.

So, if you use this particular starting point where you do not use the incompressible flow consideration, just use $\frac{\partial \tau_{ij}^{'}}{\partial x_j}$ and you continue, you will get two extra terms. The two extra terms, where does this viscosity term, this is essentially the viscous term. So, the viscous term is different in both derivations. So, where does the viscous term affect in the Reynolds stress equation? Sorry, dissipation, right? There is a viscous dissipation rate and what does viscous term do? Sorry, yeah, that is the dissipation rate, so the viscous term here is this: we split it into two parts, and then it goes into two different terms: diffusion, right? So viscosity transports turbulence, viscosity also dissipates turbulence, so we saw that it splits into two terms, right? Therefore, this particular term when if you continue to derive, you will get It will affect both the viscous dissipation rate and the viscous diffusion rate which will have much more unknowns than what we have got.

So, this affects both the viscous dissipation rate and the viscous diffusion rate. So, this leads to far too many unknowns more than what we have, and therefore, this if you continue to derive, you will get more unknowns here. So, by continuing to derive the Reynolds equation. using this star, you will get more unknowns in the viscous terms. Therefore, modeling will not use this, will not proceed in this direction with this equation.

We will continue to use what we have derived. It has less number of unknowns. So, this is just a note. You can take it as a homework. If you are enthusiastic, you can go ahead and derive.

It is not complicated. Same technique as we did. You can see that it leads to more unknowns. So, we will come back to our original Reynolds equation and see that we can derive an another equation from it, which is very useful for the both to study turbulence as well as for modeling. So, this is what we call a turbulence kinetic energy equation.

kinetic energy small k equation. So, if you look at the by definition of this k, it is also sometimes in literature referred as turbulence kinetic energy as TKE or k is defined as half of ui prime ui prime over bar , $k = \frac{1}{2}\overline{u_i^{'}u_i^{'}}$.

Now, how is this better an energy? I said this is the, obviously I have $u_i^{'}u_i^{'}$. So, that means this is sum, repeated index Einstein summation. So, this is the sum of three terms, three normal stresses that we discussed where redistribution is occurring, right? So, this is the u1 prime, u1 prime bar plus u2 prime, u2 prime bar plus u3, sorry, u3 prime, u3 prime over bar, sum of three normal stresses, the diagonal components of the Reynolds stress tensor.

$$k = \frac{1}{2}\overline{u_i^{'}\ u_i^{'}} = \frac{1}{2}\overline{u_1^{'}\ u_1^{'}\ + u_2^{'}\ u_2^{'}\ + u_3^{'}\ u_3^{'}}\ .$$

But how is this energy? This is it is giving me meter square per second square, exactly per unit mass, I need this. So, this is per unit mass turbulence, this is per unit mass. So, this will give me kg meter square per second square. So, by definition, this is your turbulence kinetic energy. So, we need an equation for it because energy is scalar.

So, one way of looking turbulence is also that I do not want to look it as directional dependency in anisotropy, I would like to look it as turbulence energy. For that it is straightforward if you have Reynolds stress equation because you see I need $u_i^{'}u_i^{'}$ average. So, I need to simply contract the indices. So, the derivation is straightforward. So, start from Reynolds stress equation, start from Reynolds stress, Reynolds stress equation which is giving me dou by dou t of I have ui prime uk prime right, uk prime plus the advection rate of change of Reynolds stresses.

ui prime uk prime equal to, I have the diffusion rate which is dou by dou xj of, I have this minus 1 by rho p prime ui prime delta kj. plus p prime uk prime delta ij that is the pressure diffusion rate term, and then you have the viscous diffusion rate, which is plus nu dou by dou xj of ui prime, uk prime and then I have the last term, which is minus ui prime, uj prime, uk prime, the triple velocity correlation term, transport due to turbulence itself, the turbulent diffusion rate, viscous diffusion rate and the pressure diffusion rate. I simply taken the divergence term out here, dou by dou xj of these three; we have already derived this. Then I have the pressure strain rate term, which is plus p prime by rho, dou i prime by dou xk, plus dou uk prime by dou xi. And then I have the production rate term which is minus of I have uk prime uj prime dou ui bar by dou xj plus ui prime uj prime dou uk bar by dou xj. And finally, the dissipation rate term which is minus 2 nu dou ui prime by dou xj dou uk prime by dou xj. This we have already derived, the transport, the diffusion rate, the redistribution rate or the pressure strain rate, pressure strain rate, production rate and dissipation rate. So, the equation is ready. The final equation is:

$$\frac{\partial \overline{u_i'u_k'}}{\partial t} + \bar{u}_j \frac{\partial \overline{u_i'u_k'}}{\partial x_j} = \frac{\partial}{\partial x_j}\underbrace{\left\{ -\frac{1}{\rho}\left(\overline{p'u_i'}\,\delta_{kj} + \overline{p'u_k'}\,\delta_{ij} \right) + \nu \frac{\partial \overline{u_i'u_k'}}{\partial x_j} - \overline{u_i'u_j'u_k'} \right\}}_{\text{Diffusion rate}} + \underbrace{\overline{\frac{p'}{\rho}\left(\frac{\partial u_i'}{\partial x_k} + \frac{\partial u_k'}{\partial x_i} \right)}}_{\text{Pressure-Strain-rate}} - \underbrace{\left\{ \overline{u_k'u_j'}\frac{\partial \bar{u}_i}{\partial x_j} + \overline{u_i'u_j'}\frac{\partial \bar{u}_k}{\partial x_j} \right\}}_{\text{production rate}}$$
$$\underbrace{- 2\nu\, \overline{\frac{\partial u_i'}{\partial x_j}\frac{\partial u_k'}{\partial x_j}}}_{\text{dissipation rate}}$$

Now, all we have to do to derive this so-called turbulence kinetic energy equation is contract the indices, set i equal to k. So, by contracting the indices that is setting i equal to k, what do I get? I get dou by dou t of ui prime ui prime over bar plus uj bar dou by dou xj ui prime ui prime equal to dou by dou xj of, now if I said i equal to k, I get two times of this.

So, I get minus i equal to k. So, I just need to write down one of the terms. So, I get minus 2 by rho p prime ui prime dou ij, right, dou ij, i equal to k, yes, delta ij term plus I have the new dou by dou xj of ui prime ui prime over bar minus ui prime ui prime uj prime, i and k we are contracting setting i equal to k. So, i prime, i prime and j prime. and then I have the pressure strain rate term.

So, p prime by rho dou ui prime by dou i equal to k becomes dou by dou xi, two of this dou ui prime by dou xi, dou ui prime by dou xi. What happens to this term? Dou ui prime by dou xi, what is it? by continuity, it is 0. So, you see, that is why I discussed the role of pressure strain rate in the context of Reynolds stresses. Because when it comes to turbulence kinetic energy equation, this term is gone.

$$\frac{\partial}{\partial t}\overline{u_i'u_i'} + \bar{u}_j \frac{\partial \overline{u_i'u_i'}}{\partial x_j} = \frac{\partial}{\partial x_j}\left\{ -\frac{2}{\rho}\overline{p'u_i'}\,\delta_{ij} + \nu \frac{\partial \overline{u_i'u_i'}}{\partial x_j} - \overline{u_i'u_i'u_j'} \right\} + 2\overline{\frac{p'}{\rho}\overset{\text{Continuity}}{\cancel{\left(\frac{\partial u_i'}{\partial x_i}\right)}}}$$

This does not mean that redistribution is active. It is just that turbulence kinetic energy is a scalar quantity. We are trying to reduce studying turbulence as a scalar and obviously, we are going to have some loss of information. Redistribution is active at the stresses because the turbulence kinetic energy definition you can see here. It is a sum of three stresses.

This redistribution is active. When these three are being computed, right? And the sum total is only the turbulence kinetic energy, but in the equation that term goes away, redistribution rate goes away. So, the modelling community will be happy, right? So, if less than less terms means much better. And therefore, the beginning of what you call the k-based models, k epsilon and k omega, would have heard about these names. So, obviously, you would like to start with an equation with less number of unknowns. So, given Reynolds stress equation and turbulence kinetic energy, you would say turbulence kinetic energy because it has less unknowns.

But remember that you are going to have some loss of information by looking into turbulence as a scalar quantity. You lose this anisotropic nature, study of the anisotropy. So, this term is gone due to the continuity here, the pressure strain rate term, an important term. And then the production rate, so it is minus, so since i is setting equal to k, this is minus 2 of this, minus 2 of its ui prime uj prime dou ui bar by dou xj. The production rate of turbulence kinetic energy here, not the stresses and the dissipation rate, which is minus 2 nu dou ui prime by dou xj, dou ui prime by dou xj, it is squared now.

$$\frac{\partial}{\partial t}\overline{u_i'u_i'} + \bar{u}_j \frac{\partial \overline{u_i'u_i'}}{\partial x_j} = \frac{\partial}{\partial x_j}\left\{ -\frac{2}{\rho}\overline{p'u_i'}\,\delta_{ij} + \nu \frac{\partial \overline{u_i'u_i'}}{\partial x_j} - \overline{u_i'u_i'u_j'} \right\} + 2\overline{\frac{p'}{\rho}\cancel{\left(\frac{\partial u_i'}{\partial x_i}\right)}} - 2\,\overline{u_i'u_j'}\frac{\partial \bar{u}_i}{\partial x_j} - 2\nu\overline{\left(\frac{\partial u_i'}{\partial x_j}\right)^2}$$

You see, now more terms are going to make sense, what each of these terms are. So now, the definition is turbulence kinetic energy k is half of ui prime ui prime bar, but now I have an equation for the twice of that. So, let us divide throughout by half also. So, setting i equal to k and dividing throughout by 2. Each term, we divide it by 2, so I get half of this.

So, I get, I can introduce here half, same thing here. This if I do it here, half of this, 2 here by 2 here, half here. So, these two terms goes away. So, giving me, since half is a constant, I can just move it inside the derivative. So, I simply get this as dou k by dou t plus, same here, half is moved inside the derivative, so I get uj bar dou k by dou xj.

Now, I am getting a transport equation for turbulence kinetic energy. So, I have now dou by dou xj of, it is p prime ui prime by rho delta ij. Plus nu, half again goes inside the derivative, making it dou k by dou xj minus half ui prime ui prime uj prime. This is the diffusion rate of turbulence kinetic energy.

This term is gone. So, I have minus ui prime uj prime. This is the production rate of turbulence kinetic energy, and then the dissipation rate of turbulence kinetic energy, dou ui prime by dou xj squared average. So, this is your equation for turbulence kinetic energy. So, where this is the diffusion rate, the diffusion rate of turbulence kinetic energy. This is your production rate of turbulence kinetic energy, and this is the dissipation rate of turbulence kinetic energy, which we use sometimes also called as epsilon. The equation for turbulent kinetic energy is:

$$\frac{\partial k}{\partial t} + \bar{u}_j \frac{\partial k}{\partial x_j} = \underbrace{\frac{\partial}{\partial x_j}\left\{ \frac{\overline{p' u_i'}}{\rho}\delta_{ij} + \nu \frac{\partial k}{\partial x_j} - \frac{1}{2}\overline{u_i' u_i' u_j'} \right\}}_{\text{Diffusion rate of TKE}} \underbrace{- \overline{u_i' u_j'} \frac{\partial \bar{u}_i}{\partial x_j}}_{\substack{\text{production rate} \\ \text{of TKE}}} \underbrace{- \nu \overline{\left(\frac{\partial u_i'}{\partial x_j}\right)^2}}_{\substack{\text{dissipation rate} \\ \text{of TKE } (\varepsilon)}}$$

The dissipation rate, this is the dissipation rate of turbulence kinetic energy, this is also symbolically used as epsilon term, this entire term is epsilon. So, some of you would have heard of this model k epsilon model. So, where k is this transport equation for k, epsilon means the dissipation rate of turbulence kinetic energy. So, this is the epsilon term here. This particular term is epsilon, the dissipation rate of turbulence kinetic energy.